Coordination Failure in a Dynamic Game with Imperfect Information: an application to Healthcare.

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Abstract.

The paper provides a theoretical analysis of the problem of the uninsured in the healthcare industry as a coordination failure in a dynamic game with imperfect information. It is argued that if sufficient information is available, coordination failure can be avoided without centralized intervention. A necessary and sufficient condition on the information structure of the game is derived for decentralized decision making to yield a unique and efficient outcome if individuals reason using as a process of iterated dominance.

JEL Classification: .

Key Words: Coordination Failure, Dominance Solvability, Best Response Conditional Action, Dominant Conditional Action, λ^*- linked information chain; λ^*- linked information net

1 Introduction

We will study the relationship between inefficiencies that can arise in a particular type of sequential game with incomplete information and the relationship of the information structure of the game and the possibility of an inefficient outcome.

The game is motivated by markets with significant economies of scale and/or markets in which significant positive network externalities are present.

2 The Model

2.1 Notation

Let the group of agents be given by $N = \{1, 2, 3, ..., n\}$, $n \geq 2$. The individuals move sequentially[1] 1 acting first followed by 2 and then by 3 and so on. For $j \in N$, $b(j) \in \{1, 0\}$ represents j's move to purchase health insurance ($b(j) = 1$) or not to purchase health insurance ($b(j) = 0$). The vector $b = (b(1), b(2), ..., b(n))$ is an action profile. We will let $b^{-j} = (b(1), b(2), ..., b(j-1))$, denote moves that have occurred before j plays. b^{-j} will be called the j-truncated action profile. Individual utilities are determined by the benefit received by the individual $\theta > 0$ (assumed to be the same for all individuals) and the price of insurance. We assume that insurance is priced at average cost with the technology satisfying economies of scale. Thus, the average cost and hence the price, p, paid by individuals for insurance declines as the number of buyers increase. In particular, we will assume p is a strictly decreasing and positive real valued function on the real line with $p(0) = \infty$ and $p(\infty) = 0$. This implies that the action profile b determines individual payoffs with the net payoff of individual j being given by:

$$f_j(b) = [\theta - p(be)]b(j).$$

This results in the payoff vector $f(b) = (f_1(b),\ f_2(b), ..., f_n(b))$.

Under our assumptions there exists a unique positive real number λ such that $\theta - p(\lambda) = 0$. λ represents a "tipping" point. If more than λ individuals "buy" then it makes sense for *all* individuals to buy insurance and if less than λ individuals buy then the *status quo* of not buying is better for all individuals.

We will assume that (i) $\lambda > 1$: some coordination between individuals is *necessary* to improve on the *status quo.* (ii) $\lambda < n$: it is possible to improve on the status quo through coordination. We will confine our analysis to the generic case[2] where λ *is not an integer.*[3] λ^* will denote $\lceil \lambda \rceil$, the integer larger than λ. Because the number of people "buying" is an integer λ^* will, effectively, be *the* tipping point of our model.

[1] This is an assumption made to allow us to adopt a simpler notation and can be easily relaxed to allow for the case where some individuals play simultaneously. Rather than assuming, as we do, that if $j' > j''$ then j' plays *after* j'', our arguments will remain valid for the case where if $j' > j''$ then j' does not play before j).

[2] The probability of λ being an integer has probability zero (is not generic) and a slight perturbation of the model will always result in the generic case.

[3] We will comment briely on the implications of λ being an integer.

2.2 The Information Structure

Every individual, j, before making a move receives a piece of information[4] $I(j,b)$represented by either an empty set (no-information) or an integer summarizing the history of purchases by individuals who have moved before j. This information, is in general, anonymous, aggregative and (possibly) asymmetric and incomplete can be described as follows:

Each $j \in N$ is associated with an unique number $k(j) \in \{0, 1, ..., j-1\}$ and the information $I(j,b)$ is given by:

$$I(j,b) = |\{i : i \leq k(j) \text{ and } b(j) = 1\}|$$

In absence of ambiguity $I(j,b)$ will be shortened to $I(j)$.

We will make two assumptions about the information structure:

A1. *$k(j)$ is common knowledge.*

A2. *Individuals moving later have at least as much information as those who move earlier (i.e., $j \geq j'$ implies $k(j) \geq k(j')$).*

Remark 1 Information is modeled as becoming available with a lag. Individual j receives a report about how many individuals who have moved before individual $k(j)+1$ have "bought" insurance and no information about how individuals after $k(j)$ have moved. Furthermore, individual j does not know which of these individuals in $\{1, 2..., k(j)\}$ have bought insurance. Note that $k(j) = 0$ *implies* $I(j,b) = \emptyset$. This tells us that j receives no information before she moves. On the other hand, $k(j) > 0$ and $I(j,b) = 0$ gives j the information that of the first $k(j)$ individuals about whose purchases j has received a report none have bought the insurance. Individual j receives a report as to how many individuals "bought" insurance. Individual j does not know which of the individuals in in $\{1, 2, ..., k(j)\}$ have bought.

We illustrate this by showing the three possible information structures for the case where $n = 3$.

Example 1. *N={1,2,3}*; $k(1) = 0$, $k(2) = 0$, $k(3) = 0$.

Example 2. *N={1,2,3}*; $k(1) = 0$, $k(2) = 0$, $k(3) = 1$.

Example 3. *N={1,2,3}*; $k(1) = 0$, $k(2) = 0$, $k(3) = 2$.

Example 4. *N={1,2,3}*; $k(1) = 0$, $k(2) = 1$, $k(3) = 2$.

In Example 4 information becomes immediately available and each individual knows the complete *aggregate* decision making history of actions that have occurred before she moves; Example 1 is the polar opposite case where the lag in the information becoming available is "more than two periods" and thus neither of the three individuals have any information when they move. Example 2 and 3 represents an intermediate cases where 1 and 2 receive no information at all . In Example 2, the information becomes available with a one period lag and individual 3 gets informed about what 1 has done before she moves. In Example 3 "more" information becomes available after a lag than in Example

[4] This includes the possibility of some individuals receiving "no information."

2. While 1 and 2 still receive no information representing a delay in the availability of information, when it does become available after 2 periods, "more" of it becomes available in Example 3 than in Example 2 and while 3 receives an aggregate report about the actions of *both* 1 and 2 in example 3, she receives a report only about the action of individual 1 in Example 1.

The information structures can be seen as being the vector $k = (k(1), k(2), ...k(n))$. Two information structure can be compared by comparing the associated k vectors with ks which are larger in a vector sense representing a greater availability of information. In our examples above, there is a clear ranking of the information structures of the four examples: as one moves from Example 1 to 2 to 3 and finally to 4 the information available to the individuals increases. In general, with three or more individuals, some comparisons between information structures may not be possible and the structures would in general be quasi-ordered.[5]

2.3 The Normal Form Game.

If $k(j) \neq 0$, a strategy for j is a $|k(j)| + 1$ dimensional binary vector $a(j) = (a_1(j), a_2(j), ..., a_{k(j)+1}(j))$ of conditional actions representing which of the two moves (buy or not buy) the individual would choose when she receives the information $I(j) \in \{0, 1, ..., k(j) + 1\}$. $a_l(j)$ the l^{th} co-ordinate of j's strategy. $a_l(j)$ is j's move $b(j)$ if the pre-requisite $I(j) = l - 1$ is satisfied.

If $I(j) = \emptyset$, then the individual's strategy is a unidimensional vector given by $a(j) = b(j)$ and represents the move that the individual would make with the information.

We will adopt the usual notation $a = (a(j), a(-j))$ to denote a strategy profile with $a(-j)$ being a possible contingency representing strategies adopted by individuals other than j. We will also use a^{-j} to denote the j-truncated contingency describing the strategies of all individuals who move before j. a^{-j} determines $I(j)$ which together with $a(j)$ determines $b(j)$.

A payoff function for the normal form game is the mapping F of strategy profiles to payoff profiles. $F(a)$ can be thought of as a composition of two functions $b = g(a)$ from strategy profiles to moves and $f(b) = (f_1(b), f_2(b), ..., f_n(b))$ from moves to individual payoffs with j's payoff being $F_j(a) = f_j(b(a))$.

Remark 2 The function g has a recursive structure: the truncated contingency a^{-j} determines the truncated profile of moves b^{-j} of all the individuals who play before j. This gives us $I(j)$ which together with $a(j)$ determines $b(j)$.

For any strategy profile a the l^{th} co-ordinate of j's strategy $a_l(j)$ is on the path of play iff I(j,b)=I(j,g(a))=l-1 (i.e., the prerequisite for the l^{th} co-ordinate is satisfied.) Otherwise, we will say $a_l(j)$ is off the path of play.

Proposition 3 is a direct consequence of the recursive structure of the game while Proposition 4 follows from our assumption that p is strictly decreasing and λ is not an integer.

[5] For instance consider the model with three individuals with information structures $(0, 0, 2)$ and $(0, 1, 1)$.

Proposition 3 *Starting from any strategy profile changes in any set of individual strategies off the path of play does not affect the payoffs of any individual* .

Proposition 4 *Starting from any strategy profile changes in any one individual j's strategy on the path of play always affects payoff of that individual.*

3 Equilibrium and Coordination Failure

Given any normal form game $\mathcal{G}$, a strategy $a(j)$ of individual j is *a best response strategy for the contingency* $a(-j)$ if and only if for all $a'(j)$, $F_j(a) = F_j(a(j), a(-j)) \geq F_j(a'(j), a(-j))$. A *pure strategy Nash equilibrium (PSNE)* can then be defined in the usual way as a strategy profile a such that for all individuals h, $a(h)$ is a best response strategy for the contingency $a(-h)$. The associated payoff of PSNEs are *pure strategy Nash equilibrium outcomes (PSNEOs).* Given that our normal form game may have different PSNEOs we will focus on analyzing conditions on the information structure of the model which under decentralized rational decision making would lead to the elimination of inefficient Nash equilibria and leave only those PSNEs whose PSNEO gives every individual the maximum utility: $(\theta - c(n))$.

We will assume that rational individuals in making their decisions will use the process of iterated dominance described below:

A strategy $a'(j)$ is dominated by a strategy $a(j)$ if j's payoff is just as large under $a(j)$ as under $a'(j)$ under every contingency and is strictly larger for some contingency.[6]

Consider our normal form game $\mathcal{G}_0$ and let $\mathcal{R}$ be a function which gives us the game $\mathcal{G}_1 = \mathcal{R}(\mathcal{G}_0)$ obtained by eliminating *all the dominated strategies* of all the individuals. Rational individuals would know that other individuals being rational would never play these dominated strategies. Thus it is possible to generate a sequence of games $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$....where $\mathcal{G}_{h+1} = \mathcal{R}(\mathcal{G}_h)$. If $\mathcal{G}_p = \mathcal{R}(\mathcal{G}_p)$ we will call the game irreducible and given that for each individual the number of strategies is finite such an irreducible game will always exist. We will analyze the sequence $\{\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m\}$ where $\mathcal{G}_m$ is the "first" irreducible game in the sequence.[7]

$\mathcal{G}_0$ is dominance solvable iff $\mathcal{G}_m$ has a unique PSNEO.

This process of iterated dominance leads us to only *one* PSNEO or to more than one. If it leads to just one PSNEO we can have greater confidence that this outcome will occur and the question arises as to whether this outcome is welfare maximizing (efficient). Since there is only one efficient outcome of the game, if there is more than one PSNEO some form of coordination failure remains a possibility.

[6] For all $a(-j)$, $F_j(a) \geq F_j(a'(j), a(-j))$ and for some $\overline{a}(-j)$, $F_j(a(j), \overline{a}(-j)) > F_j(a'(j), \overline{a}(-j))$.

[7] All games in the sequence after $\mathcal{G}_m$ areidentical to $\mathcal{G}_m$ and all games in the sequence from $\mathcal{G}_0$ to $\mathcal{G}_m$ are different from each other.

Weak Coordination Failure: Decentralized rational decision making may lead to weak coordination failure iff $\mathcal{G}_m$ has a PSNE a^* such that for some j : $F_j(a^*) \neq \theta - c(n)$.

Strong Coordination Failure: Decentralized rational decision making may lead to strong coordination failure iff $\mathcal{G}_m$ has a PSNE a^* such that for all j : $F_j(a^*) = 0$.

Under weak coordination failure the maximum possible payoff in our model is not received (in an equilibrium) by some individual. Under strong coordination failure, in some equilibrium, every individual receives the *status quo* payoff and the benefits from cooperation are completely lost.

Proposition 5 *If there exists a strategy profile $\overline{a}$ in $\mathcal{G}_m$ such that for all individuals, all co-ordinates less than or equal to the second co-ordinate are zero, then decentralized rational decision making may lead to strong coordination failure.*

Proof. In this profile $\overline{a}$ the path of play gives us the *status quo*. The change in her strategy by any individual off the path of play (Proposition 3) does not alter the outcome. A change in any individuals strategy on the path of play necessarily consists of changing the first co-ordinate of the strategy from 0 to 1. However, given that both the first and second co-ordinates of the strategies of all other individuals are 0, $\lambda^* \geq 2$, implies that such a change will reduce the payoff of the individual making the change. Thus, $\overline{a}$ is a Nash equilibrium and the *status quo* is PSNEO of $\mathcal{G}_m$. ■

While a strong coordination failure implies weak coordination failure, is the converse true? It clear that if $\mathcal{G}_0$ is not dominance solvable weak coordination failure will occur. What is the relation between the information structure of the game, dominance solvability and coordination failure? In the rest of the paper we provide a complete set of answers to these question.

4 λ^*-Linked Information Chains

Whether there is coordination failure or not depends on the amount of information that individual players, j, have prior to making their move and the extent to which the information structure of the model allows this information to filter through to players who play after j. This can be described using the following concepts of an information cover of an individual and of a λ^*-linked chain.

$$js \textit{ information covers } j' \text{ iff } j' \leq k(j).$$

Thus, j's information covers j' if j''s knowledge about the aggregate history of the game and the impact if any of j's action on this aggregate information filters through to player j.

Remark 6 *If j's information covers j' it is clear from our definition that j's information also covers all $j'' < j$.*

A β-linked information chain is an ordered set $(i_1, i_2...., i_\beta)$ of members of N such that $k(i_{p-1}) \geq i_p$ for all $p = 2, ..., \beta$.

Clearly, the existence of a β-linked chain would imply the existence of such a β'-linked chain if $\beta' < \beta$.

The following sets (defined recursively) are related to the existence of a β-linked chain:

$$
\begin{aligned}
K_\beta(1) &= \{j : k(j) \geq \beta - 1\} \\
K_\beta(2) &= \{j' : k(j') \geq \beta - 2 \text{ and there exists } j \in K_\beta(1) \text{ such that } j' \leq k(j)\} \\
&\quad .... \\
K_\beta(h) &= \{j' : k(j') \geq \beta - h \text{ and there exists } j \in K_\beta(h-1) \text{ such that } j' \leq k(j)\} \\
&\quad ... \\
K_\beta(\beta) &= \{j' : k(j) \geq 0 \text{ and there exists } j \in K_\beta(\beta - 1) \text{ such that } j' \leq k(j)\}
\end{aligned}
$$

If a β-linked chain $(i_1, i_2...., i_\beta)$ exists, $K_\beta(1), K_\beta(2), ..., K_\beta(\beta)$ are all non-empty with $i_\tau \in K_\beta(\tau)$.[8] Also, if $K_\beta(\beta) \neq \varnothing$ it is clearly possible[9], using the definition of $K_\beta(\cdot)$, to choose $i_\tau \in K_\beta(\tau)$, $\tau = \beta, \beta - 1, ..., 2, 1$ such that *a* β-linked information chain $(i_1, i_2...., i_\beta)$ of members of N is formed with $k(i_{p-1}) \geq i_p$ being true for for all $p = 2, ..., \beta$. Thus, non-emptyness of $K_\beta(\beta)$ is necessary and sufficient for the existence of a β-linked chain.

Proposition 7 *A β-linked information chain is exists iff $K_\beta(\beta) \neq \varnothing$.*

The information structure of the game completely determines the cover structure and the cover structure together with λ^* (the tipping point) determines whether a λ^*-linked information net exists. We will argue that the existence and nonexistence of a λ^*-linked net determines whether or not there is coordination failure. We illustrate this below using examples designed bring out the nature of this relationship.

Consider the four examples used earlier with $N = \{1, 2, 3\}$.

In Example 4 we had $k(3) = 2$, $k(2) = 1$, $k(1) = 0$. Thus 3 covers 2 and 3 and 2 cover 1 and a β-linked information chains for $\beta = 1, 2, 3$ exist. In this case, even with $\lambda^* = 3$ the normal form game is dominance solvable with one buying when ever he sees two people have bought (in the first round of elimination of dominated strategies), $k(.)$ being common knowledge, 2 recognizes this buying when ever he sees person 1 buying (this is the second round of iterated elimination of dominated strategies) and finally 1 recognizing that if she buys all others will also buy goes ahead and buys. This type of argument with two steps instead of three would be true if $\lambda^* = 2$[10] Thus with this information structure all the individuals would buy and coordination failure would not occur. In Example 3 we had $k(1) = 0$, $k(2) = 0$, $k(3) = 2$. If $\lambda^* = 3$ the game would not be dominance solvable but it would be dominance solvable if $\lambda^* = 2$.

[8] Note that $k(i_1) \geq \beta - 1$.

[9] If $K_\beta(\tau)$ is empty and $K_\beta(\tau + 1)$ will also be empty.

[10] Recall that $\lambda^* \geq 2$.

Two different 2-linked information chains $(3, 2)$ and $(3, 1)$ exist while a 3-linked information chain does not.[11]

In Example 2 with $k(1) = 0$, $k(2) = 0$, $k(3) = 1$.The game would be not be dominance solvable for $\lambda^* = 3$ and a 2-linked information chain does not exist. However, $(3, 1)$ is a 2-linked information chain and the game is dominance solvable with $\lambda^* = 2$.

The game in Example 1 is not dominance solvable for $\lambda \geq 2$ and contain no 2-linked or 3-linked information chains.

Examples 1 to 4 provide the intuition underlying the following theorem:

Theorem *The following statements are equivalent:*

1. $\mathcal{G}_0$ is dominance solvable.

2. $\mathcal{G}_m$ has a unique PSNEO with $F_j = \theta - c(n)$

3. Weak Coordination Failure does not occur.

4. Strong Coordination Failure does not occur.

5. A λ^*-linked information chain exists.

6. $K_{\lambda^*}(\lambda^*) \neq \varnothing$

5 Lemmas and Proofs

To prove our principal result we will need to analyze the relationship between conditional moves along the path of play and undominated strategies. To do this we introduce three concepts: Null Prerequisite (NP), BRCA (Best Response Conditional Action) and DCA (Dominant Conditional Action) in a game. A null prerequisite in a game is a prerequisite that is not possible in that game; a BRCA is a conditional action that is a best response to *some* (possible) contingency in the game and a DCA is a conditional action that represents a best response to *all* possible contingencies that can arise in the game. During the process of iterative removal of dominated strategies, co-ordinates in a strategy with NPs become irrelevant and both types of best response conditional actions are preserved. We will show that a BRCA of an individual is present in some undominated strategy of that individual and that a DCA populates all undominated strategies of the individual.

Let $\mathcal{G} \in \{\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m\}$

NP: The l^{th} co-ordinate of individual j, $a_l(j)$ has a null prerequisite (NP) in $\mathcal{G}$ iff there does not exist a strategy profile in $\mathcal{G}$ for which the prerequisite for $a_l(j)$ is satisfied.

BRCA: The entry in the l^{th} co-ordinate of individual j's strategy $a_l^*(j)$ is a BRCA for the profile $a^* = (a^*(j), a^*(-j))$ in $\mathcal{G}$ iff $a_l^*(j)$ is the l^{th} co-ordinate of a best response strategy $a^*(j)$ for the contingency $a^*(-j)$ and for the strategy profile a^*, $a_l^*(j)$ is on the path of play.

DCA: The entry in the l^{th} co-ordinate of individual j's strategy, $a_l^{**}(j)$ is a DCA in $\mathcal{G}$ iff it is a BRCA for all profiles a for which $a_l^{**}(j)$ is on the path of play.

[11] This is so because $k(2) = 0$.

Since during the process of iterated dominance strategies get removed, a contingency that does not occur in a game never occurs in any other game that follows it in the iterative process. This gives us the following lemma:

Lemma 8 *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$. If the l^{th} co-ordinate of individual j's strategy, $a_l(j)$ has a NP in $\mathcal{G}_h$ then the l^{th} co-ordinate of individual j's strategy, has a NP in $\mathcal{G}_{h+1}$.*

Remark 9 Note that by Proposition 3 and the definition of a BRCA, the conditional action $a_l^*(j)$ is on the path of play for the profile a^* implies that the prerequisite $I(j) = l - 1$ is satisfied both for the contingency $a^*(-j)$ and for the j-truncated contingency a^{*-j}.

The following lemma provides a complete characterization of undominated strategies in a game in terms of BRCAs and NPs.

Lemma 10 *(Weak Persistence Lemma) Let $\mathcal{G} \in \{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$ and let $k(j)+1 \geq l$. (i) If $a^*(j) = (a_1^*(j), a_2^*(j), ..., a_{k(j)+1}^*(j))$ is a strategy such that for all co-ordinates $l \in \{1, 2, ...a_{k(j)+1}^*(j)\}$ with a non-null pre-requisite in $\mathcal{G}$, if $a_l^*(j)$ is a BRCA, then $a^*(j)$ is undominated in $\mathcal{G}$. (ii) If a strategy $a^*(j)$ is undominated in $\mathcal{G}$ then $a^*(j) = (a_1^*(j), a_2^*(j), ..., a_{k(j)+1}^*(j))$ is such that either $a_l^*(j)$ is a BRCA of j in $\mathcal{G}$ or $a_l^*(j)$ has has a null prerequisite in $\mathcal{G}$.*

Proof. (i) Assume to the contrary that a strategy $\overline{a}(j)$ dominates $a^*(j)$. By the definition of dominance, for every possible contingency $a(-j)$ of j in $\mathcal{G}$, j's payoff must be at least as large with $\overline{a}(j)$ as with $a^*(j)$ and for some contingency it must be larger. Consider any non-null co-ordinate, l. For some contingency, by our hypothesis, $a_l^*(j)$ is a best response to this contingency and therefore if $\overline{a}_l(j) \neq a_l^*(j)$, the payoff for individual j would be lower under $\overline{a}(j)$. Hence, it must be the case that for all non-null contingencies $\overline{a}_l(j) = a_l^*(j)$. Since all other co-ordinates have null contingencies and are never on the path of play, it follows that for all contigencies $a^*(j)$ and $\overline{a}(j)$ have the same payoff. This contradicts the assumption that for some contingency the payoff for j from $\overline{a}(j)$ is larger than from $a^*(j)$.

(ii) Assume to the contrary that for some undominated strategy $a^*(j)$ there exists an $a_l^*(j)$ such that $a_l^*(j)$ is neither a BRCA of j in $\mathcal{G}$ nor has a null prerequisite in $\mathcal{G}$. Construct a strategy $a^{**}(j)$ from $a^*(j)$ by replacing *every* co-ordinate with a non-null pre-requisite for which the l^{th} co-ordinate is not a BRCA in $a^*(j)$ with a BRCA.[12] Note that for the strategy $a^{**}(j)$ that we have constructed from $a^*(j)$, all co-ordinates that have non-null prerequisites are BRCAs.[13] Since the values of the "null co-ordinates" do not matter for the

[12]This is possible because by the violation of NP at the l^{th}co-ordinate there exists some profile satisfying the pre-requisite for coordinate l. For any such contingency we can find the BRCA.

[13]Note that j has a strategy with the co-ordinates a_l^{**} as constructed. This is so because as one moves along the sequence of games $\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m$ using iterative removal of strategies,

outcome of the game, it follows that $a^{**}(j)$ dominates $a^{*}(j)$:[14] a contradiction. ■

The following is a consequence of Lemma 10:

Lemma 11 *(Strong Persistence Lemma) Let $\mathcal{G}_h \in \{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$ and $k(j)+1 \geq l$. (i) If $\widetilde{a}_l(j)$ is a DCA of j in $\mathcal{G}_h$ then for all strategies of j in $\mathcal{G}_{h+1}$, $\widetilde{a}_l(j)$ occupies the l^{th} co-ordinate. (ii) If $\widetilde{a}_l(j)$ occupies the l^{th} co-ordinate of all strategies of j in $\mathcal{G}_{h+1}$ then either the l^{th} co-ordinate j has a NP in $\mathcal{G}_h$ or $\widetilde{a}_l(j)$ is a DCA in $\mathcal{G}_h$.*

Proof. (i) Since a DCA is a BRCA it necessarily violates NP for the l^{th} co-ordinate of j in $\mathcal{G}_h$ and by Lemma 10 (i), for some undominated strategy, $\widetilde{a}_l(j)$ occupies the l^{th} co-ordinate. However, since the NP is violated for the l^{th} co-ordinate of j in $\mathcal{G}_h$, by Lemma 10 (ii), using the definition of DCA, it must be the case that *all* strategies of j in $\mathcal{G}_{h+1}$, $\widetilde{a}_l$ occupies the l^{th} co-ordinate.

(ii) Note that $\widetilde{a}_l(j)$ occupies the l^{th} co-ordinate in *all* of j's strategies in $\mathcal{G}_{h+1}$, in our case, this implies that $a_l(j)$ occupies the l^{th} co-ordinate for *some of* j's strategies in $\mathcal{G}_h$[15] and any such strategy must be undominated in $\mathcal{G}_h$. By Lemma 10(ii), $\widetilde{a}_l(j)$ is either a BRCA of j in $\mathcal{G}_h$ or has a null prerequisite in $\mathcal{G}_h$. Since $\widetilde{a}_l(j)$ occupies the l^{th} co-ordinate of all the strategies of j in $\mathcal{G}_h$ and if the pre-requisite of the l^{th} co-ordinate is non-null it follows by Lemma 10(i) that $\widetilde{a}_l(j)$ is a DCA. ■

The following sets of "buy" and "not buy" strategies for any game $\mathcal{G}$ will play an important role in our analysis by ensuring that the pre-requisites for certain specific co-ordinates are non-null and allowing us to fully exploit the weak and strong persistence lemmas. The "buy set" is a set of strategy profiles for which all individuals buy with the information that to the extent that they are aware all previous players have bought and have bought. Similarly, the "not buy" set is a set of strategy profiles for which all individuals do not buy based on the information that to the extent that they are aware all previous players have not bought bought.

$$\begin{aligned} B &= \{a : \text{For all } j,\ a_{k(j)+1}(j) = 1\} \\ NB &= \{a : \text{For all } j,\ a_0(j) = 0\} \end{aligned}$$

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contingencies present in later games are present in earlier games and therefore a BRCA in game G_{h+1} would necessarily be a BRCA in game G_h for $h = 1, 2...m$. Hence one can use, (i) of this lemma and the game $\mathcal{G}_0$ (in which there are no null prerequisites) to show the existence of a strategy with these BRCAs in game $\mathcal{G}$.

[14] Recall that conditional actions can take only two values 1 and 0. This implies that if a non-null co-ordinate is not a BRCA (i.e., it is not a best response for any contingency of the game for which it is on the path of play) then it gives a lower payoff for *all* contingencies for which this co-ordinate lies in the path of play. This is so because the BRCA for any co-ordinate on the path of play is, inder our assumptions, unique.

[15] Note that in this case, this is a logical implication because the question of existence is not in issue.

Proposition 12 *Let $\mathcal{G} \in \{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$.(i) $B \neq \varnothing$ and for all $\overrightarrow{a} \in B$, $\overrightarrow{a}$ is a PSNE. (ii) If there exists a strategy profile in $\mathcal{G}$ such that for all j, $a_l(j) = 0$ for all $l \leq 2$ in $\mathcal{G}$, then $NB \neq \varnothing$ and for all $\overleftarrow{a} \in B$, $\overleftarrow{a}$ is a PSNE.*

Proof. (i) Consider a strategy profile a in $\mathcal{G}_0$ for individuals $j \in \{1, 2....., n\}$ such that $a_l(j) = 1$ for $l = k(j)+1$ and $a_l(j) = 0$ for all $l \neq k(j)+1$.[16] (Since no strategies have been eliminated using iterated dominance in $\mathcal{G}_0$, such a profile is possible.) Clearly, $a_l(j) = 1$ for $l = k(j) + 1$, since this gives the maximum possible payoff for each individual, is a BRCA of j in $\mathcal{G}_0$ and this strategy profile is a PSNE of the game. Now, by our weak persistence lemma (Lemma 10) there exists an undominated strategy profile in $\mathcal{G}_0$ such that for all $j \in N$ in $a_l(j) = 1$ for $l = k(j) + 1$ which by definition of $\mathcal{G}_1$ is a strategy profile in $\mathcal{G}_1$. Observe that for this strategy profile in $\mathcal{G}_1$ with $a_l(j) = 1$ for $l = k(j) + 1$ is a BRCA in $\mathcal{G}_1$ and is a PSNE of $\mathcal{G}_1$,thus ensuring the existence of a strategy profile with these properties in $\mathcal{G}_2$. Thus, using our weak persistence lemma (Lemma 10) repeatedly (i) of the Lemma follows.

(ii) The the proof is similar to (i) above and is seen by using the argument in Proposition 5 and repeatedly applying our weak persistence lemma (Lemma 10).

The following is an immediate consequence of Proposition 12(i).

Corollary 13 *$\mathcal{G}_0$ is dominance solvable iff both weak and strong coordination failure do not occur.*

■

In the results that follow, assuming the existence of a strategy profile such that the first $l-1$ co-ordinates are zero and less than $(\lambda^* - 1)$ for all individuals, we will use Proposition 12 to a construct a strategy profile depending on j and l such that for any j and any l such that $k(j) \geq l - 1$ the j-truncated profile ensures that $a_l(j)$ is on the path of play. This is done by choosing the strategies of the first $(l-1)$ individuals from B and making sure that the strategies of the other individuals plating before j pass through a co-ordinate with a zero.

Lemma 14 *(Exact Contingency Lemma). Let $\mathcal{G} \in \{\ \mathcal{G}_1, ..., \mathcal{G}_m\}$, $r \leq (\lambda^* - 1)$ and let $\widehat{a}$ be any strategy profile in $\mathcal{G}$ such that for all j and for all $l \leq r$, $\widehat{a}_l(j) = 0$. Then, there exists a strategy profile a in $\mathcal{G}$ such that (i) , If $b = g(a)$ then for all j $I(b, j) = \min\{l - 1, k(j)\}$. (ii) For all $l \leq (\lambda^* - 2)$ and for all j' such that $k(j') \geq (l - 2)$, for $a_l(j') = 0$ is a BRCA in $\mathcal{G}$.*

Proof. (i) Consider the strategy $\overrightarrow{a}$ as defined in Proposition 12 (i). (Note that under this profile, *all* individuals buy conditional on all individuals before them have bought.) Consider the $\overrightarrow{a}^{\,-l}$ truncated profile (i.e. let all the individuals upto and including individual $l-1$ play the strategies in $\overrightarrow{a}$.) Using Proposition 12 (i) such a l truncated profile exists. For all $j > l - 1$, let the individuals

[16] Thus, the largest co-ordinate in each individual's strategy is 1.This tells us that each individual buys if according to the information available to him, no individual moving before him has "not bought."

play the strategies specified in the profile $\widehat{a}(j)$ (in the hypothesis of the lemma). For any profile $a^* = (\overrightarrow{a}_{\{1,2,3...l-1\}}, \widehat{a}_{\{l,l+1,...n\}})$, since for all j and for all $l \leq r$, $\widehat{a}_l(j) = 0$:

$$\begin{aligned} b_j(a) &= 1 \text{ for all } j \leq l-1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

Thus, for all j such that $k(j) \geq l-1$ the prerequisite for $a_l(j)$ is (exactly) satisfied and for all j and for all j such that $k(j) < l-1$, $I(b,j) = k(j)$.

(ii) Note that for the profile $\overrightarrow{a}$ in (i) above, for all j' such that $k(j') \geq l-1$ we have $|\{j'' : b_{j''} = 1\}| = l-1$. Thus, changing $a_l(j') = 0$ to $a_l(j') = 1$ would be on the path of play and would increase the total number of purchases from $l-1$ to l. But, since $l \leq (\lambda^* - 2)$ all players j'' after j' would have the path of play passing through either $a_l(j'') = 0$ or through $a_{l+1}(j'') = 0$. Thus this change in j's strategy would at most increase the number of purchases to at most $l \leq (\lambda^* - 1)$. This would result in reducing the individual j''s payoff from the *status quo* payoff of 0 to a negative payoff. This implies $a_l(j') = 0$ is a BRCA in $\mathcal{G}$. ■

The general reduction lemma describes conditions under which a BRCA of zero gets eliminated through dominance if the hypothesis of Lemma 14 (i) is satisfied.

Lemma 15 *(General Reduction Lemma) Let $\mathcal{G}_h \in \{\mathcal{G}_1, ..., \mathcal{G}_m\}$, $r \leq (\lambda^* - 1)$ and let $\widehat{a}$ be any strategy profile in $\mathcal{G}_h$ such that for all j and for all $l \leq r$, $\widehat{a}_l(j) = 0$. Then, (i) $a_l(j) = 1$ is a DCA in $\mathcal{G}_h$ implies there exists j' such that $k(j') \geq j$ and $a_{l+1}(j') = 1$ is a DCA in $\mathcal{G}_h$. (ii) For all j, if there exists j' such that $k(j') \geq j$ and for all $l' \geq l$, $a_{l'+1}(j') = 1$ is a DCA in $\mathcal{G}_h$ then $a_l(j) = 1$ is a DCA in $\mathcal{G}_h$.*

Proof. (i) Assume to the contrary that $a_l(j) = 1$ is a DCA in $\mathcal{G}_h$ and there does not exist j' such that $k(j') \geq j$ and $a_{l+1}(j') = 1$ is a DCA in $\mathcal{G}_h$. (i.e., using the strong persistence lemma (Lemma **??**) for all $j' > j$ such that $k(j') \geq j$ either $a_{l+1}(j')$ has a null prerequisite or there exists a strategy $\widetilde{a}_{l+1}(j')$ in $\mathcal{G}_h$ such that $\widetilde{a}_{l+1}(j') = 0$).

Following the proof of the exact contingency lemma, consider an exact contingency profile $a^* = (\overrightarrow{a}_{\{1,2,3...l-1\}}, \widehat{\widehat{a}}_{\{l,l+1,...n\}})$ where $\widehat{\widehat{a}}\ (j)$ are such that (i) for all $j' > j$, if $k(j') > j$, then $\widehat{\widehat{a}}_{l+1}(j') = \widetilde{a}_{l+1}(j') = 0$ (ii) For all $j' \geq l-1$, $j \neq j'$ and $k(j') < j$, $\widehat{\widehat{a}}_l(j') = \widehat{a}_l(j) = 0$. We know that for the profile $(a^*(-j), \overline{a}(j))$ if $\overline{a}_l(j) = 1$ exactly $l \leq \lambda^* - 1$ individuals would have purchased the good and hence $a_l(j) = 0$ would be a BRCA. This contradicts $a_l(j) = 1$ is a DCA.

(ii) Under the hypothesis of (ii) for any profile in $\mathcal{G}_h$ for which $a_l(j)$ is in the path of play will be such that for some $j' > j$ and for some $l' \geq l$ we would have $a_{l'+1}(j')$ on the path of play. Now, if $a_{l'+1}(j') = 1$ is a DCA in $\mathcal{G}_h$, by the definition of a DCA, at least λ^* individuals would buy and $a_l(j) = 1$ would be a BRCA Since this is true for any for any profile in $\mathcal{G}_h$ for which $a_l(j)$ is in the path of play, $a_l(j) = 1$ is a DCA.

The lemmas 15 (i) give us the following corollary: ■

Corollary 16 *Let $\mathcal{G}_h \in \{ \mathcal{G}_1, ..., \mathcal{G}_m\}$, $r \leq (\lambda^* - 1)$ and let $\widehat{a}$ be any strategy profile in $\mathcal{G}_h$ such that for all j and for all $l \leq r$, $\widehat{a}_l(j) = 0$. The set $\{j' : a_{l+1}(j') = 1$ is a DCA in $\mathcal{G}_h\} = \varnothing$ implies that for $r \leq (\lambda^* - 1)$ there exists a strategy profile $\widehat{\widehat{a}}$ in $\mathcal{G}_{h+1}$ such that for all j and for all $l \leq r$, $\widehat{\widehat{a}}_l(j) = 0$.*

In the next lemma we argue that if there is a strategy profile under which for all individuals and all co-ordinates less than or equal to the r^{th} co-ordinate is zero in some game $\mathcal{G}_h$, then in the game $\mathcal{G}_{h+1}$ in some strategy profile for all individuals all co-ordinates upto the $(r-1)^{\text{th}}$ co-ordinate will be zero. This tells us that a maximum possible reduction of "zeros" per round is one.

Lemma 17 *(Maximal Reduction Lemma) Let $\mathcal{G}_h \in \{ \mathcal{G}_1, ..., \mathcal{G}_m\}$, $r \leq (\lambda^* - 1)$ and let there be a strategy profile $\widehat{a}$ in $\mathcal{G}_h$ such that for all j and for all $l \leq r$, $\widehat{a}_l(j) = 0$. Then, there exists a strategy profile $\overline{a}$ in $\mathcal{G}_{h+1}$ such that $\overline{a}_l(j) = 0$ for all $l \leq r - 1$.*

Proof. Assume to the contrary that the hypothesis of the lemma is satisfied and there exists an individual j' such that for some $l \leq r - 1$, $a_l(j') = 1$, in all strategy profiles in $\mathcal{G}_{h+1}$.

For all $l \leq r - 1$ since $r \leq (\lambda^* - 2)$ we get $l \leq \lambda^* - 2$. Thus by Lemma 14 (ii) for all j', such that[17] $k(j') \geq l - 1$, $a_l(j') = 0$ is a BRCA in $\mathcal{G}_h$. By the weak persistence lemma (Lemma 10), it follows that that for each such j', $k(j') \geq l - 1$, there exists a strategy profile $a^*(j')$ in $\mathcal{G}_h$ with $a_l^*(j') = 0$ that is undominated in $\mathcal{G}_h$. This contradicts our assumption that there exists an individual j' such that $a_l(j') = 1$ in all strategy profiles in $\mathcal{G}_{h+1}$. ■

The following proposition is a direct application of the maximal reduction lemma (Lemma 17):

Proposition 18 *Let $\mathcal{G}_h \in \{ \mathcal{G}_1, ..., \mathcal{G}_m\}$. Then, $\mathcal{G}_h$ satisfies the hypothesis of maximal reduction and exact contingency lemmas (Lemmas 14 and 17) with $r = \lambda^* - h$.*

Proof. In $\mathcal{G}_0$ since all strategies are possible (i.e., no strategy has been eliminated using the iterative domination process) for j we know that for all $l \leq r = \lambda^* - 1$, there exists a contingency for j such that $a_l(j) = 0$ is a BRCA and thus by the weak persistence lemma (Lemma10), $\mathcal{G}_1$ satisfies the hypothesis of maximal reduction lemma (Lemma17) with $r = \lambda^* - 1$. We can now proceed applying the lemma sequentially. We have established that for $h = 1$, $\mathcal{G}_h$ satisfies the hypothesis of maximal reduction lemma (Lemma17) with $r = \lambda^* - h$. Applying the maximal reduction lemma (Lemma17 (i)), repeatedly it follows that $\mathcal{G}_h$ satisfies the hypothesis of maximal reduction and exact contingency lemmas (Lemmas 14 and 17) with $r = \lambda^* - h$.

The corollary uses the proposition above to impose an lower limit on the length of the sequence $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$ if the game is dominance solvable. ■

[17] If $k(j') < l - 1$, $a_l(j)$ does not exist and thus $a_l(j) = 1$ is impossible.

Corollary 19 *Consider the sequence of games* $\{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$*.* $\mathcal{G}_0$ *is dominance solvable implies* $m \geq \lambda^*$*.*

Proof. Applying Proposition 18, $\mathcal{G}_m$ satisfies the hypothesis of maximal reduction lemma with $r = \lambda^* - m$. For $\mathcal{G}_0$ to be dominance solvable (corollary 13) in $\mathcal{G}_m$ all individuals must be buying and for individual 1, since $k(1) = 0$ we must have $a_1(1) = 1$ for all strategies of 1 in $\mathcal{G}_m$. Thus, $\mathcal{G}_m$ can satisfy the hypothesis of maximal reduction lemma (Lemma17) with $r = \lambda^* - m$ only if $\lambda^* - m \leq 0$. ■

To prove our theorem, with each game $\mathcal{G}_h \in \{\mathcal{G}_0, \mathcal{G}_1, ..., \mathcal{G}_m\}$ we will associate two sets of individuals, $\overline{K}(\mathcal{G}_h)$ and $\overline{\overline{K}}(\mathcal{G}_h)$ as follows:

$$\begin{aligned}\overline{K}(\mathcal{G}_h) &= \{j' : a_{\lambda^*-h}(j') = 1 \text{ is a DCA in } \mathcal{G}_h\} \\ \overline{\overline{K}}(\mathcal{G}_h) &= \{j' : a_{\lambda^*-h'}(j') = 1 \text{ is a DCA in } \mathcal{G}_h \text{ for all } h' \leq h \text{ and } k(j') \geq \lambda^* - h'\}\end{aligned}$$

The next proposition establishes that $\overline{K}(\mathcal{G}_h)$ being non-empty is a necessary condition for dominance solvability of $\mathcal{G}_0$.

Proposition 20 *Let* $\mathcal{G}_h \in \{\ \mathcal{G}_0, \mathcal{G}_1,\ ...., \mathcal{G}_{\lambda^*}\}$ *then* $\mathcal{G}_0$ *is dominance solvable implies* $\overline{K}(\mathcal{G}_h) \neq \varnothing$ *for all* $h \in \{0, 1...., \lambda^* - 1\}$

Proof. By Proposition 18 $\mathcal{G}_h$ satisfies the hypothesis of maximal reduction lemma (Lemma17) with $r = \lambda^* - h$. Corollary 16 implies that if $\overline{K}(\mathcal{G}_h) = \varnothing$ then $\mathcal{G}_{h+1}$ would also satisfy the hypothesis of the reduction lemma (Lemma15) with $r = \lambda^* - h$ and by Lemma15 , $\{a_{\lambda^*-h+1}(j') = 1 \text{ is a DCA in } \mathcal{G}_{h+1}\} = \varnothing$. Using the reduction lemma repeatedly it follows that $\mathcal{G}_m$ would satisfy the hypothesis of maximal reduction lemma (Lemma17) with $r = \lambda^* - h \geq 1$. This violates the assumption of dominance solvability of $\mathcal{G}_0$. ■

In the next Lemmas we link $K_{\lambda^*}(\cdot)$ to $\overline{K}(\cdot)$ and hence demonstrate the connection between the dominance solvability of $\mathcal{G}_0$ to the existence of a λ^*-linked chain.

Lemma 21 *Consider the sequence of games* $\mathcal{G}_0$*,* $\mathcal{G}_1$*, ...,* $\mathcal{G}_m$ *and let* $\mathcal{G}_0$ *be dominance solvable. Then,* $K_{\lambda^*}(1) \neq \varnothing$ *and* $K_{\lambda^*}(1) = \overline{K}(\mathcal{G}_0) = \overline{\overline{K}}(\mathcal{G}_0)$

Proof. Assume to the contrary that $K_{\lambda^*}(1) = \{j : k(j) \geq \lambda^* - 1\} = \varnothing$. In $\mathcal{G}_0$ since all strategies are possible (i.e., no strategy has been eliminated using the iterative domination process) and since for all j, $k(j) < \lambda^* - 1$, for all j and l we know that for $k(j) \geq l - 1$, there exists a contingency for j such that $a_l(j) = 0$ is a BRCA in $\mathcal{G}_0$. Thus, $\overline{K}(\mathcal{G}_0) = \varnothing$. By Proposition 20 this contradicts the dominance solvability of $\mathcal{G}_0$. Thus, we get $K_{\lambda^*}(1) \neq \varnothing$.

Moreover, in $\mathcal{G}_0$ since all strategies are possible (i.e., no strategy has been eliminated using the iterative domination process) for j we know that for all $l \leq r = \lambda^* - 1$, there exists a contingency for j such that $a_l(j) = 0$ is a BRCA and for all j' such that $k(j') \geq l - 1$, $a_l(j') = 1$ is a BRCA for all $l \geq \lambda^*$. Hence, for all contingencies and for all j' such that $k(j') \geq l - 1$, $a_l(j') = 1$ is a DCA in $\mathcal{G}_0$ for any $l \geq \lambda^*$. Thus, $\overline{K}(\mathcal{G}_0) = K_{\lambda^*}(1) = \overline{\overline{K}}(\mathcal{G}_0)$. ■

Lemma 22 *Consider the sequence of games $\mathcal{G}_0$, $\mathcal{G}_1$, ...,$\mathcal{G}_m$ and let $\mathcal{G}_0$ be dominance solvable and if for some $h \in \{2, 3, ..., m\}$, let $K_{\lambda^*}(h) = \overline{K}(\mathcal{G}_{h-1}) = \overline{\overline{K}}(\mathcal{G}_{h-1}). \neq \varnothing$. If $m > h - 1$ then: (i) $K_{\lambda^*}(h+1) \neq \varnothing$ and (ii) $K_{\lambda^*}(h+1) = \overline{K}(\mathcal{G}_h) = \overline{\overline{K}}(\mathcal{G}_h)$.*

Proof. (i) Assume to the contrary that $K_{\lambda^*}(h+1) = \{j' : k(j') \geq \lambda^* - h$ and there exists $j \in K_{\lambda^*}(h-1)$ such that $j' \leq k(j)\} = \varnothing$. By the hypothesis of the lemma, $K_{\lambda^*}(h-1) = \overline{K}(\mathcal{G}_{h-2}) \neq \varnothing$. Thus, the set $K_{\lambda^*}(h) = \{j' : k(j') \geq \lambda^* - h$ and there exists $j \in \overline{K}(\mathcal{G}_{h-2})$ such that $j' \leq k(j)\}$ is empty. By Lemma 15, $\overline{K}(\mathcal{G}_{h-1}) = \varnothing$. Proposition 20, this contradicts the assumption that $\mathcal{G}_0$ is dominance solvable. Thus, $K_{\lambda^*}(h) \neq \varnothing$.

To complete the proof we need to show that $K_{\lambda^*}(h) = \overline{K}(\mathcal{G}_{h-1}) = \overline{\overline{K}}(\mathcal{G}_{h-1})$. We will first show that (a) $\overline{K}(\mathcal{G}_{h-1}) \subseteq K_{\lambda^*}(h)$. (b) $K_{\lambda^*}(h) \subseteq \overline{K}(\mathcal{G}_{h-1})$.

(a) By the general reduction lemma (Lemma 15) $j \in \overline{K}(\mathcal{G}_{h-1})$ implies that there exists $j' \in \overline{K}(\mathcal{G}_{h-2}) = K_{\lambda^*}(h-1)$ such that $k(j') \geq j$. By definition this implies $j \in K_{\lambda^*}(h)$. Thus, $\overline{K}(\mathcal{G}_{h-1}) \subseteq K_{\lambda^*}(h)$.

(b) If $j \in K_{\lambda^*}(h)$ then $k(j) \geq \lambda^* - h + 1$ and there exists $j' \in K_{\lambda^*}(h-1) = \overline{\overline{K}}(\mathcal{G}_{h-2})$ such that $k(j') \geq j$. Thus we have for such a j', $a_{\lambda^*-h'+1}(j') = 1$ is a DCA in $\mathcal{G}_{h-2}$ for all $h' \leq h$. Hence, by the strong persistence lemma $a_{\lambda^*-h'+1}(j') = 1$ is a DCA in $\mathcal{G}_{h-1}$ for all $h' \leq h$. Thus, If $j \in K_{\lambda^*}(h)$ then $k(j) \geq \lambda^* - h + 1$ and there exists j' such that $k(j') \geq j$ and $a_{\lambda^*-h'+1}(j') = 1$ is a DCA in $\mathcal{G}_{h-1}$ for all $h' \leq h$. By Lemma **??** (ii), $j \in \overline{K}(\mathcal{G}_{h-1})$.

Since by definition $\overline{\overline{K}}(\mathcal{G}_{h-1}) \subseteq \overline{K}(\mathcal{G}_{h-1})$, to complete the proof we need to argue $K_{\lambda^*}(h) = \overline{K}(\mathcal{G}_{h-1}) \subseteq \overline{\overline{K}}(\mathcal{G}_{h-1})$. ■

Proof. Assume to the contrary that for some j', $j' \in K_{\lambda^*}(h) = \overline{K}(\mathcal{G}_{h-1})$ and $j' \notin \overline{\overline{K}}(\mathcal{G}_{h-1})$. Now, $j' \in \overline{K}(\mathcal{G}_{h-1})$ and $j' \notin \overline{\overline{K}}(\mathcal{G}_{h-1})$ implies that $k(j') \geq \lambda^* - h$[18] and by the strong persistence lemma that

$$j' \notin K_{\lambda^*}(h-1) = \overline{\overline{K}}(\mathcal{G}_{h-2}) \tag{one}$$

Since $j' \in K_{\lambda^*}(h)$, there exists $j \in K_{\lambda^*}(h-1)$ such that $k(j) \geq j'$. But, $j \in K_{\lambda^*}(h-1)$ implies $j'' \in K_{\lambda^*}(h-2)$ and $k(j'') > j$. Thus, using $k(j'') \geq j > k(j) \geq j'$. But $j'' \in K_{\lambda^*}(h-2)$, $k(j'') > j'$ together with $k(j') \geq \lambda^* - h$ imply $j' \in K_{\lambda^*}(h-1)$. This contradicts (**??**). ■

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Proof. Of Theorem 1: Necessity follows from the repeated application of Lemma 22 and sufficiency from Lemma 22 and Lemma 4 (ii). ■

6 Conclusion.

In the presence of Economies of Scale /Positive network Externalities Rational Sequential Decision Making in the presence of incomplete information can

[18] Since we already know that $a_l(j') = 1$ is a DCA for $l = \lambda^* - h$, it must be the case that $a_l(j') = 0$ is a BRCA for some $l' \geq \lambda^* - h + 1$. In other words, $k(j') \geq \lambda^* - h$.

produce Coordination Failure. Higher the valuation of the good by households, lower the cost of production (i.e., smaller λ^*) and "more informed" the consumers are about the aggregate history of purchases the more likely that this type of market failure can be avoided. Three types of intervention will mitigate this distortion: (a) Consumer mandates compelling people to buy the good. (b) Price Controls setting a ceiling on the price at minimum possible Average Cost. (c) Making aggregate information about the history of purchases available among consumers as completely and quickly as possible.Since the provision of only *aggregate* information need be a policy tool under the third of the three interventions above it both protects anonymity of households and is the least coercive and intrusive of the three policies.