Improving Nash by Correlation in Quadratic Potential Games*

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Abstract

We consider a class of quadratic potential games and prove that (*simple symmetric*) *coarse correlated equilibria* (as introduced by Moulin and Vial 1978) can strictly improve upon the Nash equilibrium payoffs (that can not be improved upon by correlated equilibrium *a la* Aumann). We fully characterise the structure of the coarse correlated equilibrium that achieves the maximum improvement possible in this class of games and apply our characterising result to a few specific economic models including duopoly and public goods.

Keywords: Simple correlation device, Coarse correlated equilibrium, Quadratic games, Duopoly models.

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1 INTRODUCTION

We know that correlated equilibrium, as introduced by Aumann (1974, 1987), improves upon the Nash equilibrium in strategic form games, barring a few exceptions. Unfortunately, one of the most fundamental models of strategic markets, that of oligopoly, serves as one such exception. Liu (1996) and Yi (1997) proved that the only correlated equilibrium for an oligopoly is the Nash equilibrium, a result that Neyman (1997) and Ui (2008) generalised for a specific class of *potential* games.

Although correlation *a la* Aumann may not achieve anything more than the Nash outcome, as one reckons, a coarsening of the set of correlated equilibria may exist in certain oligopoly models. Indeed, we do have such a concept in the literature, introduced by Moulin and Vial (1978), called the *coarse correlated equilibrium*[1], that has recently evoked interests in several contexts (Young 2004, Roughgarden 2009, Forgó 2010 and Ray and Sen Gupta 2012 among others).

A correlation device is a probability distribution over the outcomes in the game. The best interpretation of correlation *a la* Aumann is that of a fictitious mediator who uses a correlation device. The mediator first selects and sends private messages (strategies to play) to the players according to the probability distribution; then, the players play the original game. A direct correlated equilibrium (Aumann, 1974, 1987) is a mediator whose recommendations the players find optimal to follow obediently, that is, playing the recommended strategies forms a Nash equilibrium in the extended game. The interpretation of the device in the weaker concept of correlation introduced by Moulin and Vial (1978) is that of a solicitor who asks the players to either commit to the device or to play any strategy of their own. If the players accept the device, they do not play the game anymore and simply receive the payoffs from the outcome chosen by the solicitor according to the probability distribution. A correlation device is called a coarse correlated equilibrium if it is in no player's interest to choose any alternative strategy of his own, given that other players choose to commit to the device.

As one rightly anticipates, such a coarse notion may improve upon the Nash equilibrium when correlated equilibrium fails to do so. Indeed, Gerard-Varet and Moulin (1978) proved that for duopoly games (satisfying certain mild assumptions on payoff functions) Nash equilibrium can be *locally improvable* by using this concept introduced by Moulin and Vial (1978)[2]. From the analysis by Gerard-Varet and Moulin (1978), we do learn under which condition in a class of specific duopoly games, Nash equilibrium can be improved upon; however, the specific definition they used for improvement involves sequences of devices with support close to the Nash equilibrium. Also, we do not know whether or not such devices necessarily has to use a support be *local* or close to the Nash equilibrium.

[1] Although this notion is due to Moulin and Vial (1978), they have not named the equilibrium concept. They have called such a correlation device a *correlation scheme*. Young (2004) and Roughgarden (2009) called this equilibrium the *coarse correlated equilibrium*, while Forgó (2010) called it the *weak correlated equilibrium*.

[2] Forgó *et al* (2005) also recently used Moulin and Vial's notion of correlation in environmental games.

In this paper, we analyse coarse correlation in a well-studied two-player game with quadratic payoffs, that we call the *quadratic game*. There are many examples of such games in the literature, including duopoly models with a quadratic profit function and the *search game* (Diamond 198?),

The purpose of this paper is two-fold. First, the quadratic game, being a potential game, does not possess any correlated equilibrium other than the Nash equilibrium (Neymann 1997); hence, one may ask if there exists coarse correlated equilibria for this class of games. Second, one may actually be interested in improving upon the Nash equilibrium payoff by coarse correlation in these games without using a local argument as in Gerard-Varet and Moulin (1978).

For our purpose, we, in this paper, consider a specific form of correlation devices that we call a Simple Symmetric Correlation Device (k-SSCD) to analyse coarse correlation. The device (Ganguly and Ray 2005, Ray and Sen Gupta 2012) is so-called as the discrete probability distribution of the device is symmetric and the support of it is finite. Using the notion of coarse correlation introduced by Moulin and Vial we fully characterise an equilibrium concept that we call k-Simple Symmetric Coarse Correlated Equilibrium (k-SSCCE).

We consider and characterise Moreover, we find the maximum possible improvement by such a 2-SSCCE using our construction. We thus extend the result by Gerard-Varet and Moulin (1978), Our constructions do provide specific devices and also do not depend on any *local* arguments.

Correl. equil. (Aumann) were invented explicitely as a way to improve upon the inefficient non cooperative outcome by means of a mild coordination device. The improvement or its absence is easy enough to check when the game has a small finite number of strategies, but there is no systematic approach to compute the best corr. equilibria when the strategy sets are of dimension 1 or more. The situation is similar to the comps of mixed strategy eq. : hard in general with infinite spaces. The only systematic result is negative: in a potential game, corr. equ. do not allow any improvement, in fact they do not add any new equ. (Neyman 9?).

CoarseCorr equ. (moulin Vial) were invented explicitely as a way to improve the inefficient non cooperative outcome when the regular corr. equ. is useless, by means of a stronger, but still non invasive, coordination device. see Moulin Vial for simple bimatrix examples. But computing such improvements when the strategy sets are of dimension 1 or more is still difficult, and in fact there is only one available result for the case of 1 dimensional strategy spaces, characterizing the possibility of some improvement, i.e.e, a local analysis (GVM78). Here we propose, for the 1st time, and in a reasonably rich class of games, a complete answer to the Q: can the Nash eq. of this game be improved by a CCequ., and if so, by how much, and by what coordination device?

Our class of games is limited on three counts: only two players, symmetric, and quadratic payoffs. But these restrictions still allow us to capture simple versions of several benchmark economic games of strategy: Cournot and Bertrand duopoly, provision of a public good, search game (Diamond). And the answers are interesting and surprising.

Here explain two of our models, say a search game with numerical values, and maybe a public good provision game, and describe the optimal lottery

The paper is organised as follows. We present the game and the concepts in the next section. Section 3 compiles all our results and Section 4 illustrates them using specific games from the literature while Section 5 concludes. The proofs are in the Appendix.

2 SET-UP

2.1 Correlation and Coarse Correlation

The following definitions are borrowed from our parallel work (Ray and Sen Gupta 2012) on this research agenda and are presented here for the sake of completeness. Fix any finite normal form game, $G = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, with set of players, $N = \{1, ..., n\}$, finite pure strategy sets, $S_1, ..., S_n$ with $S = \prod_{i \in N} S_i$, and payoff functions, $u_1, ..., u_n$, $u_i : S \to \Re$, for all i. A normal form game, G, can be extended by using a direct correlation device.

Definition 1 *A (direct) correlation device μ is a probability distribution over S.*

A direct correlation device can be used to define two concepts. The device can be a correlated equilibrium (Aumann 1974, 1987) or a coarse correlated equilibrium (Moulin and Vial 1978) if certain conditions are met in an appropriately chosen extended game. Formally, with the notation $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$,

Definition 2 *μ is a (direct) correlated equilibrium of the game G if for all i, for all s_i, $t_i \in S_i$, $\sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i}) u_i(t_i, s_{-i})$..*

If players commit to the correlation device μ, the expected payoff for any player i is given by $\sum_{s \in S} \mu(s) u_i(s) = E_\mu u_i$ (say). With the notation for the *marginal* probability distributions $\mu_i'(s_{-i}) = \sum_{s_i \in S_i} \mu(s_i, s_{-i})$ over $s_{-i} \in S_{-i}$, for any player $i \in N$, the coarse correlated equilibrium formally is defined as below.

Definition 3 *μ is a (direct) coarse correlated equilibrium of the game G if for all i, for all $t_i \in S_i$, $\sum_{s \in S} \mu(s) u_i(s) \geq \sum_{s_{-i} \in S_{-i}} \mu_i'(s_{-i}) u_i(t_i, s_{-i})$.*

From the system of inequalities in the above definitions[3], it is clear that indeed a coarse correlated equilibrium is a coarsening of the concept of correlated equilibrium. Also, it is obvious that any Nash

[3] One may also use strict inequalities in the above two definitions for the corresponding stronger notions. Indeed, Gerard-Varet and Moulin (1978) used such a strong notion of coarse correlated equilibrium in their definition of improvement over the Nash equilibrium in any duopoly model.

equilibrium and any convex combination of several Nash equilibria of any given game G, *corresponds* to a correlated and hence, a coarse correlated equilibrium.

2.2 Quadratic Games

In this paper, we consider several 2-person symmetric normal form games with continuum of strategies and a quadratic payoff function, for each player. We will denote the levels of strategic choices for the two players respectively by x_1 and x_2, with both $x_i \in I$, a closed interval of $\mathbb{R}_+$, not necessarily bounded. We consider a general form of symmetric quadratic payoff functions for the players; each payoff is a polynomial of degree 2 in the two strategies x_1, x_2, given by, $u_1(x_1, x_2) = ax_1 + bx_2 + cx_1x_2 + dx_1^2 + ex_2^2$, and $u_2(x_1, x_2) = ax_2 + bx_1 + cx_1x_2 + dx_2^2 + ex_1^2$, respectively. The only restriction we maintain is $d < 0$, so that the payoffs are concave in own strategy. We do not specify the sign of the other parameters at this stage.

For the rest of the paper, this two-person game will be called the *quadratic game*. The quadratic game is a potential game with a smooth and concave potential function, P, given by, $P(x_1, x_2) = a(x_1 + x_2) - cx_1x_2 - d(x_1^2 + x_2^2)$. Therefore, using the result by Neymann (1997), this game has a unique correlated equilibrium *a la* Aumann that coincides with the Nash equilibrium of the game.

The unique symmetric Nash equilibrium of the quadratic game

We look for possible CC-improvements of the Nash equilibrium (or equilibria) of the game, therefore the first step is to describe precisely these Nash equilibrium (a) and their associated payoffs. A full analysis with general interval I and the parameters involves too many cases, so we will restrict attention to 3 classes of games corresponding to the economic examples we have in mind. In each class the critical parameters a, c, d, will be strictly positive, and the sign of the remaining parameters b, e will not matter.

2.3 Simple Correlation Devices

We now consider a specific form of correlation devices, called *simple* devices (Ganguly and Ray 2005, Ray and Sen Gupta 2012) to analyse coarse correlation for the quadratic game. Although the strategy set for a player in this game is a continuum, a simple device involves only finitely many strategies for each player, i.e., the support of the probability distribution in the direct correlation device is finite. We use a class of simple devices (as in Ray and Sen Gupta 2012) by imposing a mild restriction, to analyse possible improvement over Nash by coarse correlation in the quadratic game. We consider symmetric probability distributions and the same strategy levels for both the players. These restrictions are fairly innocuous in this context as the quadratic game is entirely symmetric. Formally, the specific form of the device we consider in this paper is defined below.

Definition 4 *A k-Simple Symmetric Correlation Device (k-SSCD), $[P; q_c]$, is a symmetric probability* *distribution matrix, P, over $q_c \times q_c$, where, $q_c = (q_1, q_2, ..., q_k)$, with $q_i \geq 0$ and $P = \{(p_{ij})_{i=1,2...,k;j=1,2...,k}\}$ with each $p_{ij} \in [0,1]$, $p_{ij} = p_{ji}$ and $\sum_{ij} p_{ij} = 1$.*

The interpretation of a k-SSCD, $[P; q_c] = [\{(p_{ij})_{i=1,2...,k;j=1,2...,k}\}; (q_i)_{i=1,2...,k}]$, is that the players are given a choice to commit to the device. If both players commit, the device will then pick the strategies q_i and q_j for the two players respectively, with probability p_{ij}; the players do not play the game, however, get the payoffs, respectively, $u_1(q_i, q_j)$ and $u_2(q_i, q_j)$ that correspond to the chosen strategy profile (q_i, q_j). Thus, if both players commit to the device, the expected payoffs for the two players are the same (by symmetry) and is given by $E_P u = E_P u_1 = E_P u_2 = (a+b)(\pi.q_c) - (d+e)(\pi.q_c^{(2)}) - c q_c P q_c$.

As explained in Ray and Sen Gupta (2012), any k-SSCD is a coarse correlated equilibrium (Moulin and Vial 1978) if the device is accepted by both players, that is, if neither of the players wishes to unilaterally deviate to choose any other possible strategy in the game, while the other commits to the device. Note that, although the device, $[P; q_c]$, involves only finitely many strategies, the deviation for a player is however not restricted; any strategy $x \geq 0$ (even outside the domain, q_c, of the device) can be played by a player if he doesn't commit to the device. The deviant faces the *marginal* probability distribution $ṕ$ over $q_j \in q_c$ which is given by $ṕ(q_j) = \sum_{q_i \in q_c} p(q_i, q_j)$. Let $E_P(u \mid x)$ denote the expected payoff of any deviating player (by symmetry) by playing x. Clearly, by symmetry, $E_P(u \mid x) = \sum_{q_j \in q_c} ṕ(q_j) u_1(x, q_j) = (a - c\pi.q_c)x - dx^2 + b(\pi.q_c) - e(\pi.q_c^{(2)})$. As mentioned, the equilibrium condition requires that the device be accepted by both players. The equilibrium condition is defined formally as,

Definition 5 *A k-SSCD, $[P; q_c]$, is called a k-Simple Symmetric Coarse Correlated Equilibrium (k-SSCCE) if both the players commit to the device, i.e., given that the other player is committing to the device, a player does not deviate to play any other strategy $x \geq 0$.*

Clearly, the Nash equilibrium of the quadratic game can be viewed as a device with probability 1 on the Nash equilibrium quantity, x^*, and is trivially a k-SSCCE. This fact is observed in the above characterisation, satisfying the condition weakly. Hence, the unique Nash and correlated equilibrium *a la* Aumann of the quadratic game is a k-SSCCE. The question now is whether it is possible to improve upon the Nash equilibrium using a k-SSCCE. We analyse this issue in the next section.

3 RESULTS

In this section, we analyse

We look for a general coarse correlated equilibrium (CCE) L_0 improving upon the symmetric Nash equilibrium (x^*, x^*). We observe first that it is enough to consider symmetric lotteries, if we are only looking for symmetric improvements (each player gets the same utility boost), or for the best possible improvement of $u_1 + u_2$ over $2u^*$. This is because our game is symmetric, so if L_0 is a CCE improving upon (x^*, x^*), then the lottery L_0^s obtained from L_0 by exchanging coordinates, is also an improving CCE, and so is the symmetric lottery $\frac{1}{2}(L_0 + L_0^s)$.

In the computations below, we replace L_0 by L, translated from L_0 by (x^*, x^*): $pro_L(A) = pro_{L_0}(A - (x^*, x^*))$. Note that we will need to make sure that its support is contained in $I - \{x^*\}$. We write L^m for the marginal distribution of L on $\mathbb{R}$ (as L is symmetric, there is only one).

If the random variable (Z_1, Z_2) has distribution L, the expected profit of either player when playing the lottery is given by,

$u_1(L) = aE_L[x^* + Z_1] + bE_L[x^* + Z_2] + cE_L[(x^* + Z_1) \cdot (x^* + Z_2)] + dE_L[(x^* + Z_1)^2] + eE_L[(x^* + Z_2)^2]$

$\Rightarrow u_1(L) = u^* + (a + b + 2(c + d + e)x^*)E_{L^m}[Z_1] + cE_L[Z_1 \cdot Z_2] + (d + e)E_{L^m}[Z_1^2]$,

where we use the symmetry of L in the last step.

Therefore, our CCE L_0 improves over (x^*, x^*) if and only if,

$\Delta = (a + b + 2(c + d + e)x^*)E_{L^m}[Z_1] + (d + e)E_{L^m}[Z_1^2] + cE_L[Z_1 \cdot Z_2] > 0$

Next we write the CC equilibrium conditions when player 1 uses a pure strategy and player 2 follows L_0^m. Player 1 maximizes over I,

$u_1(x_1 \otimes L^m) = ax_1 + bE_{L^m}[x^* + Z_1] + cx_1E_{L^m}[x^* + Z_1] + dx_1^2 + eE_{L^m}[(x^* + Z_1)^2]$

$= \{(a + cx^* + cE_{L^m}[Z_1])x_1 + dx_1^2\} + (bx^* + ex^{*2}) + (b + 2ex^*)E_{L^m}[Z_1] + eE_{L^m}[Z_1^2]$

Thus, L provides a CC-improvement of the Nash equilibrium (x^*, x^*) if and only if Δ is positive, and $\max_{x_1 \in I} u_1(x_1 \otimes L^m) \leq u^* + \Delta$. Clearly these two inequalities depend only upon the three moments, $E_L[Z_1]$, $E_L[Z_1^2]$ and $E_L[Z_1 \cdot Z_2]$. It will be convenient to rewrite them with the simplifying notation as follows, $\alpha = E_L[Z_1]$, $\beta = E_L[Z_1^2]$ and $\gamma = E_L[Z_1 \cdot Z_2]$:

$\Delta = (a + b + 2(c + d + e)x^*)\alpha + (d + e)\beta + c\gamma > 0$ (3)

$\max_{x_1 \in I}\{(a + cx^* + c\alpha)x_1 + dx_1^2\} + (bx^* + ex^{*2}) + (b + 2ex^*)\alpha + e\beta \leq u^* + \Delta$

$\Leftrightarrow \max_{x_1 \in I}\{(a + cx^* + c\alpha)x_1 + dx_1^2\} \leq ax^* + (c + d)x^{*2} + (a + 2(c + d))\alpha + d\beta + c\gamma$ (4).

Note that we have omitted the additional requirement that the support of the lottery L_0 be contained in $\mathbb{R}_+^2$. We come back to this in the next section.

3.1 Improvement

We are interested in the improvement over the Nash equilibrium payoff, u_{NE}, in the quadratic game, using a k-SSCCE. For the sake of completeness, we present the obvious notion of improvement (similar to Definition 2 in Gerard-Varet and Moulin (1978, page 128)).

Definition 6 *A k-Simple Symmetric Coarse Correlated Equilibrium (k-SSCCE) improves upon the Nash equilibrium of the quadratic game if* $E_P u > u_{NE}$.

We now show that it is enough to seek improving CCEs in the very small class of symmetric lotteries L_0 with binary support $x^* + z$, $x^* + z'$ for each player, and the following form,

	$x^* + z$	$x^* + z'$
$x^* + z'$	$\frac{\lambda}{2}$	$\frac{\lambda'}{2}$
$x^* + z$	$\frac{\lambda'}{2}$	$\frac{\lambda}{2}$

, where $\lambda + \lambda' = 1$ and $\lambda, \lambda' \geq 0$.

We call the translated lottery $L = \begin{array}{|c|c|} \hline \frac{\lambda}{2} & \frac{\lambda'}{2} \\ \hline \frac{\lambda'}{2} & \frac{\lambda}{2} \\ \hline \end{array}$, a *simple* 2×2 *lottery*.

Lemma 1 *(i) For any symmetric distribution L we have,* $\beta \geq \gamma; \beta + \gamma \geq 2\alpha^2$ *(5).*

(ii) Equality $\beta = \gamma$ *holds if and only if L is diagonal; equality* $\beta + \gamma = 2\alpha^2$ *holds if and only if* $Z_1 + Z_2$ *is constant.*

(iii) Every (α, β, γ) *satisfying (5) is achieved by a simple* 2×2 *lottery.*

Proof of (i), (ii)**:** To check $\beta \geq \gamma$ we use the symmetry of L,

$\beta = \frac{1}{2}(E_L[Z_1^2] + E_L[Z_2^2]) = E_L[\frac{1}{2}(Z_1^2 + Z_2^2)] \geq E_L[Z_1 \cdot Z_2]$.

Inequality $\beta + \gamma \geq 2\alpha^2$ follows similarly,

$\beta + \gamma = E_L[\frac{1}{2}(Z_1^2 + Z_2^2) + Z_1 \cdot Z_2] = \frac{1}{2}E_L[(Z_1 + Z_2)^2]$

$\alpha = \frac{1}{2}(E_L[Z_1] + E_L[Z_2]) \Rightarrow 2\alpha^2 = \frac{1}{2}(E_L[Z_1 + Z_2])^2$, hence the desired conclusion because $E_L[X^2] \geq (E_L[X])^2$. Statement (ii) is straightforward.

Proof of (iii)**:** We look for z, z', λ, such that the corresponding simple 2×2 lottery achieves a given profile (α, β, γ) meeting (5),

$\frac{1}{2}(z + z') = \alpha$; $\frac{1}{2}(z^2 + z'^2) = \beta$; $\frac{\lambda}{2}(z^2 + z'^2) + \lambda' z z' = \gamma$ (6).

Set $S = z + z'$, $P = zz'$, so that $S = 2\alpha$, $P = 2\alpha^2 - \beta$, and z, z' are the roots of $Y^2 - 2\alpha Y + 2\alpha^2 - \beta = 0$. These roots are real if and only if, $\alpha^2 \geq 2\alpha^2 - \beta \Leftrightarrow \beta \geq \alpha^2$. The latter is a consequence of (5). Next, we can find λ in $[0, 1]$ satisfying the third equality in (6) if and only if,

$zz' \leq \gamma \leq \frac{1}{2}(z^2 + z'^2) \Leftrightarrow 2\alpha^2 - \beta \leq \gamma \leq \beta$,

and the proof of (iii) is complete. ■

Suppose we can find α, β, γ, satisfying (5) and the two inequalities (3), (4), then the corresponding simple lottery L described in the above proof is a CC-improvement of the Nash equilibrium, provided the support of L_0, translated from L_0 by $-(x^*, x^*)$, remains in I, that is to say, z, $z' \in I - \{x^*\}$.

As z, z' are the roots of the polynomial $Y^2 - 2\alpha Y + 2\alpha^2 - \beta$, this amounts to,

$\alpha \in I - \{x^*\}$ and $Y^2 - 2\alpha Y + 2\alpha^2 - \beta \geq 0$ for $Y = I_\varepsilon - x^*$, $\varepsilon = +, -$ (7).

So when we look for the best feasible CC-improvement by means of simple 2×2 lotteries, if property (7) does not bind, the result will automatically be the best feasible CC-improvement by any lottery L_0. But if the inequality does bind, we cannot so conclude.

The next statement gathers our general results.

Proposition 1 *i) Consider the program*

$\max_{\alpha,\beta,\gamma}\{a+b+2(c+d+e)x^*\}\alpha+(d+e)\beta+c\gamma$ *(8)*

under the constraints (5) and (4) on α, β, γ. *If its solution* α^*, β^*, γ^* *meets the constraint (7), then the corresponding simple* 2×2 *lottery gives the best possible CC-improvement of the Nash equilibrium* (x^*,x^*).

ii) Consider the program (8) under the constraints (5), (4) and (7) on α, β, γ. *Its solution gives the best possible CC-improvement of the Nash equilibrium* (x^*,x^*) *by a simple* 2×2 *lottery.*

Note that the optimal value of $\Delta(\alpha,\beta,\gamma)$ must be non negative, for the trivial lottery concentrated at (x^*,x^*) (the original Nash equilibrium) meets all the constraints and corresponds to $\alpha=\beta=\gamma=0$.

4 EXAMPLES

In this section, we illustrate our results and the constructions using several examples of quadratic games from the literature.

4.1 Class 1

Payoffs $u_1(x_1,x_2)=ax_1+bx_2-cx_1x_2-dx_1^2+ex_2^2$, and $I=\mathbb{R}_+$.

The two economic examples are (*explain*)

- Cournot Duopoly with linear demand and decreasing marginal costs,

  $u_1(x_1,x_2)=(P-A(x_1+x_2))x_1-(Bx_1-Mx_1^2)$.

  So $a=P-B, c=A, d=A-M, b=e=0$.

- Provision of a public good with decreasing marginal benefits and increasing marginal costs

  $u_1(x_1,x_2)=\{A(x_1+x_2)-B(x_1+x_2)^2\}-(Cx_1+Dx_1^2))$.

  Here $a=A-C, c=2B, d=B+D$

The best reply function is $br_i(x_j)=\frac{1}{2d}(a-cx_j)$, and we distinguish two cases:

Case 1: If $c<2d$, the unique eqilibrium is (x^*,x^*) where $x^*=\frac{a}{2d+c}$, and corresponding payoff,

$u^*=\frac{a}{(2d+c)^2}(a(d-e)+b(2d+c))$. (2)

Case 2: If $c>2d$, we add to the equilibrium above two asymmetric Stackelberg equilibria ($x_i=0, x_j=\frac{a}{2d}$) with payoffs $(\frac{a}{4d}(2b+\frac{a}{4d}e),\frac{a^2}{4d})$.

[*Note: we want to compare the best symmetric CC-improvement to the total payoff at those Stackelberg eqs.*]

4.2 Class 2

Payoffs, $u_1(x_1, x_2) = ax_1 + bx_2 + cx_1x_2 - dx_1^2 + ex_2^2$, and $I = \mathbb{R}_+$.

The economic example is Bertrand duopoly, $u_1(x_1, x_2) = (D - Ax_1 + Bx_2)x_1$. Here $a = D$, $c = B$, $d = A$.

The best reply function is $br_i(x_j) = \frac{1}{2d}(a + cx_j)$, and if $c > 2d$ the game has no Nash equilibrium, because I is unbounded. Even if I is bounded, the Nash equilibrium has both players using the largest feasible strategy. If $c+e-d \geq 0$ this is clearly Pareto optimal, hence not CC-improvable. If $c+e-d < 0$, requiring e negative and large, the Nash equilibrium would give negative payoffs and may be improvable. But we dismiss this case for lack of any relevant example.

So in this class we focus on the case $c < 2d$, the unique Nash eqilibrium is then (x^*, x^*), where $x^* = \frac{a}{2d-c}$, and the payoffs are,

$u^* = \frac{a}{(2d-c)^2}(a(d-e) + b(2d-c))$.

[Note: maybe in this class we should always assume $b = e = 0$, as it is our only example.]

4.3 Class 3

Payoffs $u_1(x_1, x_2) = -ax_1 + cx_1x_2 - dx_1^2$, and $I = [0, 1]$. We assume $b = e = 0$ for simplicity of the analysis. The economic example is Diamond's search game (*explain*), with payoffs,

$u_1(x_1, x_2) = Cx_1x_2 - (Ax_1 + Dx_1^2)$, where $a = A$, $c = C$, $d = D$.

The best reply function is $br_i(x_j) = median\{0, 1, \frac{1}{2d}(cx_j - a)\}$. If $c > a + 2d$, we have three Nash equilibria, $(0, 0)$, (x^*, x^*), $x^* = \frac{a}{c-2d}$, and $(1, 1)$. The latter, $(1, 1)$, is Pareto optimal, hence not CC-improvable. If $c = a + 2d$, we still have $(1, 1)$ as a Nash equilibrium, and thus unimprovable.

The only interesting case in this class is thus when $c < a + 2d$, then the unique Nash equilibrium is $(0, 0)$ with zero payoffs for both players.

4.4 Duopoly

General model : $x^* = \frac{-a}{2d+c}$, $u^* = \frac{a}{(2d+c)^2}[a(e-d) - b(2d+c)]$

5 Cournot Duopoly with linear demand and quadratic costs

Here $u_1(x_1, x_2) = (P - A(x_1 + x_2))x_1 - (Bx_1 - Mx_1^2)$, and $I = [0, \infty[$.

So $a = P - B$, $b = e = 0$, $c = A$, $d = A - M$.

From GVM78 we know that there is no CC improvement if $|A| \leq A - M \Leftrightarrow M < 0$, i.e., if marginal costs increase.

So we assume decreasing (but not too decreasing) marginal costs, $A > M > 0$,and develop the computations with the parameters a, c, d, all strictly positive. Recall $x^* = \frac{a}{2d+c}$ (denoted x for simplicity). We will find that the CC-improvement is entirely described by the two parameters x^* and $\theta_1 = \frac{c}{d}$, and the relative improvement $\frac{\Delta^*}{u^*}$ only depends upon θ_1, i.e., upon $\frac{M}{A}$.

Theorem *The optimal CC-improvement ratio is*

$\frac{\Delta^*}{u^*} = \frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1)$ for $\theta_1 \geq 1.17$,

$\frac{\Delta^*}{u^*} = (\theta_1 - 1) + \frac{2(2-\theta_1)\sqrt{\theta_1 - 1}}{2\sqrt{\theta_1 - 1} + \theta_1 + 2} - \frac{8(\theta_1 - 1)}{(2\sqrt{\theta_1 - 1} + \theta_1 + 2)^2}$ for $1 \leq \theta_1 \leq 1.17$.

Interpretation: draw the graph and downplay the role of the little piece before 1.17.

If the random variable (Z_1, Z_2) has distribution L, the expected profit of either player when playing the lottery is

$u_1(L) = aE_L[x^* + Z_1] + bE_L[x^* + Z_2] - cE_L[(x^* + Z_1) \cdot (x^* + Z_2)] - dE_L[(x^* + Z_1)^2] - eE_L[(x^* + Z_2)^2]$

$\Rightarrow u_1(L) = u^* + (b - a\frac{2e+c}{2d+c})\alpha - c\gamma - (d+e)\beta$, where we use the symmetry of L in the last step.

Therefore, our CCE L_0 improves over (x^*, x^*) if and only if:

$\Delta = (b - a\frac{2e+c}{2d+c})\alpha - (d+e)\beta - c\gamma > 0 \quad (1)$

Next, we write the CC equilibrium conditions when player 1 uses a pure strategy and player 2 follows L_0^m:

$\max_{x_1 \in I} u_1(x \otimes L^m) = ax_1 + bE_{L^m}[x^* + Z_1] - cx_1 E_{L^m}[x^* + Z_1] - dx_1^2 - eE_{L^m}[(x^* + Z_1)^2]$

$\Rightarrow \max_{x_1 \in I} u_1(x \otimes L^m) = (a - c(x^* + E_{L^m}[Z_1]))x_1 - dx_1^2 + b(E_{L^m}[x^* + Z_1])) - eE_{L^m}[(x^* + Z_1)^2]$

We therefore have two cases:

Case 1: $\frac{2ad}{2d+c} \leq c\alpha$

Then the coefficient $a - c(x^* + \alpha) = \frac{2ad}{2d+c} - c\alpha$ is non positive, therefore player 1's optimal pure strategy is $x_1 = 0$, and the CC equilibrium condition is: $b(x^* + E_{L^m}[Z_1])) - eE_{L^m}[(x^* + Z_1)^2] \leq u^* + \Delta$, which is rearranged as,

$\frac{ac}{2d+c}\alpha + d\beta + c\gamma \leq \frac{a^2 d}{(2d+c)^2} \quad (2)$

Case 2: $c\alpha \leq \frac{2ad}{2d+c}$

Now, $a - c(x^* + \alpha) \geq 0$, so player 1's optimal pure strategy is $x_1 = \frac{1}{2d}(\frac{2ad}{2d+c} - c\alpha)$ and the CC equilibrium condition is, $\frac{1}{4d}(\frac{2ad}{2d+c} - cE_{L^m}[Z_1])^2 + b(x^* + E_{L^m}[Z_1])) - eE_{L^m}[(x^* + Z_1)^2] \leq u^* + \Delta$, which is rearranged as,

$\frac{c^2}{4d}\alpha^2 + d\beta + c\gamma \leq 0 \quad (3)$

We sum up our results so far.

Lemma 1 *The symmetric Nash equilibrium is improved by the CCE* L_0 *if and only if* $\Delta > 0$ *(1) and*

either $\{\frac{2ad}{2d+c} \leq c\alpha$ *and inequality (2)*$\}$

or $\{c\alpha \leq \frac{2ad}{2d+c}$ *and inequality (3)*$\}$

Step 1: We have $-\Delta(\alpha,\beta,\gamma) = cx\alpha + d\beta + c\gamma$,

so we are looking for, $\min_{\alpha,\beta,\gamma} cx\alpha + d\beta + c\gamma$,

such that

$2\alpha^2 - \beta \leq \gamma \leq \beta$

and either (2) or **(3),** and finally the support conditions,

$-x \leq \alpha$ and $\beta \leq 2\alpha^2 + 2\alpha x + x^2$. (4)

Case 1: we assume (3). We perform our computations as if this was the "right' choice, then we will compare the result with that under assumption (2) (Case 2 below). Assumption (3) gives the following constraints on α:

$\alpha \leq \frac{2d}{c}x$ and $\frac{c^2}{4d}\alpha^2 + d\beta + c\gamma \leq 0$.

We start by eliminating γ for fixed α, β:

$2\alpha^2 - \beta \leq \gamma \leq \min\{\beta, -\frac{c}{4d}\alpha^2 - \frac{d}{c}\beta\}$. (5)

So we must choose $\gamma = 2\alpha^2 - \beta$, reducing the problem to $\min_{\alpha,\beta} cx\alpha + 2c\alpha^2 - (c-d)\beta$,

where, by (5), α, β are such that,

$\alpha^2 \leq \beta$, $(2 + \frac{c}{4d})\alpha^2 \leq \frac{c-d}{c}\beta \Leftrightarrow \frac{(8d+c)c}{4d(c-d)}\alpha^2 \leq \beta$

and (4). Note that $\frac{(8d+c)c}{4d(c-d)} > 1$ so we can ignore the left inequality above.

Fixing α, we must take β as large as possible, i.e., $\beta = 2\alpha^2 + 2x\alpha + x^2$, which gives a one dimensional problem

$\min_\alpha cx\alpha + 2c\alpha^2 - (c-d)(2\alpha^2 + 2\alpha x + x^2) = \min_\alpha 2d\alpha^2 + (2d-c)x\alpha - (c-d)x^2$ (6)

under constraints

$\alpha \leq \frac{2d}{c}x$ and $-x \leq \alpha$ and $\frac{c^2+8d^2}{4d(c-d)}\alpha^2 - 2x\alpha \leq x^2$ (7), (the right one coming from $\frac{(8d+c)c}{4d(c-d)}\alpha^2 \leq 2\alpha^2 + 2x\alpha + x^2$).

The unconstrained optimum in (6) is $\alpha^* = -\frac{2d-c}{4d}x$.

It is clearly larger than $-x$, irrespective of the sign of $2d-c$. It meets the left inequality in (7) if and only if, $-\frac{2d-c}{4d} \leq \frac{2d}{c} \Leftrightarrow \theta_1 \leq 4$ (recall $\theta_1 = \frac{c}{d} > 1$).

Subcase 1.1: $\theta_1 \leq 4$ The right inequality in (7) for $\alpha = -\frac{2d-c}{4d}x$ boils down to,

$(c^2 + 8d^2)(2d-c)^2 \leq 32cd^2(c-d) \Leftrightarrow \theta_1^4 + 32 \leq 4\theta_1^3 + 20\theta_1^2$,

which is true in an interval $[\mu, 4]$ with $\mu \simeq 1.17$.

Subsubcase 1.1.1: $1.17 \leq \theta_1 \leq 4$

The optimal value is $\alpha^* = \frac{\theta_1 - 2}{4}x$ and the optimal surplus is,

$\Delta = \frac{(2d-c)^2}{8d}x^2 + (c-d)x^2 = \frac{\theta_1^2 + 4\theta_1 - 4}{8}dx^2$.

Comparing with $u^* = dx^2$ gives, $\frac{\Delta}{u^*} = \frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1)$, (8)

which starts at 25% for $\theta_1 = 1.17$, is as large as 100% for $\theta_1 = 2$, and 350% for $\theta_1 = 4$.

Subsubcase 1.1.2: $1 < \theta_1 \leq 1.17$

Now the optimal α is the negative root of $\frac{c^2+8d^2}{4d(c-d)}\alpha^2 - 2x\alpha = x^2$, because the unconstrained optimum in (6) is negative. This root is $\alpha^* = -\frac{2\sqrt{\theta_1-1}}{2\sqrt{\theta_1-1}+\theta_1+2}x$, and the optimal relative CC-improvement is,

$\frac{\Delta}{u^*} = (\theta_1 - 1) + \frac{2(2-\theta_1)\sqrt{\theta_1-1}}{2\sqrt{\theta_1-1}+\theta_1+2} - \frac{8(\theta_1-1)}{(2\sqrt{\theta_1-1}+\theta_1+2)^2}$, which is concave and increasing in $[1, 1.17]$. It is zero at $\theta_1 = 1$ with an infinite derivative, and reaches 19.5% at $\theta_1 = 1.1$, then 25% at $\theta_1 = 1.17$.

Subcase 1.2: $\theta_1 \geq 4$ Now our candidate for α is $\alpha^* = \frac{2d}{c}x$, which is optimal if it meets the right inequality in (7). This amounts to,

$\frac{c^2+8d^2}{c-d}\frac{d}{c^2} - \frac{4d}{c} \leq 1 \Leftrightarrow 8 + 4\theta_1 \leq 2\theta_1^2 + \theta_1^3$

and is clearly true for $\theta_1 \geq 4$. Thus the optimal surplus is,

$\Delta = (c-2d)\frac{2d}{c}x^2 - \frac{8d^3}{c^2}x^2 + (c-d)x^2$,

implying,

$\frac{\Delta}{u^*} = 1 + \theta_1 - \frac{4}{\theta_1} - \frac{8}{\theta_1^2}$ (9).

Case 2: we assume (2)

We have the additional constraints on α,

$\frac{2d}{c}x \leq \alpha$ and $cx\alpha + d\beta + c\gamma \leq dx^2$.

We must find the minimum of $cx\alpha + d\beta + c\gamma$. Note that the first constraint in (4) is always true, so we only have to take care of $\beta \leq 2\alpha^2 + 2\alpha x + x^2$.

We start as before by eliminating γ for fixed α, β:

$2\alpha^2 - \beta \leq \gamma \leq \min\{\beta, -x\alpha - \frac{d}{c}\beta + \frac{d}{c}x^2\}$.

We must choose $\gamma = 2\alpha^2 - \beta$, reducing the problem, as before, to

$\min_{\alpha,\beta} cx\alpha + 2c\alpha^2 - (c-d)\beta$

with constraints

$\alpha^2 \leq \beta$ and $2\alpha^2 + x\alpha - \frac{d}{c}x^2 \leq \frac{c-d}{c}\beta$ and $\beta \leq 2\alpha^2 + 2\alpha x + x^2$ (10).

We must choose $\beta = 2\alpha^2 + 2\alpha x + x^2$ as in case 1. Thus we are left with the same minimization problem (6), under $\frac{2d}{c}x \leq \alpha$ and the two constraints coming from (10) above. The first one ($\alpha^2 \leq 2\alpha^2 + 2\alpha x + x^2$) is clear as $\alpha > 0$, the second one gives,

$\frac{2d}{c}\alpha^2 + \frac{2d-c}{c}x\alpha \leq x^2$. (11)

The unconstrained optimum in (6), $\alpha = -\frac{2d-c}{4d}x$, is optimal as in case 1 if it meets the above constraints, which is true if, $\frac{2d}{c} \leq -\frac{2d-c}{4d} \Leftrightarrow \theta_1 \geq 4$, and inequality (11) is then fine.

Subcase 2.1: $\theta_1 \geq 4$

As in subcase 1.1.1, the ratio $\frac{\Delta^*}{u^*}$ is given by (8).

Subcase 2.2: $1 \leq \theta_1 \leq 4$

To minimize $2d\alpha^2 + (2d-c)x\alpha$, as required in (6), the optimal choice is now $\alpha = \frac{2d}{c}$, just as in subcase 1.2, which gives the ratio $\frac{\Delta^*}{u^*}$ in (9).

Step 2 Gathering our results we see that the optimal ratio under assumption **(2)** is

$\frac{\Delta}{u^*} = 1 + \theta_1 - \frac{4}{\theta_1} - \frac{8}{\theta_1^2}$ for $1 \leq \theta_1 \leq 4$; $\frac{\Delta}{u^*} = \frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1)$ for $4 \leq \theta_1$,

where for θ_1 close to 1 the above ratio is negative.

And under assumption **(3)** it is,

$\frac{\Delta}{u^*} = \frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1)$ for $1.17 \leq \theta_1 \leq 4$; $\frac{\Delta}{u^*} = 1 + \theta_1 - \frac{4}{\theta_1} - \frac{8}{\theta_1^2}$ for $4 \leq \theta_1$.

In fact, $\frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1) \geq 1 + \theta_1 - \frac{4}{\theta_1} - \frac{8}{\theta_1^2}$ for all $\theta_1 \geq 0$, because the left function is convex, the right one is concave, and they are equal and equally sloped at $\theta_1 = 4$. Moreover the function $1 + \theta_1 - \frac{4}{\theta_1} - \frac{8}{\theta_1^2}$ is negative for $\theta_1 \leq 2$, so for $1 \leq \theta_1 \leq 1.17$ the relevant function is the one in subsubcase 1.1.2,

$\frac{\Delta}{u^*} = (\theta_1 - 1) + \frac{2(2-\theta_1)\sqrt{\theta_1 - 1}}{2\sqrt{\theta_1 - 1} + \theta_1 + 2} - \frac{8(\theta_1 - 1)}{(2\sqrt{\theta_1 - 1} + \theta_1 + 2)^2}$.

5.1 Example

$u_1 = x_1 - 2x_1x_2 - \frac{3}{2}x_1^2$, we have $x = \frac{1}{5}$ and $\theta_1 = \frac{4}{3}$, which fits this subsubcase 1.1.1.

$\alpha^* = \frac{(\theta_1 - 2)x}{4} - \frac{1}{30}$, $\beta^* = 2\alpha^2 + 2x\alpha + x^2 = \frac{26}{900}$, $\gamma^* = 2\alpha^2 - \beta = -\frac{2}{75}$.

$z = \alpha + \sqrt{\beta - \alpha^2} = \frac{2}{15}$ and $z' = \alpha - \sqrt{\beta - \alpha^2} = -\frac{1}{5} = -x^*$.

$\lambda = \frac{\gamma - zz'}{\frac{1}{2}(z^2 + z'^2) - zz'} = 0 \Rightarrow \lambda' = \frac{1}{2}$.

The ratio $\frac{\Delta}{u^*} = \frac{1}{8}\theta_1^2 + \frac{1}{2}(\theta_1 - 1)\frac{7}{18} = 39\%$.

The optimal lottery needs, $x + z = \frac{1}{3}$, $x + z' = 0$, $\lambda' = \frac{1}{2}$.

	0	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{1}{2}$	0
0	0	$\frac{1}{2}$

6 Provision of Public good

$u_1(x_1, x_2) = A(x_1 + x_2) - B(x_1 + x_2)^2 - Cx_1$ and $I = [0, \infty[$.

Comparing with the general model, we have, $a = A - C$, $b = A$, $c = 2B$, $d = e = B$. We develop the computations with a,b,c,d,e all strictly positive. First we need to say that this is a case with $c = 2d$, special in the sense that we have infinitely many Nash equilibria, covering the interval $x_1 + x_2 = \frac{a}{2d}$; $x_{NE} = x = \frac{a}{2d+c} = \frac{a}{4d}$. The total utility is constant and of course we look for an improvement that is symmetric, so our benchmark is the utility of the symmetric Nash equilibrium, that is: $u^* = \frac{ab}{4d}$. Note that this knife edge case is without loss of generality, as nearby problems with c either above or below $2d$ would still make CC-improvement possibles.

We are looking for,

$Max\ \Delta(\alpha, \beta, \gamma) = (b - a)\alpha - 2d\beta - 2d\gamma$

such that,

$2\alpha^2 - \beta \leq \gamma \leq \beta$

and either (9) or (10), and finally the support conditions, $-x \leq \alpha$ and $\beta \leq x^2 + 2\alpha x + 2\alpha^2$.

Note that irrespective of (9) or (10), the constraints on γ are only lower bounds, so the optimal γ is $2\alpha^2 - \beta$, and then β disappears from the objective function, and thus our new objective function is a function of α and is given by, $\Delta = (b-a)\alpha - 4d\alpha^2$. We only have to check that the constraints on β are compatible.

7 Case 1

We assume (10), which gives us the following constraints,

$-\alpha \leq x \leq \alpha$ and $5\alpha^2 \leq \beta \leq x^2 + 2\alpha x + 2\alpha^2 \Leftrightarrow 3\alpha^2 - 2x\alpha \leq x^2 \Leftrightarrow (\alpha - x)(3\alpha + x) \leq 0$, therefore the system reduces to,

$\frac{-x}{3} \leq \alpha \leq x$. We may further write the two constraints as $0 \leq \alpha \leq x$, as a negative α will make Δ negative.

The unconstrained optimum is, $\alpha^* = \frac{b-a}{8d} > 0$.

Clearly, α^* satisfies the constraint iff, $\theta_2 \leq 3$, where $\theta_2 = \frac{b}{a} > 1$.

7.1 $1 \leq \theta_2 \leq 3$

For $\alpha^* = \frac{b-a}{8d}$, we have the optimal surplus, $\Delta = \frac{(b-a)^2}{16d}$, and the ratio, $\frac{\Delta}{u*} = \frac{\theta_2}{4} + \frac{1}{4\theta_2} - \frac{1}{2}$, which gives a maximum of 33% improvement at $\theta_2 = 3$.

7.2 $\theta_2 \geq 3$

Our candidate for α is $\alpha^* = x = \frac{a}{4d}$, and we have the optimal surplus, $\Delta = \frac{ab-2a^2}{4d}$, and thus the ratio, $\frac{\Delta}{u^*} = 1 - \frac{2}{\theta_2}$, i.e. for a very large θ_2, the relative CC-improvement goes to 100%.

8 Case 2

We assume (9). The following are the constraints,

$\alpha \geq x$, $\alpha^2 \leq \beta \leq x^2 + 2\alpha x + 2\alpha^2$ and $\frac{a}{2}\alpha + d\beta + 2d(2\alpha^2 - \beta) \leq \frac{a^2}{16d}$. Considering the constraint on β, we can reduce the third constraint to $\alpha^2 \leq x^2$.

Thus our constraints are $\alpha \geq x$ and $\alpha^2 \leq x^2$, which implies $\alpha = x = \frac{a}{4d}$, which is the same as in the Subcase 1.2. Thus the optimal surplus is $\Delta = \frac{ab-2a^2}{4d}$, with the relative CC-improvement ratio $\frac{\Delta}{u^*} = 1 - \frac{2}{\theta_2}$. Note here that the only constraint on θ_2 is that it should be ≥ 2, as otherwise the ratio becomes negative.

9 Example

$u_1(x_1, x_2) = 2x_1 + 3x_2 - 2x_1^2 - 2x_2^2 - 4x_1x_2$

$\theta_2 = \frac{b}{a} = \frac{3}{2}$, which fits in subcase 1.1.

$\alpha^* = \frac{1}{16}, \beta^* = \frac{13}{128}, \gamma^* = \frac{-3}{32}, z = \frac{3}{8}, z^{'} = \frac{-1}{4} = -x.$

The ratio, $\frac{\Delta}{u^*} = \frac{1}{24} = 4.16\%$.

	$\frac{5}{8}$	0
$\frac{5}{8}$	0	$\frac{1}{2}$
0	$\frac{1}{2}$	0

Part I

Class 2

The model here is,

$u_1(x_1, x_2) = ax_1 + bx_2 + cx_1x_2 - dx_1^2 + ex_2^2$, and $I = \mathbb{R}_+$.

10 Economic Example: Bertrand Duopoly

$u_1(x_1, x_2) = (D - Ax_1 + Bx_2)x_1$

When $c > 2d$ the game has no Nash equilibrium (NE), because I is unbounded. Even if I is bouded, the NE has both the players using the largest feasible strategy, and is clearly Pareto optimal, hence not CC-improvable. Thus the only interesting case in this class is when $c < 2d$; the unique NE is then given by $x^*_{i=1,2} = x = \frac{a}{2d-c}$, and the NE payoff is $u^* = dx^2$.

We use the notation $(\alpha, \beta, \gamma) = (E_L[Z_1], E_L[Z_1^2], E_L[Z_1 \cdot Z_2])$.

The CC payoff is $u_1(L) = u^* + x^*c\alpha + c\gamma - d\beta$. Therefore, our CCE, L_o, improves upon the Nash if and only if $\Delta = x^*c\alpha + c\gamma - d\beta > 0$. Next we write the CCE conditions when Player 1 plays a pure strategy x_1 and Player 2 follows L_o^m:

$$u_1(x_1 \otimes L_o^m) = ax_1 - dx_1^2 + cx_1E_{L^m}[x^* + Z_1] = (2dx^* + c\alpha)x_1 - dx_1^2 \qquad (1)$$

, and the CCE condition is $\max u_1(x_1 \otimes L_o^m) \leq u_1(L) = u^* + \Delta$. Finally we must ensure the support conditions $-x^* \leq \alpha$ and $\beta \leq x^{*2} + 2\alpha x^* + 2\alpha^2$. Note that under our assumption $c \leq 2d$, inequality $\alpha \geq -x^*$implies $\alpha \geq -\frac{2dx^*}{c}$, i.e., the coefficient of x_1 in (1) is non negative, which in turn gives, the optimal pure strategy for Player 1 as,

$x_1 = \frac{1}{2d}(\frac{2ad}{2d-c} + c\alpha)$, with the $\max u_1(x_1 \otimes L_o^m) = \frac{1}{4d}(2dx^* + c\alpha)^2$.

Thus the CCE condition is, $\frac{c^2}{4d}\alpha^2 + d\beta - c\gamma \leq 0$.

Thus, we have the following CC-improvement problem,

$\max \Delta(\alpha, \beta, \gamma) = cx\alpha - d\beta + c\gamma$

such that,

$\max\{2\alpha^2 - \beta, \frac{c}{4d}\alpha^2 + \frac{d}{c}\beta\} \leq \gamma \leq \beta,$

$-x \leq \alpha$ and $\beta \leq x^2 + 2\alpha x + 2\alpha^2.$

For a fixed α,β, we must choose $\gamma = \beta$, which reduces our problem to,

$\max cx\alpha + (c - d)\beta,$

under the constraints,

$\alpha^2 \leq \beta$ and $\frac{c}{4d}\alpha^2 \leq \frac{c-d}{c}\beta \quad (2)$

$-x \leq \alpha$ and $\beta \leq x^2 + 2\alpha x + 2\alpha^2.$

Check that $c \leq d$ delivers an impossibility: if $c < d$ we get $\alpha = \beta = \gamma = \Delta = 0$; similarly $c = d$ gives $\alpha = 0$ then $\Delta = 0$. So we assume from now on $c > d$.

Note that $\frac{c^2}{4d(c-d)} > 1$, and therefore we ignore the first inequality of (2), and thus the constraint on β is,

$\frac{c^2}{4d(c-d)}\alpha^2 \leq \beta \leq x^2 + 2\alpha x + 2\alpha^2$.

We must choose β as large as possible, i.e., $\beta = x^2 + 2\alpha x + 2\alpha^2$, thereby leading us to the problem in one dimension:

$\max_\alpha (3c - 2d)x\alpha + 2(c - d)\alpha^2 + (c - d)x^2$

under the constraints,

$-x \leq \alpha$

$\frac{c^2}{4d(c-d)}\alpha^2 \leq x^2 + 2\alpha x + 2\alpha^2.$

Note that the improvement Δ is positive for $\alpha = 0$, corresponding to $\beta = \gamma = x^{*2}$, i.e., the lottery L_0 choosing $(2x^*, 2x^*)$ or $(0, 0)$, each with probability $\frac{1}{2}$. As the coefficients of α and α^2 in the function to maximize are postive, we can ignore negative values of α, and we are left with the problem,

$\max_\alpha (3c - 2d)x\alpha + 2(c - d)\alpha^2 + (c - d)x^2$

under the constraint, $[\frac{c^2 - 8d(c-d)}{4d(c-d)}]\alpha^2 - 2x\alpha - x^2 \leq 0$

in which the optimal solution α^* is the largest value permitted by the constraint. We have, $1 < \frac{c}{d} = \theta_3 < 2$, and the constraint on α is satisfied if $c^2 - 8d(c - d) \leq 0$, which holds true, if and only if $2(2 - \sqrt{2}) \simeq 1.17 \leq \theta_3 < 2$. In this case any positive α meets the constraint, and there are CC-improvements of any size! This is not too surprising because as soon as $\frac{c}{d} > 1$ the efficient surplus in the original game is unbounded.

Note that for $1 < \theta_3 < 1.17$, the optium improvement is bounded, and the optimal α is the positive root of $[\frac{c^2 - 8d(c-d)}{4d(c-d)}]\alpha^2 - 2x\alpha - x^2 = 0$, which is $\alpha^* = \frac{2x\sqrt{\theta_3 - 1}}{2 - \theta_3 - 2\sqrt{\theta_3 - 1}}$.

The optimal surplus, $\Delta = dx^2[\frac{2(3\theta_3-2)\sqrt{\theta_3-1}}{2-\theta_3-2\sqrt{\theta_3-1}} + \frac{8(\theta_3-1)^2}{[2-\theta_3-2\sqrt{\theta_3-1}]^2} + (\theta_3 - 1)$.

10.1 Example

This game illustrates the case where CC can improve as much as we want.

$u_1(x_1, x_2) = x_1 - x_1^2 + \frac{6}{5}x_1x_2$ ($a = 1$, $c = \frac{6}{5}$ and $d = 1$)

We have $x = 1.25$ and $u^* = 1.5625$. Take for instance $\alpha = 10$, hence $\beta = \gamma = 226.5625$. The corresponding improvement is, $(3c - 2d)x\alpha + 2(c - d)\alpha^2 + (c - d)x^2 = 60.3125$, and the corresponding lottery L_0 chooses $(22.5, 22.5)$ or $(0, 0)$, each with probabilities $\frac{1}{2}$.

11 Class 3 and the Diamond's search game

The symmetric payoffs in this class of games is given by, $u_1(x_1, x_2) = -ax_1 + cx_1x_2 - dx_1^2$, and $I = [0, 1]$. We assume $b = e = 0$, for simplicity of analysis. The economic example relevant for Class 3 is the Diamond's search game, with the symmetric payoffs, $u_1(x_1, x_2) = Cx_1x_2 - (Ax_1 + Dx_1^2)$, where, $a = A$, $c = C$ and $d = D$. Assuming player j plays x_j, the best response of player i is given by $br_i(x_j) = \frac{1}{2d}(cx_j - a)$. If player j plays $x_j = 0$, $br_i(x_j) = 0$, and if player j plays $x_j = 1$, $br_i(x_j) = 1$. The best response function is $br_i(x_j) = median(0, \frac{1}{2d}(cx_j - a), 1)$. If $c > a + 2d$, we have three Nash equilibria, $(0, 0)$, (x^*, x^*) where $x^* = \frac{a}{c-2d}$, and $(1, 1)$. If $c = a+2d$, we have $(1, 1)$ as Nash equilibrium, and hence un-improvable. If $c < a+2d$, the unique Nash equilibrium is $(0, 0)$, with zero payoffs for both players. This is the only interesting case.We look for a CC-equilibirum with positive payoff.

Suppose we have $u_i(x_1, x_2) \geq 0$, $i = 1, 2$, for some x_1,x_2, with at least one strict inequality. Set $s = x_1 + x_2$ and $p = x_1x_2$. Note the following: $s^2 \geq 4p$ and $s \geq 2p$, the latter is true because $(x_1, x_2) \in [0, 1]$. Then, $u_1(x_1, x_2) + u_2(x_1, x_2) > 0$ is arranged as follows,

$-as - ds^2 + 2p(d + c) > 0 \Leftrightarrow 2p(d + c) > ds^2 + as \Rightarrow 2p(d + c) > d(4p) + a(2p) \Rightarrow c > d + a$, which implies that $(0, 0)$ is not Pareto optimal. Thus, the Nash equilibrium payoff, $u^* = (0, 0)$, is Pareto optimal if and only if $c \leq a + d$, and therefore un-improvable by coarse correlation. Thus we assume,

$$a + d < c < a + 2d \qquad (1).$$

Let $\widetilde{u}$ be the symmetric Pareto-optimal payoff in our symmetric game, given by,

$\widetilde{u} = \max_{[0,1]}\{-ax + (c - d)x^2\} = c - a - d$, because the maximum value, $x = 1$.

Thus, the efficient surplus is $2\widetilde{u}$.

Let us have a symmetric lottery L, with parameters, $(\alpha, \beta, \gamma) = (E_L[Z_1],\ E_L[Z_1^2],\ E_L[Z_1.Z_2])$, and the symmetric expected payoff accruing to the players by playing the lottery is given by, $u_1(L) = -a\alpha + c\gamma - d\beta = \Delta$, because $u^* = (0, 0)$.

From before, we have, $\beta \geq \gamma$ and $\beta + \gamma \geq 2\alpha^2$. Moreover, the support $\{z, z'\}$ of L must be in $[0, 1]$, imposing new conditions on α, β, γ. Firstly, $0 \leq \alpha \leq 1$. From the proof of (iii) in Lemma 1, we have, $S = z + z' = 2\alpha$ and $P = zz' = 2\alpha^2 - \beta$. We need the polynomial $Y^2 - SY + P \geq 0$, at $Y = 0, 1$. For, $Y = 0$, $P \geq 0 \Rightarrow 2\alpha^2 \geq \beta$, and for $Y = 1$, $1 - S + P \geq 0 \Rightarrow 1 + 2\alpha^2 - \beta \geq 2\alpha$. Let us define $(w)_- = \min\{w, 0\}$, thus, we have the following conditions on α, β:

$$\beta \leq 2\alpha^2 + (1 - 2\alpha)_- \qquad (2).$$

For the symmetric lottery L to be a CCE, we have the following equilibrium condition which needs to be satisfied,

$\max_{[0,1]} u_1(x \otimes L^m) = (-a + c\alpha)x_1 - dx_1^2 \leq u_1(L)$, where x_1 the strategy to which the deviant player deviates. Following is the optimisation problem for the Diamond search game,

$\max_{\alpha,\beta,\gamma} \Delta = -a\alpha + c\gamma - d\beta$

subject to the following constraints,

$\beta \geq \gamma$

$\beta + \gamma \geq 2\alpha^2$

$\beta \leq 2\alpha^2 + (1 - 2\alpha)_-$

$\max_{[0,1]} u_1(x \otimes L^m) = (-a + c\alpha)x_1 - dx_1^2 \leq -a\alpha + c\gamma - d\beta \qquad (3)$

and $0 \leq \alpha \leq 1$.

Let us now define a simple diagonal lottery L^*, selecting $(0, 0)$ and $(1, 1)$, each with a probability of $\frac{1}{2}$. Thus, $\alpha = \beta = \gamma = \frac{1}{2}$, and the support conditions and (2) hold true. Given these values of α, β, γ, we have $u_1(L) = \Delta = \frac{1}{2}(c - a - d)$, which is a total of $(c - a - d) = \widetilde{u}$, i.e. 50% of the efficient surplus. For L^* to be a CCE, we need L^* to satisfy (3). For, $\alpha = \beta = \gamma = \frac{1}{2}$, the inequality (3) is,

$\max_{[0,1]}(-a + \frac{c}{2})x_1 - dx_1^2 \leq \frac{1}{2}(c - a - d)$. We have the following two cases,

Case (a): $\frac{a}{c} \geq \frac{1}{2}$

In this case, the coefficient of x_1 is negative and therefore the maximum $x_1^* = 0$, $\max_{[0,1]}(-a + \frac{c}{2})x_1 - dx_1^2 = 0$, thereby satisfying (3), because $c > a + d$.

Case (b): $\frac{a}{c} \leq \frac{1}{2}$

We have the coefficient of x_1 to be positive, and the maximum $x_1^* = \frac{c-2a}{4d}$, and the $\max_{[0,1]}(-a + \frac{c}{2})x_1 - dx_1^2 = \frac{(c-2a)^2}{16d}$. Assuming $\frac{a}{c} = a'$ and $\frac{d}{c} = d'$, the inequality (3) becomes,

$$a'^2 + 2d'^2 + 2a'd' - a' - 2d' + \tfrac{1}{4} \leq 0 \qquad (4),$$

which is the interior of an ellipse containing around 90% of a trapezium in a',d', defined by (1) and $a' \leq \frac{1}{2}$.

Thus, the '50% of efficient surplus' performance of the L^* is very good in this class of games.

Lemma 2 *Whenever the lottery L^* is a CCE, it is the optimal one, so the* 50% *share is the optimal improvement.*

Proof. Assume $d' > 0$, and compute, $\max_{[0,1]}(-a + c\alpha)x_1 - dx_1^2 = f(\alpha)$. Then ,

$\max_{[0,1]}[c\{(-a' + \alpha)x_1 - d'x_1^2\}] = cf(\alpha)$. For $0 \leq \alpha \leq a'$, the coefficient of x_1 is negative, and, $f(\alpha) = 0$.

For, $a' \leq \alpha \leq 1$, the coefficient of $x_1$is positive. The function above is therefore maximised at $x_1^* = \frac{\alpha - a'}{2d'}$, with $f(\alpha) = \frac{(\alpha - a')^2}{4d'}$.

Thus, we have the following optimisation problem (ignoring the restriction $0 \leq \alpha \leq 1$ for a while),

$\max_{\alpha,\beta,\gamma} -a\alpha + c\gamma - d\beta$,

under the constraints,

$\gamma \leq \beta \leq 2\alpha^2 + (1 - 2\alpha)_-$

$\beta + \gamma \geq 2\alpha^2$

and $f(\alpha) + a'\alpha \leq \gamma - d'\beta$ (cancelling c on both sides).

Fixing α, β, we have the following, condition on γ, $\max\{2\alpha^2 - \beta,\ f(\alpha) + a'\alpha + d'\beta\} \leq \gamma \leq \beta$, and we choose the largest value of $\gamma = \beta$ (optimal). Therefore our optimisation problem reduces in terms of α, β, as follows,

$\max -a'\alpha + (1 - d')\beta$ (ignore c, without loss of generality),

under the constraints,

$f(\alpha) + a'\alpha \leq (1 - d')\beta$

and $\alpha^2 \leq \beta \leq 2\alpha^2 + (1 - 2\alpha)_-$.

Fixing α, the optimal β is $\beta = 2\alpha^2 + (1 - 2\alpha)_-$, which leads us to the follwong problem in α,

$\max -a'\alpha + 2(1 - d')\alpha^2 + (1 - d')(1 - 2\alpha)_-$,

under the constraints,

$g(\alpha) = f(\alpha) + a'\alpha - (1 - d')[2\alpha^2 + (1 - 2\alpha)_-] \leq 0$

and $0 \leq \alpha \leq 1$.

Let us discuss the shape of the function, Δ, to be maximised on $[0, 1]$.

(i) α between 0 to $\frac{1}{2}$; $1 - 2\alpha > 0$, and therefore, $(1 - 2\alpha)_- = 0$.

Thus, the maximising function is a parabola,

$y_1 = -a'\alpha + 2\alpha^2(1 - d') = 2(1 - d')[\alpha - \frac{a'}{4(1-d')}] - \frac{a'^2}{2(1-d')}$.

At $\alpha = 0$, $y_1 = 0$, and at $\alpha = \frac{1}{2}$, $y_1 = \frac{1-a'-d'}{2} > 0$. The slope of the parabola,$\frac{\partial y_1}{\partial \alpha} = -a' + 4\alpha(1 - d')$, is

$-a'$ at $\alpha = 0$, and at $\alpha = \frac{1}{2}$, it is $2 - a' - 2d' > 0$.

(ii) α between $\frac{1}{2}$ to 1; $1 - 2\alpha < 0$, and therefore, $(1 - 2\alpha)_- = 1 - 2\alpha$.

Thus, the maximising function is a parabola,

$y_2 = -a'\alpha + 2\alpha^2(1-d') + (1-d')(1-2\alpha) = 2(1-d')[\alpha - (\frac{2+a'-2d'}{4(1-d')})]^2 + (1-d') - \frac{(2+a'-2d')^2}{8(1-d')}$.

At $\alpha = 1$, $y_2 = 1 - a' - d' > 0$, and the slope of the parabola, $\frac{\partial y_2}{\partial \alpha} = -(a' - 2d' + 2) + 4\alpha(1-d')$, which

takes a value of $-a'$ at $\alpha = \frac{1}{2}$ and at $\alpha = 1$, the slope is positive at $2 - a' - 2d'$.

The function, Δ, to be maximized (draw a figure) looks as follows on $[0,1]$: from 0 to $\frac{1}{2}$, an upward parabola starting at 0 with negative slope $-a'$ and ending at $\frac{1}{2}$ with positive value $\frac{1}{2}(1-a'-d')$, and positive slope $2-(a'+2d')$. From $\frac{1}{2}$ to 1, an upward parabola starting at $\frac{1}{2}$ with negative slope $-a'$ and ending at 1 with positive value $(1-a'-d')$, and positive slope $2-(a'+2d')$.

	0			$\frac{1}{2}$			1
Δ	0	$\searrow$	$\nearrow$	$\frac{1}{2}(1-a'-d')$	$\searrow$	$\nearrow$	$(1-a'-d')$

Thus, we know that from 0 to $\frac{1}{2}$, the function takes a value of $\frac{1}{2}(1-a'-d')$ at $\alpha = \frac{1}{2}$; but from $\frac{1}{2}$ to 1, the function first falls and then rises to $1-a'-d'$. This further implies that in $\frac{1}{2} < \alpha \leq 1$, there is

a range of values of α, where the function Δ is less than $\frac{1}{2}(1-a'-d')$. Clearly, the function in $\frac{1}{2} < \alpha \leq 1$, takes the value $\frac{1}{2}(1-a'-d')$ twice, thus finding the roots of the function, $-a'\alpha + 2\alpha^2(1-d') + (1-d')(1-2\alpha) = \frac{1}{2}(1-a'-d')$ we have $\alpha = \frac{1}{2}$ and $\frac{1}{2} + \frac{a'}{2(1-d')}$. Thus, the objective function is below $\frac{1}{2}(1-a'-d')$ for $\alpha \leq \alpha^* = \frac{1}{2} + \frac{a'}{2(1-d')}$.

Therefore the optimality of L^* can only be defeated if there is some α in $[\alpha^*, 1]$ such that $g(\alpha) \leq 0$. We know that L^* gives a payoff which is 50% of the efficient outcome $(= \frac{1-a'-d'}{2})$. For L^* to be optimal, there should not be any α which generates $\Delta > \frac{1}{2}\widetilde{u}$, i.e. the optimality of L^* can defeated if there is some α in $[\alpha^*, 1]$ such that $g(\alpha) \leq 0$.

To proceed with the discussion, we first need to compare α^* with a', so as to check the value of $f(\alpha)$ and thereby determine $g(\alpha)$. Rearranging (1), we have the following restriction on a', $1 - 2d' \leq a' \leq 1 - d'$, thus the maximum $a' = 1 - d'$, with $\alpha^* = 1$. If we assume $a' \geq \alpha^*$, we have $d' \leq 0$, therefore not possible. Thus, $\alpha^* = \frac{1}{2} + \frac{a'}{2(1-d')} \geq a'$, and therefore, $f(\alpha) = \frac{(\alpha-a')^2}{4d'}$.

Thus, in $[\alpha^*, 1]$, we have,

$$g(\alpha) = \frac{(\alpha-a')^2}{4d'} + a'\alpha - (1-d')(2\alpha^2 - 2\alpha + 1). \qquad (5)$$

Rearranging and simplifying the terms, we have,

$g(\alpha) = \alpha^2(\frac{1-8d'+8d'^2}{4d'}) + \alpha(\frac{2a'd'+4d'-a'-4d'^2}{2d'}) + (\frac{a'^2+4d'^2-4d'}{4d'})$, which represents a parabola.

From here we can have two cases:

Case 1: $g(\alpha^*) < 0$

L^* is not optimal because a small increase in α raises Δ and meets the constraints.

Case 2: $g(\alpha^*) > 0$

When $g(\alpha^*) > 0$, at α^*, the L^* is not a CCE. But there can be a lottery L^* in $\frac{1}{2} \leq \alpha \leq \alpha^*$, where L^*

is a CCE.

Consider $g(\alpha^*) > 0$ and $a' \leq \frac{1}{2}$. Clearly, $\alpha^* = \frac{1}{2} + \frac{a'}{2(1-d')} > a'$, $a' \leq \frac{1}{2}$, thus (5) gives us

the $g(\alpha)$ in $[\frac{1}{2}, 1]$. At $\alpha = \frac{1}{2}$, $g(\alpha = \frac{1}{2}) = \frac{1+4a'^2-4a'+8a'd'-8d'(1-d')}{16d'}$, the sign of which depends on the

numerator, which is the same as (4), which we assume. Therefore, $g(\frac{1}{2}) \leq 0$. At $\alpha = 1$,

$g(\alpha = 1) = \frac{1+a'^2-2a'+4a'd'-4d'+4d'^2}{4d'}$. The numerator $(1-a')^2 + 4a'd' + 4d'^2 - 4d' > 0$, for $a' \leq \frac{1}{2}$ and $\frac{1-a'}{2} < d' < 1-a'$. Thus, $g(1) > 0$. Therefore, $g(\frac{1}{2}) \leq 0$, $g(\alpha^*) > 0$ and $g(1) > 0$, and g is a quadratic

and therefore single-peaked) function. Thus g is positive in $[\frac{1}{2}, 1]$, and at $\alpha = \frac{1}{2}$, L^* is a CCE.

Now consider $g(\alpha^*) > 0$ and $a' \geq \frac{1}{2}$. Here, we can still conclude that L^* is optimal if $g(\alpha)$ is a

downward parabola in $[\alpha^*, 1]$, which will be true if in the function of $g(\alpha)$, the coefficient of α^2 is

negative, i.e.,$\frac{1-8d'+8d'^2}{4d'} \leq 0 \Rightarrow 8d'(1-d') \geq 1 \iff d' \in [\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2} + \frac{\sqrt{2}}{4}]$. Notice that the ellipse given

by (4) intersects the axis $a' = 0$ precisely at those two values, and is entirely in the region of

$d' \leq \frac{1}{2} + \frac{\sqrt{2}}{4}$. Thus, it conincides with (4), which we assume to hold true for L^* to be a CCE.

Thus, L^* is the optimal CCE. ■

12 CONCLUSION

In this paper, we have analysed the notion of coarse correlation in a model where correlation *a la* Aumann does not offer anything more than the Nash equilibrium. We have defined and characterised an equilibrium notion called $k-$SSCCE for this quadratic game and constructed specific coarse correlated equilibria $k-$SSCCEs that improve upon the Nash payoffs.

We note that Gerard-Varet and Moulin (1978) also used similar devices and the solution concept to show *local* improvement upon the Nash equilibrium in specific duopoly models. In their main theorem, they were able to improve upon the Nash equilibrium of a class of duopoly games satisfying certain assumptions, using a *public* (actually, *diagonal*) 2-SSCD. However, their notion of improvement involved a specific criterion using sequences of devices close to the Nash point.

We however extended the result by Gerard-Varet and Moulin (1978) in this paper, as we constructed specific equilibria to improve upon the Nash payoff using the natural definition of improvement, without referring to any *local* arguments and were able to find the maximum possible improvement using our construction.

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