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MECHANISM DESIGN BY OBSERVANT AND INFROMED PLANNERS

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1 Introduction

The Gibbard-Satterthwaite Theorem is a fundamental result in the theory of mechanism design. It states that if a planner has to provide dominant strategy incentives for agents to reveal their private information, then this can accomplished only by allowing some agent (called the *dictator*) to always get his most preferred alternative. An assumption which is crucial for the result, is that the domain of preference ordering is complete. An extensive literature on *restricted domains* has emerged whose objective is to escape the strongly negative conclusion of the Gibbard-Satterthwaite Theorem by assuming that the domain of admissible preferences is restricted. For instance, if preferences are assumed to be *single-peaked*, then the median voter rule provides appropriate incentives for all agents to be truthful. If money is introduced in the model and preferences are assumed to be *quasi-linear*, then the rich theory of Groves-Clarke transfers applies and numerous possibility results exist. Other examples of restricted domains include *economic environments* where agents' preferences are assumed to be continuous and convex and environments where the objective function of the planner is *stochastic* and agents' preferences satisfy von-Neumann-Morgenstern axioms.

In this paper, we focus attention on a model which is related by quite different in spirit to therestricted domain model. We refer to this model as one with a *partially observant* planner. The idea is that ex-ante, an agent can have any preference ordering. However, *after* realization, the planner is able to observe some feature of these preferences. For instance, in a model of committee voting, the planner may be able to observe that voter 1's most preferred candidate is x, voter 2's least preferred candidate is y, voter 3 prefers w to z and so on. Thus, the planner has some (ex-post) information on preferences which could be based on commonly known ideological positions, personal dislikes etc. The mechanism, however, in keeping with the standard assumption has to designed ex-ante, i.e. before the realization of preferences. On the other hand, in the restricted domains model, which we refer to as the *partially informed* planner model, the planner has some ex-ante information on the structure of preferences. We believe that the observant planner model is a realistic one and worthy of attention. It is particularly plausible in the standard voting model where it may be unnatural to impose structure such as single-peakedness, convexity or cardinal-valuedness.

The observant planner model is also related to the complete information implementation model. In the latter, agents know each others preferences perfectly but the planner is completely ignorant. The problem here is to design a mechanism which will allow the planner to collate reports from each agent to infer something about the true state of the world. In the observant planner model, *partial* information about each voters' preference is not only common knowledge amongst the other voters but is also known to the planner.

The analysis in the observant planner model differs in crucial respects from that in the informed planner model. In the former, the domain of preferences remains complete unlike that in the informed planner model. However, the incentive compatibility condition is weaker. In particular we require only that no voter can gain by misrepresenting his preferences only for thos misrepresentations which are consistent with observed information. Suppose that the planner knows that voter i's peak is x. Then it must the case that the agent cannot do better than truth-telling than by announcing any other preference whose peak is x. The analysis, in the two models is thus independent of each other.

In the paper, we assume that the observant planner can observe the peak of each voters' preference ordering. We provide a complete characterization of incentive compatible social choice functions under a range assumption. We contrast this case with that of a restricted domain model where the planner has some ex-ante information about peaks. In particular, it is known that each voter's peak lies in some pre-specified set which is a subset of the set of alternatives.

Our results are as follows. In the observant planner case we show that if a range condition is satisfied, a social choice condition is incentive compatible if and only if, for every vector peaks, there is a voter and a set of alternatives over which this voter is a dictator. The choice of the voter who dictates and the set over which he does so could depend on what the planner observes. In the informed planner model on the other hand, incentive compatibility (or strategy-proofness) implies that there is a dictator over the range of the social choice function. There are therefore significant possibility results in the observant planner case unlike in the informed planner case. We also demonstrate that the dictatorship result in the informed planner is rather delicate and depends critcally on our assumption that the planner only has *a priori* information on voter peaks. We show by means of an example that if the planner had information on the alternatives which were ranked first and second, then non-dictatorial possibility results exist.

Although our results are quite intuitive they are not very easy to prove. There does appear to be a way to apply the Gibbard-Satterthwaite Theorem directly. A special feature of these models is that the "effective" domain of preferences are voter specific. The induction technique of coalescing or cloning voters used in various proofs (for example, Sen (2001)) can no longer be used. We develop a completely novel induction technique which can be used to provide yet another proof of the Gibbard Satterthwaite Theorem.

This paper is organized as follows. Section 2 lays out the basic notation while the next two sections discuss the observant and informed planner models. The last section concludes.

2 Basic Notation

The set $I = \{1, \cdots, N\}$ is the set of individuals or voters. The set of alternatives is the set A with $|A| = m$. Elements of A will be denoted by a, b, c, d etc. Let

$I\!\!P$ denote the set of strict orderings[1] of the elements of A. A typical preference ordering will be denoted by P_i where aP_ib will signify that a is preferred (strictly) to b under P_i. A preference profile is an element of the set $I\!\!P^N$. Preference profiles will be denoted by $P, \bar{P}, P'$ etc and their i-th components as $P_i, \bar{P}_i, P'_i$ respectively with $i = 1, \cdots, N$. Let $(\bar{P}_i, P_{-i})$ denote the preference profile where the i-th component of the profile P is replaced by $\bar{P}_i$.

For all $P_i \in I\!\!P$ and $k = 1, \cdots, m$, let $r_k(P_i)$ denote the k th ranked alternative in P_i, i.e., $r_k(P_i) = a$ implies that $|\{b \neq a | bP_ia\}| = k - 1$. For all P_i, the alternative $r_1(P_i)$ will be referred to as the *peak* of P_i.

For all $P_i \in I\!\!P$ and $B \subset A$, max (P_i, B) will denote the maximal element in B according to P_i.

3 The Partially Observant Planner

We assume that each individual i's preferences P_i are drawn from the set $I\!\!P$. The objectives of the planner are described by a social choice function defined below.

Definition 3.1 *A Social Choice Function (SCF) f is a mapping $f : I\!\!P^N \to A$.*

The preference ordering of voter i is i's private information. However, once it has been realized, each voter's peak $r_1(P_i)$ can be observed by the planner. Since the vaule of a SCF f at a preference profile could depend on more than voters' peaks at that profile, preferences have to be elicited from voters. The appropriate incentive constraints to ensure truth telling in this setting are described below.

Definition 3.2 *A SCF f is strategy-proof* (SP*) if, for all $i \in I$, $P_i \in I\!\!P$, $P_{-i} \in I\!\!P^{N-1}$, there does not exist $P'_i \in I\!\!P$, such that*

- $r_1(P_i) = r_1(P'_i)$ *and*

[1] A strict ordering is a complete, transitive and antisymmetric binary relation

- $f(P_i', P_{-i}) P_i f(P_i, P_{-i})$

Thus, incentive compatibility requires that no can individual profit from deviating from truth telling when these deviations are consistent with the information held by the planner. Observe that SP* differs from the standard notion of strategy-proofness only in this respect. (The latter condition does not require that the P_i' in the definition above satisfies the condition $r_1(P_i) = r_1(P_i')$). Of course, a SCF which is strategy-proof also satisfies SP*.

Our goal is to characterize the SCFs which satisfy SP*. We first note a familiar definition.

DEFINITION 3.3 *A SCF f is dictatorial in the range of f, denoted by R^f, if there exists an individual i such that for all profiles $P \in I\!\!P^N$, we have $f(P) = max$ (P_i, R^f).*

Dictatorial SCFs play a central role in the theory of strategy-proof SCFs. A dictatorial SCF is strategy-proof SCF. Moreover, according to the well-known Gibbard-Satterthwaite Theorem, a strategy-proof SCF which has a range of at least three alternatives, is dictatorial. The example below shows that this is no longer true under SP*.

EXAMPLE 3.1

For each $a \in A$, let $B^a \subset A$. Define a SCF f as follows.

For all $P \in I\!\!P^N$, $f(P) = \max\ (P_2, B^{r_1(P_1)})$

Thus voter 1 "offers" a set of outcomes for voter 2 to choose from. This set depends on voter 1's observable peak. Note that this SCF is SP*. It is also non-dictatorial. In fact, by choosing the set B^a to be either a singleton or the whole set A, the domain of preference profiles $I\!\!P^N$ can be partitioned arbitrarily into two sets, one over which voter 1 gets his maximum and the other over which 2 gets his maximum.

The set of all SP* SCFs can be obtained by suitably generalizing the example above.

Let f be a SCF. Let $a_1, a_2, \cdots, a_N \in A$. The set $R^f(a_1, a_2, \cdots, a_N)$ is defined as follows.

$$R^f(a_1, a_2, \cdots, a_N) = \{f(P)|r_1(P_i) = a_i, i = 1, 2, \cdots, N\}.$$

Thus $R^f(a_1, a_2, \cdots, a_N)$ is the range of f when the peak of P_i is constrained to be a_i for $i = 1, 2, \cdots, N$.

For all $P \in \mathbb{P}^N$, we will let $r_1(P)$ denote the vector $(r_1(P_1), r_1(P_2), \cdots, r_1(P_N))$.

We are ready for the characterization result.

THEOREM **3.1** *Let f be a SCF. Assume that for all $a_1, a_2, \cdots, a_N \in A$,*

$|R^f(a_1, a_2, \cdots, a_N)| \geq 3$. Then f is SP if and only if there exist maps $\phi^1 : A^N \to N$ and $\phi^2 : A^N \to 2^A$* [2] *such that*

$$\textit{for all } P \in \mathbb{P}^N,\ f(P) = \textit{max}\ (P_{\phi^1(r_1(P))}, \phi^2(r_1(P))$$

Proof: Sufficiency

This follows easily from the observation that at no profile P, can a voter can change the identity of the individual $\phi^1(r_1(P))$ nor the set $\phi^2(r_1(P))$ by changing his preference announcement. Of course, player $\phi^1(r_1(P))$ gets his best alternative in the feasible set by telling the truth and can only be worse-off by misrepresenting his preferences.

Necessity

We will prove the result by induction on N.

[2] We let 2^A denote the set of all non-empty subsets of A

Consider the case where there is only one voter say voter 1. Let f be a SCF satisfying SP*. The function ϕ^1 is the constant function whose value is 1. For all $a_1 \in A$, let $\phi^2(a_1) = R^f(a_1)$. An immediate consequence of SP* is that for all $P_1 \in I\!P$, $f(P_1) = \max\ (P_1, R^f(r_1(P_1))$. This establishes the result in the case where $N = 1$.

We now complete the induction step. Pick an integer $N > 1$ and assume that the result is valid for all integers $K < N$. Let $f : I\!P^N \to A$ be a SCF satisfying SP* and such that for all $a_1, a_2, \cdots, a_N \in A$, $|R^f(a_1, a_2, \cdots, a_N)| \geq 3$.

For all $a_i \in A$, let $I\!P^{a_i}$ denote the set of all orderings whose peak is a_i. For all $a_1, \cdots, a_N \in A$, let $D(a_1, \cdots, a_N)$ denote the Cartesian product of the sets $I\!P^{a_i}$ with $i = 1, \cdots, N$. Let $g : D(a_1, \cdots, a_N) \to A$ be the restriction of f to the set $D(a_1, \cdots, a_N)$, i.e. for all $P \in D(a_1, \cdots, a_N)$, $g(P) = f(P)$. Clearly g satisfies SP*. We will show that g is dictatorial over the range of g which we will denote simply as R^g. This will enough to prove the theorem because of the following observations (i) the sets $D(a_1, \cdots, a_N)$ obtained as $a_1, \cdots, a_N$ vary, form a partition of the set of profiles $I\!P^N$ (ii) we can let the values of $\phi^1(a_1, \cdots, a_N)$ and $\phi^2(a_1, \cdots, a_N)$ be the the identity of the dictator and the range respectively, of the appropriate function g.

Let $i \in I$ and $P_i \in I\!P^{a_i}$. Let

$$R^g(P_i) = \{x | x = g(P_i, P_{-i}) \text{ for some } P_{-i} \in D_{-i}(a_1, \cdots, a_N)\}$$ [3]

Thus $R^g(P_i)$ is the range of g when voter i's preference is fixed at P_i. Our objective is to show that there exists some i and P_i such that $|R^g(P_i)| \geq 3$. For future reference we let B denote the (unrestricted) range of g. (By assumption $|B| \geq 3$.

[3]The set $D_{-i}(a_1, \cdots, a_N)$ is defined in the obvious way: for all $P_i \in D_{-i}(a_1, \cdots, a_N)$, we have $(P_i, P_{-i}) \in D(a_1, \cdots, a_N)$.

Lemma 3.1 Let $x \in B$ and $P_i \in \mathbb{P}^{a_i}$ be such that $x = \max(P_i, B)$. Let $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$ be such that for all $j \neq i$, we have $x = r_2(P_i)$ whenever $x \neq a_j$. Then $g(P_i, P_{-i}) = x$.

Proof: Since $x \in B$, there exists $\bar{P} \in D(a_1, \cdots, a_N)$ such that $g(\bar{P}) = x$. Now pick a voter $j \neq i$ and switch his preference ordering from $\bar{P}_j$ to P_j. Suppose $g(P_j, \bar{P}_{-j}) = w$. If xP_jw, then j will gain by announcing $\bar{P}_j$ instead of his true preference P_j at profile $(P_j, \bar{P}_{-j})$. If wP_jx, then $w = a_j$ and $w\bar{P}_jx$. Then j will be better off by announcing P_j instead of announcing his true preference $\bar{P}_j$ at profile $\bar{P}$. Therefore $w = x$. Repeating this argument for all $j \neq i$, we obtain $g(\bar{P}_i, P_{-i}) = x$. Suppose $g(P_i, P_{-i}) = w$. If $w \neq x$, then xP_iw because $x = \max$ (P_i, B). But then i will gain by announcing $\bar{P}_i$ rather than his true preference P_i at the profile (P_i, P_{-i}). Therefore $g(P_i, P_{-i}) = x$. ■

According to Lemma 3.1, the maximal element in B according to P_i is the outcome under g of the profile where all other voters rank this element as "high as possible". We also record a trivial corollary of the Lemma.

Corollary An immediate consequence of Lemma 3.1 is that if $x = \max$ (P_i, B), then $x = R^g(P_i)$.

Lemma 3.2 Let $P_i, \bar{P}_i \in \mathbb{P}^{a_i}$ be such that $\max(P_i, B) = \max(\bar{P}_i, B)$. Then $R^g(P_i) = R^g(\bar{P}_i)$.

Proof: Suppose not. Let $\max(P_i, B) = \max(\bar{P}_i, B) = x$. In view of Lemma 3.2, there must exist $y \neq x$ such that $y \in R^g(P_i)$ but $y \notin R^g(\bar{P}_i)$. Construct $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$ as follows. For all $j \neq i$

- if $x = a_j$, then $r_2(P_j) = y$
- if $y = a_j$, then $r_2(P_i) = x$
- if x and y are both distinct from a_j, then $r_2(P_j) = y$ and $r_3(P_j) = x$.

In other words, under P_j, y is better than x whenever possible. In addition, they are also both ranked as "high as possible".

We first claim that $g(P_i, P_{-i}) = y$. In order to see this, note that since $y \in R^g(P_i)$, there exists $P'_{-i} \in D_{-i}(a_1, \cdots, a_N)$ such that $g(P_i, P'_{-i}) = y$. Now pick a arbitrary voter $j \neq i$ and let $g(P_i, P_j, P'_{i,j}) = w$. Oberve that if yP_jw, then j would gain by announcing P'_j instead of his true preference P_j at the profile $(P_i, P_j, P'_{i,j})$. On the other hand, if wP_jy, then $w = a_j$ and once again SP* would be violated because j would gain by announcing P_j instead of P'_j at (P_i, P'_i). Therefore $w = y$ and applying this argument repeatedly, we claim that $g(P_i, P_{-i}) = y$.

We claim next that $g(\bar{P}_i, P_{-i}) = x$. To see this consider all voters $j \neq i$ and let $\bar{P}_j \in I\!P^{a_j}$ be the ordering obtained by just reversing the ranking of x and y whenever possible leaving preferences over all other alternatives undisturbed. If either x or y coincide with a_j, $\bar{P}_j = P_j$. It follows from Lemma 3.1 that $g(\bar{P}) = x$ (since all voters other than i rank x as high as possible). Now change voter j's ordering from $\bar{P}_j$ to P_j. The new outcome can only be either x or y. But it cannot be y because $y \notin R^g(\bar{P}_i)$. Proceeding to the end of the sequence and repeating the same argument, we obtain $g(\bar{P}_i, P_{-i}) = x$.

But this will violate SP* because voter i whose true preference is P_i will be strictly better off by announcing $\bar{P}_i$ in the profile $(\bar{P}_i, P_{-i})$. ■

Lemma 3.3 Either $|R^g(P_i)| = 1$ or $|R^g(P_i)| \geq 3$.

Proof: Suppose not. Suppose there exists $x, y, z \in B$ with $x, y \in R^g(P_i)$ and $z \notin R^g(P_i)$. In view of the Corollary, we can assume without loss of generality that $x = \max(P_i, B)$. Also applying Lemma 3.2, we can assume that $r_2(P_i) = x$, $r_3(P_i) = z$ and $r_4(P_i) = y$ if $x \neq a_i$ and $r_2(P_i) = z$ and $r_2(P_i) = y$ if $x = a_i$. Now choose $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$ such that for all $j \neq i$,

- if $a_j = y$, then $r_2(P_j) = z$
- if $a_j = z$, then $r_2(P_j) = y$
- if $a_j \neq y, z$, then $r_2(P_j) = z$ and $r_3(P_j) = y$.

We claim that $g(P_i, P_{-i}) = y$. We use an argument identical to one we employed in the previous Lemma. Let $\bar{P}_{-i} \in D_{-i}(a_1, \cdots, a_N)$ be obtained by reversing wherever possible, the alternatives y and z in the preferences of all voters $j \neq i$ in P_{-i}. Since $y \in R^g(P_i)$ and y is ranked as "high as possible", it follows (by using an argument in Lemma 3.2) that $g(P_i, \bar{P}_i) = y$. Now progressively switch the preferences of all voters $j \neq i$ from $\bar{P}_j$ to P_j. All along this sequence, the outcome is either y or z because they are the only two alternatives over which preferences are changing. But the outcome can never be z anywhere along the sequence because $z \notin R^g(P_i)$. Therefore $g(P_i, P_{-i}) = y$.

Let $P_i' \in I\!\!P^{a_i}$ be obtained by switching, (if possible), y and z in the ordering P_i. Notice that at the profile (P', P_{-i}), z is ranked as "high as possible" in the ranking in all the voters'preferences. Since $z \in B$, it follows from the argument used in Lemma 3.1 and elsewhere that $g(P_i', P_{-i}) = z$. But then SP* is violated because zP_iy and $g(P_i, P_{-i}) = y$. This proves the lemma. ■

LEMMA 3.5 It cannot be the case that for all $i \in I$ and $P_i \in I\!\!P^{a_i}$, we have $|R^g(P_i)| = 1$.

Proof: Suppose not. Pick $P_i \in I\!P^{a_i}$ and let $R^g(P_i) = x$. For all $j \neq i$, pick $\bar{P}_j \in I\!P^{a_j}$ such that x is ranked "as low as possible" i.e. if $x = a_j$, then $r_1(\bar{P}_j) = x$; otherwise x is ranked last in $\bar{P}_j$. It follows immediately from the definition of the set $R^g(P_i)$ that $g(P_i, \bar{P}_{-i}) = x$. Let $\bar{P}_i \in I\!P^{a_i}$ be such that x is ranked "as low as possible". Now pick some $j \neq i$. Since $|R^g(\bar{P}_j| = 1$ by hypothesis and since $g(P_i, \bar{P}_{-i}) = x$, it follows that $g(\bar{P}) = x$. Let $\tilde{P} \in D(a_1, \cdots, a_N)$ be an arbitrary profile. By changing voter preferences progressively from the $\bar{P}$ profile to the $\tilde{P}$ profile and applying SP* repeatedly, we can conclude that $g(\tilde{P}) = x$. But this implies that $B = \{x\}$ which contradicts our assumption that $|B| \geq 3$. ■

We know from Lemma 3.5 that there exists $i \in I$ and $P_i \in I\!P^{a_i}$ such that $|R^g(P_i)| \geq 3$. Let $h : D_{-i}(a_1, \cdots, a_N) \to A$ be defined as follows. For all $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$

$$h(P_{-i}) = g(P_i, P_{-i})$$

Clearly h is a SCF defined over a society of $N-1$ voters. It is trivial to check that h satisfies SP*. Moreover, since the range of h is $R^g(P_i)$, it has at least three elements. We can therefore apply the induction hypothesis to conclude that that there exists a voter, say j where $j \in I - \{i\}$ such that for all $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$, we have

$$g(P_i, P_{-i}) = h(P_{-i}) = \max(P_j, R^g(P_i))$$

Here $\phi^2(a_1, .., a_{i-1}, a_{i+1}, .., a_N) = \text{range } h = R^g(P_i)$.
We complete the proof by showing that j is also the dictator in g.

In order to do this let $\bar{P}_i \in I\!P^{a_i}$ be such that it involves a switch of two alternatives which were ranked consecutively in P_i. In other words, there exists x and y such that $r_k(P_i) = x$ and $r_{k+1}(P_i) = y$ for some integer k and $r_k(\bar{P}_i) = y$ and $r_{k+1}(\bar{P}_i) = x$. The ranking of all alternatives other than x and y are unchanged in $\bar{P}_i$ relative to P_i. Suppose that there exists $P'_{-i} \in D_i(a_1, \cdots, a_N)$ such that $g(P_i, P'_{-i}) \neq g(\bar{P}_i, P'_{-i})$. In order for SP* not to be violated it must be the case that $g(P_i, P_{-i}) = x$ and $g(\bar{P}_i, P_{-i}) = y$. Since j dictates in h, it follows that $x = \max(P_j, R^g(P_i))$. Furthermore, x is not the peak of P_j; otherwise $R^g(P_i) = \{x\}$ which contradicts our hypothesis that $|R^g(P_i)| \geq 3$. In view of Lemma 3.4, there are two cases to consider.

Case A $|R^g(\bar{P}_i)| \geq 3$.

It follows from the induction hypothesis that there exists a voter $k \neq i$ such that for all $P_{-i} \in D_i(a_1, \cdots, a_N)$, we have

$$g(\bar{P}_i, P_{-i}) = \max\,(P_k, R^g(\bar{P}_i))$$

It is clear that k must be distinct from j; otherwise $g(P_i, P'_{-i}) = g(\bar{P}_i, P'_{-i}) = x$ which violates our hypothesis. Assume therefore that k and j are distinct. Now pick $\bar{P}_{-i} \in D_i(a_1, \cdots, a_N)$, such that $r_2(\bar{P}_j) = y$ and $r_2(\bar{P}_k) = x$. (Note that the peaks of P_j and P_k cannot lie in the sets $R^g(P_i)$ and $R^g(\bar{P}_i)$ respectively. If they did then the cardinality of these sets be one instead of at least three as we have assumed.) Then $g(P_i, \bar{P}_{-i}) = y$ and $g(\bar{P}_i, \bar{P}_{-i}) = x$ and SP* would be violated since xP_iy by assumption.

Case B $|R^g(\bar{P}_i)| = 1$.

An immediate consequence of this assumption is that $g(\bar{P}_i, P_{-i}) = y$ for all $P_{-i} \in D_i(a_1, \cdots, a_N)$.

Since $|R^g(P_i)| \geq 3$, there must exist $z \in R^g(P_i)$ distinct from both x and y. Since x and y are contiguous in P_i, z must be either worse than both x and y or better than both x and y according to P_i.

Suppose that the former is true. Pick $\bar{P}_{-i} \in D_{-i}(a_1, \cdots, a_N)$ such that $r_2(\bar{P}_j) = z$. It follows that $g(P_i, \bar{P}_{-i}) = z$. On the other hand $g(\bar{P}_i, \bar{P}_{-i}) = y$. Since $g(\bar{P}_i, \bar{P}_{-i}) = y$ and yP_iz, it follows that SP* will be violated. Now suppose instead that z is better than both x and y according to P_i. This impies that $z\bar{P}_iy$. Picking

$\bar{P}_{-i}$ as before, we observe that voter i will be better off announcing P_i instead of his true preference $\bar{P}_i$ at the profile $(\bar{P}_i, \bar{P}_{-i})$.

We have established that $g(P_i, P_{-i}) = g(\bar{P}_i, P_{-i})$ for all $P_{-i} \in D_{-i}(a_1, \cdots, a_N)$. By considering a sequence of switches of contiguous alternatives, we can demonstrate the same equality for all $\bar{P}_i \in \mathbb{P}^{a_i}$. But $g(P_i, P_{-i}) = \max(P_j, R^g(P_i))$. Therefore, the set $R^g(P_i)$ does not depend on P_i. Writing it as R^g, we have

$$g(P) = \max(P_j, R^g) \text{ for all } P \in D(a_1, \cdots, a_N)$$

Thus voter j dictates in g over the range R^g which proves the result. ■

Remark 3.1 Example 3.1 is a special case of the characterization in Theorem 3.1 where $N = 2$, $\phi^1(a_1, a_2) = 2$ and $\phi^2(a_1, a_2) = B^{a_1}$ for all (a_1, a_2).

Remark 3.2 The proof of Theorem 3.1 has a special feature. The usual induction proofs of such propositions, such as the proof of the Gibbard-Satterthwaite Theorem in Sen (2001) employ the technique of coalescing or cloning voters in the induction step. This is done in order to define a SCF on a society of lower cardinality with the appropriate properties (strategy-proofness and unanimity). This makes the induction step relatively straightforward but entails the additional cost of having to establish the Theorem in the non-trivial case of $N = 2$. In the current setting, the cloning technique does not work because the peaks of all voters in the function g may be different. In order to define a SCF in a society of $N - 1$ voters we use a projection technique. Most of the effort in proving the result goes into showing that there exists a SCF induced on a N_1 society which satisfies the range requirement. Some of the methods here are reminiscent of the arguments developed in Barberà and Peleg (1990). In fact the object $R^g(P_i)$ can be interpreted as the *option set* offered by voter i to the voters $I - \{i\}$. However, a pleasant aspect of our approach is that the induction can begin at $N = 1$ which is a trivial case. We note that this approach can be used to give yet another proof of the Gibbard-Satterthwaite Theorem.

Remark 3.3 Note that for all $a_1, a_2, \cdots, a_N \in A$, it must be true that $a_i \notin \phi^2(a_1, a_2, \cdots, a_N)$ where $\phi^1(a_1, a_2, \cdots, a_N) = i$. In other words, if i is the dictator when the vector of peaks is $(a_1, a_2, \cdots, a_N)$, then the set of alternatives over which voter j is allowed to choose from cannot include j's peak. If it did, then $R^f(a_1, a_2, \cdots, a_N)$ would be a singleton consisting of this peak which would violate the assumption that this range has at least three alternatives.

Remark 3.4 We have noted (Remark 3.3) that the assumption that

$R^f(a_1, a_2, \cdots, a_N) \geq 3$ can be restrictive. It is however not difficult, though rather clumsy to relax this assumption. If $|R^f(a_1, a_2, \cdots, a_N)| = 1$, then the associated g function is constant. If $|R^f(a_1, a_2, \cdots, a_N)| = 2$, then the associated g function is defined by a *committee*. The latter comprises a set of winning coalitions which satisfy a monotonicity property. Suppose $R^f(a_1, a_2, \cdots, a_N) = \{x, y\}$. For any profile $P \in D(a_1, a_2, \cdots, a_N)$, we have $g(P) = x$ if and only if the set of voters who prefer x to y is a winning coalition.

Remark 3.5 We can provide some intuition for Theorem 3.1 and Remark 3.4. In order to characterize f which stisfies SP* we simply partition the domain $\mathbb{P}^N$ into sets with the property that while the planner cannot distinguish between two profiles in the same element of the partition, he can do so between profiles in different elements of the partition. The problem then "essentially" reduces to finding conventional strategy-proof SCFs over domains which constitute each element of the partition (we are not being precise here - formal arguments are required which we have not provided). It is, however, still not a trivial problem to solve (one might, for instance, be tempted to believe that the Gibbard-Satterthwaite Theorem could be applied to each "sub-domain", i.e. the problem over a particular element of the partition). This is because preferences are restricted in each of the sub-domains. In particular, we are not free to choose peaks of voter preferences.

4 The Partially Informed Planner

In this section, we consider the case where the planner has some ex-ante information about the peaks of individual preferences. Our objective is to contrast both the formulation and the results here with those of the previous model.

For all $i \in I$, let $A_i \subset A$. The set A_i is the set of admissible peaks of voter i. The planner thus has some ex-ante information about voter preferences.

Let $D_i = \{P_i \in \mathbb{P} | r_1(P_i) \in A_i\}$. We shall let D denote the Cartesian product of the sets $D_1, D_2, \cdots D_N$. Elements of the set D_i and D wil be referred to as an admissible preference for voter i and an admissible preference profile respectively. Finally, let D_{-i} denote the Cartesian product of the sets $D_1, \cdots, D_{i-1}, D_{i+1}, \cdots, D_N$.

Definition 4.1 *A Social Choice Function (SCF) is a mapping $f : D \to A$.*

Definition 4.2 *A SCF f is manipulable by voter i at (admissible) profile P via (admissible) ordering P_i' if*

$$f(P', P_{-i}) P_i f(P)$$

A SCF f is strategy-proof if it is not manipulable by an voter at any admissible profile.

The defintions above are completely standard in the restricted domain literature. In contrast to the observant planner model, the domain of preferences here are restricted but the incentive compatibility condition is stronger.

As in the previous section, let R^f denote the range of the SCF f.

THEOREM **4.1** *Let $f : D \to A$ be a SCF with $|R^f| \geq 3$. Then f is strategy-proof if and only if it is dictatorial.*

Proof: The sufficiency part of the result is, of course, trivial. We prove only the necessity part. The proof uses the same ideas as the proof of Theorem 3.1. However, many of the details are far more subtle.

We will prove the result by induction on N, the number of voters. The result is obvious in the case $N = 1$. In order to establish the induction step, we assume that the result holds for all societies of size K where K is an integer strictly less than some positive integer N.

Let $f : D \to A$ be a strategy-proof SCF with $|R^f| \geq 3$. For all $i \in I$ and $P_i \in D_i$, let

$$R^f(P_i) = \{x | x = f(P_i, P_{-i}) \text{ for some } P_{-i} \in D_{-i}\}$$

LEMMA 4.1 Let $P_i, \bar{P}_i \in D_i$ be such that $\max(P_i, R^f) = \max(\bar{P}_i, R^f)$. Then $R^f(P_i) = R^f(\bar{P}_i)$.

Proof: Let $\max(P_i, R^f) = \max(\bar{P}_i, R^f) = x$. We first show that $x \in R^f(P_i)$ and $x \in R^f(\bar{P}_i)$.

Since $x \in R^f$, there exists $\tilde{P} \in D$ such that $f(\tilde{P}) = x$. Let $f(P_i, \tilde{P}_{-i}) = w$. If $x \neq w$, then xP_iw. But then voter i will manipulate at $(P_i, \tilde{P}_{-i})$ via $\tilde{P}_i$. Therefore $f(P_i, \tilde{P}_{-i}) = x$ so that $x \in R^f(P_i)$. By an identical argument, $x \in R^f(\bar{P}_i)$.

Suppose that the Lemma is false. Then there must exist y distinct from x such that $y \in R^f(P_i)$ but $y \notin R^f(\bar{P}_i)$. Let $\bar{P}_{-i} \in D_{-i}$ be such that $f(P_i, \bar{P}_i) = y$.

Now construct $P_{-i} \in D_{-i}$ by raising y and x "as high as possible" in $\bar{P}_j$ for each $j \neq i$. In other words, if $y \in A_j$, then $r_1(P_j) = y$ and $r_2(P_j) = x$. If $y \notin A_j$, then $r_1(P_j) = r_1(\bar{P}_j)$, $r_2(P_j) = y$ and $r_3(P_j) = x$. Now progressively switch preferences of all voters $j \neq i$ from $\bar{P}_j$ to P_j. It follows from standard arguments that strategy-proofness implies that each stage the outcome remains y, i.e $f(P_i, P_{-i}) = y$.

Our goal is to show that $f(\bar{P}_i, P_{-i}) = x$. We do this in a sequence of steps.

Step 1 For all $j \neq i$, we construct $P'_j \in D_j$ by interchanging x and y in P_j. If it is not possible to do this, i.e $y \in A_j$ but $x \notin A_j$, we let $P_j = P'_j$. We claim that $f(P_i, P'_{-i}) \in \{x, y\}$. To see this, change the preferences of some $j \neq i$ from P_j to P'_j in the profile (P_i, P'_{-i}). Since $f(P_i, P_{-i}) = y$ and the only preference reversal between P_j and P'_j is that between x and y, it follows from standard strategy-proofness arguments that the new outcome can only be either x or y. Moreover the same argument holds as we progressively change preferences from P_{-i} to P'_{-i} which establishes that $f(P_i, P'_{-i}) \in \{x, y\}$.

Step 2 We claim that in fact, $f(P_i, P'_{-i}) = x$. Since $x \in R^f(P_i)$, there exists $\tilde{P}_{-i} \in D_{-i}$ such that $f(P_i, \tilde{P}_{-i}) = x$. Let $r_1(\tilde{P}_j) = b_j$ for all $j \neq i$. Similarly, let $r_1(P'_j) = a_j$ for all $j \neq i$. By a standard strategy-proofness argument, we can assume without loss of generality that for all $j \neq i$, if b_j is distinct from a_j, x and y, then $r_4(P'_j) = b_j$. [4] Similarly we can assume without loss of generality that if a_j, b_j, x and y are all distinct, then $r_3(\tilde{P}_j) = y$ and $r_4(\tilde{P}_j) = a_j$. Moreover, we can also assume without loss of generality that the ranking of all alternatives other than these four agree in $\tilde{P}_j$ and P'_j for all $j \neq i$. Now, suppose that contrary to our claim, $f(P_i, P'_{-i}) = y$. Start with the profile (P_i, P'_{-i}) and progressively (in some sequence) switch preferences of all voters $j \neq i$ to $\tilde{P}_j$. Note that at some stage the outcome must change because $f(P_i, \tilde{P}_{-i}) = x$ by assumption. Let k

[4]If b_j is indeed distinct from the other three alternatives, then b_j is worse than these alternatives in P'_j and can be shuffled around as long as it remains below these three alternatives, without affecting the outcome $f(P_i, P'_{-i})$ which we have established is either x or y.

be the first voter in the sequence such that outcome differs from y. Note that by strategy-proofness, this outcome can only be b_k. Moreover, if the outcome switches to z further along the sequence when, say, voter t switches preferences from P'_t to $\tilde{P}_t$. By strategy-proofness this is possible only if $a_t = b_k$ and $z = b_t$. We can conclude therefore that $f(P_i, \tilde{P}_{-i}) = b_k$ for some voter $k \neq i$. But this contradicts the assumption that $f(P_i, \tilde{P}_{-i}) = x$. Therefore $f(P_i, P'_{-i}) = x$.

STEP 3 We claim that $f(\bar{P}_i, P'_{-i}) = x$. Suppose instead that $f(\bar{P}_i, P'_{-i}) = z$ where z is distinct from x. Since $x = \max(\bar{P}_i, R^f)$ by assumption, $x\bar{P}_i z$. Therefore i will manipulate at $(\bar{P}_i, P'_{-i})$ via P_i. This completes the step.

STEP 4 We claim that $f(\bar{P}_i, P_{-i}) = x$. Recall that for all $j \neq i$, P'_j is obtained from P_j by interchanging x and y (which are contiguous in P_j). Now start with the profile $(\bar{P}_i, P'_{-i})$ and switch preferences of all $j \neq i$ progressively from P'_j to P_j. It follows from strategy-proofness that the outcome at each point in the sequence is either x or y. But it cannot be y because $y \notin R^f(\bar{P}_i)$ by hypothesis. Therefore $f(\bar{P}_i, P_{-i}) = x$.

We now complete the proof of the Lemma. Since $f(P_i, P_{-i}) = y$ and $f(\bar{P}_i, P_{-i}) = x$ where xP_i, y, voter i can manipulate. ■

LEMMA 4.2 $|R^f(P_i)| = 1$ or $|R^f(P_i)| \geq 3$.

Proof: Suppose not. Suppose there exists $x, y, z \in R^f$ with $x, y \in R^f(P_i)$ and $z \notin R^f(P_i)$.

We have shown in the proof of the earlier lemma that if an alterntive is maximal in R^f according to P_i, then it belongs to $R^f(P_i)$. We can therefore assume without loss of generality that $x = \max(P_i, R^f(P_i))$. Furthermore, applying Lemma 4.1 we can assume that x and z are contiguous i.e. there exists an integer k such that $r_k(P_i) = x$ and $r_{k+1}(P_i) = z$. Since $y \in R^f(P_i)$, there exists $P_{-i} \in D_{-i}$ such that

$f(P_i, P_{-i}) = y$. Moreover, using arguments that we have used repeatedly earlier, we can assume that for all voters $j \neq i$

- if $y \in A_j$, then $y = r_1(P_i)$ and $z = r_2(P_i)$
- if $y \notin A_j$, then $y = r_2(P_i)$ and $z = r_3(P_i)$.

For all $j \neq i$, let P_j' be obtained by switching (if possible) y and z in P_j. Progressively switch preferences of all voters $j \neq i$ in the profile (P_i, P_{-i}) from P_j to P_j'. It follows from strategy-proofness that at each stage of this procedure, the outcome can only be either y or z. But it cannot be z because $z \notin R^(P_i)$. Therefore $f(P_i, P_{-i}') = y$.

Let P_i' be the preference ordering obtained by switching x and z in P_i. Since both z and x are strictly better than y according to P_i, strategy-proofness implies that $f(P', P_{-i}) = y$. The proof of the Lemma is completed by showing that $f(P', P_{-i}')$ must, in fact, be z.

In order to establish this, note that $z = \max\ (P', R^f)$, so that $z \in R^f(P_i')$. We can now mimic the arguments in Step 2 of the previous lemma and we sketch them briefly. Since $z \in R^f(P_i')$, there exists $\bar{P}_{-i} \in D_{-i}$ such that $f(P_i', \bar{P}_{-i}) = z$. Moreover we can assume that z and y are contiguous (with z better than y whenever possible) in $\bar{P}_j$, $j \neq i$ and both these alternatives are ranked as high as possible. Now progressivly switch preferences from $\bar{P}_{-i}$ to P_{-i}'. The critical difference between the $\bar{P}_j$ and P_j' is that their peaks could differ. By making suitable (innocuous) assumptions regarding the ranking of alternatives other than y and z, it is possible to conclude that if $f(P_i', P_{-i}') \neq z$, then it must be the case that $f(P_i', P_{-i}') = a_k$ where $a_k = r_1(P_k')$ for some voter $k \neq i$. But this will contradict our earlier conclusion that the outcome at this profile is y. ■

LEMMA 4.3 It cannot be the case that for all $i \in I$ and $P_i \in D_i$, we have $|R^f(P_i)| = 1$.

The proof of this Lemma is identical to that of Lemma 3.4 and is omitted. The idea of the proof is as follows. If $R^f(P_i)$ is a singleton say x, then the outcome of every f, at every profile where voter i's preference is P_i must be x. In particular, the outcome where all voters $j \neq i$ rank x last, must also be x. Now fix the preferences of some $j \neq i$ and repeat the argument to conclude that x is the outcome even when it is ranked last by all voters. But strategy-proofness would then immediately imply that that the outcome at all profiles is x contradicting the assumption that the range of f has at least three elements.

It follows from Lemmas 4.2 and 4.3 that there exists a voter i and an ordering $P_i \in D_i$ such that $|R^f(P_i)| \geq 3$. Define the function $h : D_{-i} \rightarrow A$ as follows.

$$h(P_{-i}) = f(P_i, P_{-i}) \text{ for all } P_{-i} \in D_{-i}$$

It is trivial to check that h is strategy-proof. Moreover we have shown that $|R^h| \geq 3$. Therefore we can apply the induction hypothesis and infer that there exists $j \neq i$ such that for all $P_{-i} \in D_{-i}$,

$$f(P_i, P_{-i}) = \max\ (P_j, R^f(P_i))$$

In other words, if voter i's preference is P_i, then voter j dictates over $R^f(P_i)$. In order to complete the proof of the Theorem, it is required only to show that voter j continues to dictate even when i changes his preference. We can use the same arguments that we used to show the same thing in the proof of Theorem 3.1. We briefly sketch the details.

Consider a switch of two contiguous elements, say x and y in P_i does not change the outcome for any $P_{-i} \in D_{-i}$. Suppose that xP_iy and let the new preferences of i be denoted by $\bar{P}_i$. Suppose that $|R^f(\bar{P}_i)| \geq 3$. Applying the induction hypothesis, it follows tht there exists a voter k who dictates over $R^f(\bar{P}_i)$. If the outcome is to change when i changes his preferences, it must be true that j and k are distinct. But if they are then it is easy to derive a contradiction. Consider a profile for all players other than i where voter j'and k' s maximal elements in $R^f(P_i)$ and $R^f(\bar{P}_i)$ are y and x respectively. Then i would manipulate aat this profile when his true preferences are P_i by announcing $\bar{P}_i$. So suppose $|R^f(\bar{P}_i)| = 1$ and suppose that for some profile of preferences of the other voters, the outcome changes from x to y (this is the only change consistent with strategy-proofness). There must exist some z in $R^f(P_i)$ distinct from x and y. Pick a profile for voters other than i where z is maximal in R^f for voter j. There are two cases to consider. One is where x and y are bothbetter than z under P_i and the other is when x and y are both worse than z under P_i. (Recall that x and y are contiguous in P_i.) Since $R^g(\bar{P}_i)$ is a singleton, the outcome at the profile where i's preferences are $\bar{P}_i$ must be y. Suppose that yP_iz. Then i will manipulate when his preferences are P_i via $\bar{P}_i$.

The proof is now completed by observing that every ordering for voter i can be obtained by a sequence of switches elements starting from P_i. ∎

Remark 4.1 Like the proof of Theorem 3.1, it does not seem possible to provide a proof of Theorem 4.1 by the cloning method. Once again the difficulty is that the domain resrictions are voter-specific.

The dictatorship result does not hold if the planner has information about more than just top-ranked alternatives. We illustrate this with an example.

Example 4.1

Let $I = \{1, 2\}$ and $A = \{a, b, c, d\}$.

Suppose that the planner has the following information regarding the preferences of voter 1. He knows that if 1 ranks a first, then he ranks b second. There are no other restrictions regarding the ranking of alternatives. In other words, among the 12 possible pairs of first and second alternatives, exactly two, viz. a is first and

c second and a is first and d second are infeasible. There are no other restrictions on the preferences of other voters.

We claim that there is exists a non-dictatorial SCF in this setting. The outcome at any profile is voter 1's top ranked alternative if this alternative is b, c or d. If it is a then the outcome is the alternative in the pair $\{a, b\}$ which is higher ranked in voter 2's preferences.

It is easy to check that this SCF is strategy-proof. Voter 1 is not getting his peak only in the case where his peak is a. In this case he might get his second ranked alternative b. However, since 2 prefers b to a, there is no way for 1 to do better and get a.

This example appeared originally in Aswal, Chatterji and Sen (2001) where it was employed for a different purpose. We note that there are other ways to extend the spirit of the domain restrictions from the tops case analysed in this section to the more case. For instance, we may require that an alternative is never ranked in the kth position and so on.

5 Conclusion

This paper addresses the problem of mechanism design in two different models. The first is one where the planner is able to observe the actual realization of voter peaks. The second is one where the planner only has ex-ante information about possible peaks. The paper defines and characterizes ncentive compatible social choice functions in both settings. The class of such mechanisms is much larger in the observant planner case where the mechanism designer's information is ex-post.

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