Discussion Papers in Economics

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October 2008

Discussion Paper 08-07

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# Characterization of the Walrasian Equilibria of the Assignment Model *

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October 2008

**Abstract**

We study the assignment model where a collection of indivisible goods are sold to a set of buyers who want to buy at most one good. We characterize the extreme and interior points of the set of Walrasian equilibrium price vectors for this model. Our characterizations are in terms of demand sets of buyers. Using these characterizations, we also give a unique characterization of the minimum and the maximum Walrasian equilibrium price vectors. Also, necessary and sufficient conditions are given under which the interior of the set of Walrasian equilibrium price vectors is non-empty. Several of the results are derived by interpreting Walrasian equilibrium price vectors as potential functions of an appropriate directed graph.

JEL Classification: D50, C69, D44

*We thank Rakesh Vohra for useful comments. The first author thanks the Center of Economic Research (CentER) of Tilburg University and the Netherlands Organisation for Scientific Research (NWO) for financial support.

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# 1 Introduction

The classical Arrow-Debreu model (Arrow and Debreu, 1954) for studying competitive equilibrium assumes goods to be divisible (commodities). But economies with indivisible goods are common in many types of markets such as housing markets, job markets, and auctions with goods like spectrum licenses. This paper investigates economies with indivisible goods under the assumption that buyers have unit demand, i.e., every buyer can buy at most one good, and quasi-linear utility functions. The unit demand assumption is common, for example, in settings of housing and job markets. Even though buyers can buy at most one good, they have valuations (possibly zero) for every good.

In this model, the existence of a Walrasian equilibrium is guaranteed, and the set of Walrasian equilibrium price vectors form a complete lattice (Shapley and Shubik, 1972). In this paper, we are concerned with a *verification* problem. Suppose the seller announces a price vector, and every buyer submits his demand set, the set of all goods that give him the maximum payoff at the announced price vector. Then, we are concerned with the following verification questions given that the *only information available is the demand set of each buyer*:

1. *How can one verify if the announced price vector is a Walrasian equilibrium price vector?*

2. *How can one verify if the announced price vector is an extreme point or an interior point or the maximum or the minimum points in the set of Walrasian equilibrium price vectors?* [1]

To answer the first question, we show that a price vector is a Walrasian equilibrium price vector if and only if no set of goods is *overdemanded* and no set of goods is *underdemanded* at that price vector. Whether a set of goods is overdemanded or underdemanded can be verified using only demand set information of buyers. This characterization of Walrasian equilibrium price vector is pivotal in answering the other questions.

Concerning the second question, we show that every Walrasian equilibrium price vector is a *potential* of an appropriate directed graph. These potentials form a lattice, and we characterize the extreme points of this lattice in terms of shortest paths in the underlying directed graph. This characterization along with the characterization of a Walrasian equilibrium price vector enables us to characterize the extreme points of the Walrasian equilibrium price lattice. These characterizations also require verifications that can be done using demand set information of buyers only. Our characterization shows that at the extreme points of the set of Walrasian equilibrium price vectors no subset of a weakly overdemanded set of

[1] Given the complete lattice structure of the Walrasian equilibrium price vector space, the minimum and the maximum Walrasian price vectors are well defined.

goods is weakly underdemanded and no subset of a weakly underdemanded set of goods is weakly overdemanded. Similarly, we characterize the minimum and the maximum Walrasian equilibrium price vector.

Finally, we show that a price vector is an interior point of the Walrasian equilibrium price lattice if and only if the demand set of every buyer is a singleton and no two buyers have the same good in their demand sets. Notice that such demand sets are minimally informative, in the sense that every buyer's demand set consists of only the good he is allocated. Thus, the characterization shows that the *only* Walrasian equilibrium price vectors where demand set of every buyer consists of the good he is allocated, are the interior Walrasian equilibrium price vectors. However, the interior of the Walrasian equilibrium price lattice may be empty. We show that an interior Walrasian equilibrium price vector exists if and only if there is a unique efficient allocation and the number of buyers exceeds the number of goods.

In summary, we characterize the entire Walrasian equilibrium price vector set using demand set information of buyers only. Further, we provide necessary and sufficient conditions for the interior of the Walrasian equilibrium price vector space to be non-empty, i.e., Walrasian equilibrium price vector space to be full-dimensional.

We show an application of some of our main results. The application is in the design of iterative auctions for this model. If the valuations of buyers are private information, the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) is efficient and strategy-proof, where the payment of a buyer is his externality on other buyers. However, the VCG mechanism is a direct mechanism, requiring buyers to directly reveal their values on goods. In many practical settings, iterative auctions (i.e., ascending or descending price auctions) like the English (ascending price) auction, which generates the same outcome as the VCG outcome, is a preferred mechanism due to various reasons (Cramton, 1998). Although the English auction is known to mimic the outcome of the second-price Vickrey auction for the single good case, the extension of the English auction to the assignment model is not trivial. Demange et al. (1986) and Sankaran (1994) design such auctions. One can also design descending auctions that mimic the outcome of the VCG mechanism - see for example Mishra and Parkes (2008).

An interesting feature of these iterative auctions is that these are procedures to search for a Walrasian equilibrium for the assignment model (See de Vries et al. (2007) for a detailed discussion). Iterative auctions that search for the minimum Walrasian equilibrium price vector inherit the incentive properties of the VCG mechanism. [2] Using our results, we give a broad class of iterative auctions for this model. Every iterative auction in this class terminates at the minimum Walrasian equilibrium price vector. The auctions in Demange et al. (1986); Sankaran (1994); Mishra and Parkes (2008) fall into this class. Analogously, one can design

[2]Iterative auctions do not necessarily have a dominant strategy equilibrium but they have an ex post equilibrium in which the outcome of the VCG mechanism is implemented.

iterative auctions that terminate at the maximum Walrasian equilibrium price vector under truthful bidding behavior of buyers - Sotomayor (2002) is an example of such a descending price auction. Though truthful bidding is not an equilibrium in these auctions, these auctions are interesting *algorithms* to compute a Walrasian equilibrium. We give a broad class of iterative auctions that terminate at the maximum Walrasian equilibrium price vector under truthful bidding of buyers. The auction of Sotomayor (2002) falls into this class. Thus, our results serve to unify existing iterative auctions under one umbrella. [3]

The literature in the assignment model is long - for a survey, see Roth and Sotomayor (1990). The initial literature focuses on the structure of the set of Walrasian equilibria (Shapley and Shubik, 1972), its strategic properties (Leonard, 1983; Demange and Gale, 1985), and the relation with the core of an appropriate cooperative game (Shapley and Shubik, 1972; Roth and Sotomayor, 1988; Balinski and Gale, 1990; Quint, 1991). The studies of the core for our model is complementary to the study of Walrasian equilibria, since the core and the set of Walrasian equilibria are equivalent (Shapley and Shubik, 1972). However, this literature does not answer the verification question we address in this work.

There is also a literature that is concerned with the computation of Walrasian equilibrium prices using auction-like processes (Crawford and Knoer, 1981; Demange et al., 1986; Sankaran, 1994; Sotomayor, 2002). The notion of overdemanded and underdemanded sets of goods, which we use in our characterizations, has been used in this literature. Demange et al. (1986) use the notion of overdemanded goods to design an ascending auction that terminates at the minimum Walrasian equilibrium price vector. Analogously, Sotomayor (2002) uses the notion of underdemanded goods to design a descending auction that terminates at the maximum Walrasian equilibrium price vector. Both papers do not make any connection between these notions. Gul and Stacchetti (2000) consider a model where they allow a buyer to buy more than one good and having *gross substitutes* valuations. In such a model, a Walrasian equilibrium price vector is guaranteed to exist (Kelso and Crawford, 1982), and the set of Walrasian equilibrium price vectors form a complete lattice (Gul and Stacchetti, 1999). For such a model, they provide a generalization of Hall's theorem (Hall, 1935), which results in a necessary condition for a Walrasian equilibrium. Therefore, they do not characterize the set of Walrasian equilibrium price vectors.

## 2 The Model

There is a set of indivisible goods $N = \{0, 1, \ldots, n\}$ for sale to a set of buyers $M = \{1, \ldots, m\}$. Each buyer can be assigned to at most one good. The good 0 is a dummy good which can be assigned to more than one buyer. Denote $N_0 = N \setminus \{0\}$ as the set of real goods. The

[3] There are some iterative auctions in the literature which converge to a Walrasian equilibrium *approximately*, e.g., the auction in Crawford and Knoer (1981) and an auction in Demange et al. (1986). These auctions do not fall into our broad class of iterative auctions.

value of buyer $i \in M$ on good $j \in N$ is $v_{ij}$, assumed to be a non-negative real number. Every buyer has zero value on the dummy good. A **feasible allocation** $\mu$ assigns every buyer $i \in M$ a good $\mu_i \in N$ such that no good in $N_0$ is assigned to more than one buyer. Notice that a feasible allocation assigns every buyer a good (maybe the dummy good), but some goods may not be assigned to any buyer. We say good $j \in N$ is **unassigned** in $\mu$ if there exists no buyer $i \in M$ with $\mu_i = j$. Let $\Gamma$ be the set of all feasible allocations. An **efficient allocation** is a feasible allocation $\mu^* \in \Gamma$ satisfying $\sum_{i \in M} v_{i\mu_i^*} \geq \sum_{i \in M} v_{i\mu_i}$ for all $\mu \in \Gamma$.

A price vector $p \in \mathbb{R}_+^{n+1}$ assigns every good $j \in N$ a nonnegative price $p(j)$ with $p(0) = 0$. We assume quasi-linear utilities. Given a price vector $p$, the payoff of buyer $i \in M$ on good $j \in N$ at price vector $p$ is $v_{ij} - p(j)$. The **demand set** of buyer $i$ at price vector $p$ is $D_i(p) = \{j \in N : v_{ij} - p(j) \geq v_{ik} - p(k) \ \forall \ k \in N\}$.

Definition 1 *A* **Walrasian equilibrium** *(WE) is a price vector $p$ and a feasible allocation $\mu$ such that*

$$\mu_i \in D_i(p) \qquad \text{for all } i \in M \tag{\textbf{WE-1}}$$

*and*

$$p(j) = 0 \qquad \text{for all } j \in N \text{ that are unassigned in } \mu. \tag{\textbf{WE-2}}$$

*If $(p, \mu)$ is a WE, then $p$ is a* **Walrasian equilibrium price vector** *and $\mu$ is a* **Walrasian equilibrium allocation***.*

It is well known that a Walrasian equilibrium allocation is efficient and that the set of WE price vectors, which is non-empty, form a complete lattice (Shapley and Shubik, 1972). This implies the existence of a unique minimum WE price vector ($p^{min}$) and a unique maximum WE price vector ($p^{max}$).

The lattice corresponding to the WE price vectors is of a special shape - known as the "45 degree"-lattice (Quint, 1991). In case of three buyers and two goods with values $v_{11} = 5,\ v_{12} = 3,\ v_{21} = 3, v_{22} = 4,\ v_{31} = 2,\ v_{32} = 2$, the lattice shape of the WE price vector set is shown in Figure 1. Notice that the boundary of the lattice is defined by lines that are either parallel or at 45 degrees to the axes.

We define **demanders** of a set of goods $S \subseteq N_0$ at price vector $p$ as $U(S, p) = \{i \in M : D_i(p) \cap S \neq \emptyset\}$. We define the **exclusive demanders** of a set of goods $S \subseteq N_0$ at price vector $p$ as $O(S, p) = \{i \in M : D_i(p) \subseteq S\}$. Clearly, for every $p$ and every $S \subseteq N_0$, we have $O(S, p) \subseteq U(S, p)$. We denote the cardinality of a finite set $S$ as $\#S$. Given a price vector $p$, define $N^+(p) = \{j \in N : p(j) > 0\}$. By definition $0 \notin N^+(p)$ for any $p$.

Definition 2 *A set of goods $S$ is* **(weakly) overdemanded** *at price vector $p$ if $S \subseteq N_0$ and $\#O(S, p)(\geq) > \#S$.*


Figure 1: Lattice nature of WE prices

The notion of overdemanded sets of goods can be found in Demange et al. (1986) and Sankaran (1994), who use it as a basis for the design of ascending auctions for our model. For settings where a buyer can buy more than one good, the notion of overdemanded goods has been generalized in Gul and Stacchetti (2000) and de Vries et al. (2007), who also use it as a basis for the design of ascending auctions for general models.

Definition 3 *A set of goods $S$ is* **(weakly) underdemanded** *at price vector $p$ if $S \subseteq N^{+}(p)$ and $\#U(S,p)(\leq) < \#S$.*

The notion of underdemanded sets of goods can be found in Sotomayor (2002), who uses it to design descending auctions for our model.[4] Both concepts give us an idea about the imbalance of supply and demand in the economy, albeit differently. A measure of total demand on a set of goods is obtained by counting the number of exclusive demanders of these goods in the notion of sets of overdemanded goods and by counting the number of demanders of these goods in the notion of sets of underdemanded goods. However, the dummy good is never part of a set of overdemanded goods and zero priced goods, which always includes the dummy good, are never part of sets of underdemanded goods. In some sense, the existence of sets of overdemanded (underdemanded) goods at a price vector indicates that there is excess demand (supply) in the economy. Since both overdemanded and underdemanded sets of goods may exist at a given price vector, excess demand and excess supply can exist simultaneously in the economy.

[4] There is a slight difference between our definition of underdemanded goods and the definition in Sotomayor (2002). Sotomayor (2002) assumes the existence of a dummy buyer who demands every good with zero price and who can be allocated more than one good. Then, a set of goods $S$ is underdemanded in Sotomayor (2002) at a price vector $p$ if *every* good in $N$ is demanded by a buyer (possibly the dummy buyer), $S \subseteq N^{+}(p)$ and $\#U(S,p) < \#S$.

## 3 Walrasian Equilibrium Characterization

In this section, we give a characterization of the Walrasian equilibrium price vectors. Our characterization is based on the notions of sets of overdemanded and underdemanded goods. Define $M^+(p) = \{i \in M : 0 \notin D_i(p)\}$ for any price vector $p$. Notice that $M^+(p) = O(N_0, p)$. Now, consider the following lemmas.

Lemma **1** *Suppose no set of goods is overdemanded. Then there exists a feasible allocation in which every buyer is assigned a good from his demand set.*

*Proof*: Since $N_0$ is not overdemanded, $\#N_0 \geq \#O(N_0, p) = \#M^+(p)$. Consider $S \subseteq M^+(p)$. Let $T = \cup_{i \in S} D_i(p)$. Since $0 \notin T$ and $T$ is not overdemanded, we get $\#T \geq \#O(T, p) \geq \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every buyer $i$ in $M^+(p)$ can be assigned a good in $D_i(p)$, and every buyer in $M \setminus M^+(p)$ can be assigned the dummy good 0, which is in his demand set. ■

Lemma **2** *Suppose no set of goods is underdemanded. Then there exists a feasible allocation in which every good in $N^+(p)$ is assigned to a buyer who is a demander of that good.*

*Proof*: Since $N^+(p)$ is not underdemanded, $\#N^+(p) \leq \#U(N^+(p), p) \leq \#M$. Consider $T \subseteq N^+(p)$. Let $S = U(T, p)$. Since $T$ is not underdemanded, $\#T \leq \#U(T, p) = \#S$. Using Hall's theorem (Hall, 1935), there is a feasible allocation in which every good in $N^+(p)$ can be assigned to a buyer who is a demander of that good, and the remaining buyers can be assigned the dummy good. ■

The absence of only overdemanded or only underdemanded sets of goods cannot guarantee a WE price vector. For instance, consider an example with a single good and three buyers with values 10, 6, and 3. A WE price is any price between 6 and 10. At any price higher than 10, the good is not overdemanded but it is not a WE price. Similarly, at any price between 3 and 6, the good is not underdemanded but it is not a WE price. In some sense, Lemma 1 says that condition (**WE-1**) in Definition 1 is satisfied in the absence of overdemanded goods, but condition (**WE-2**) may be violated. Similarly, Lemma 2 says that condition (**WE-2**) in Definition 1 is satisfied in the absence of underdemanded goods, but condition (**WE-1**) may be violated. However, the WE prices can be precisely characterized by the absence of both overdemanded and underdemanded sets of goods.

Theorem **1** *A price vector $p$ is a WE price vector if and only if no set of goods is overdemanded and no set of goods is underdemanded at $p$.*

*Proof*: The proof is in the appendix. ■

The characterization in Theorem 1 shows that given a price vector and the demand sets of buyers, it is possible to check if the given price vector is a WE price vector by checking for the existence of overdemanded and underdemanded sets of goods. In some sense this is a generalization of Hall's theorem (Hall, 1935) for our model.

Theorem 1 gives another definition of a Walrasian equilibrium price vector. But, in contrast to Definition 1, the characterization in Theorem 1 does not require to compute a feasible allocation to check if a price vector is a WE price vector. Theorem 1 uses only demand set information of buyers to characterize the WE price vectors. Moreover, Theorem 1 is the basis of all our results.

Notice that absence of overdemanded goods requires that there is no excess demand in a weak sense, since we only count the exclusive demanders in checking for overdemanded goods. Similarly, absence of underdemanded goods requires that there is no excess supply in a weak sense, since zero priced goods are not counted while checking for underdemanded goods. Theorem 1 assures the existence of a Walrasian equilibrium at a price vector if there is neither excess demand nor excess supply. This provides a direct economic interpretation of our result.

We now use Theorem 1 to characterize the minimum and the maximum WE price vectors. Let $K(p)$ contain all goods that are not part of any weakly overdemanded set at $p$ and $L(p)$ contain all goods that are not part of any weakly underdemanded set at $p$, i.e.,

$$K(p) = \{j \in N_0: \text{ for all } S \ni j,\ S \text{ is not weakly overdemanded at } p\}$$

and

$$L(p) = \{j \in N_0: \text{ for all } S \ni j,\ S \text{ is not weakly underdemanded at } p\}.$$

The following two lemmas characterize the sets $L(p)$ and $K(p)$ when $p$ is a WE price vector.

LEMMA **3** *At a Walrasian equilibrium price vector $p$ it holds that $L(p) = \{j \in N_0\colon p(j) = p^{min}(j)\}$.*

*Proof*: The proof is in the appendix. ■

LEMMA **4** *At a Walrasian equilibrium price vector $p$ it holds that $K(p) = \{j \in N_0 : p(j) = p^{max}(j)\}$.*

*Proof*: The proof is similar to the proof of Lemma 3. ■

So, the set $K(p)$ denotes the set of goods whose prices are at the maximum WE price and the set $L(p)$ denotes the set of goods whose price are at the minimum WE price. Our main result in this section uses these two lemmas.

THEOREM **2** *A price vector $p$ is equal to $p^{min}$ if and only if no set of goods is overdemanded and no set of goods is weakly underdemanded at $p$. Similarly, a price vector $p$ is equal to $p^{max}$ if and only if no set of goods is underdemanded and no set of goods is weakly overdemanded at $p$.*

*Proof*: Suppose $p = p^{min}$. By Theorem 1, no set of goods is overdemanded and no set of goods is underdemanded at $p^{min}$. By Lemma 3, $L(p^{min}) = N_0$. By definition of $L(p^{min})$, no set of goods is weakly underdemanded at $p^{min}$.

Suppose no set of goods is overdemanded and no set of goods is weakly underdemanded at $p$. Then, $L(p) = N_0$, and again by Lemma 3 $p = p^{min}$.

A similar proof using Lemma 4 proves that $p = p^{max}$ if and only if no set of goods is overdemanded and no set of goods is weakly underdemanded at $p$. ■

The characterization of the minimum WE price vector gives an idea about the existence of overdemanded and weakly underdemanded sets of goods in other regions of the price vector space.

COROLLARY **1** *If $p \ngeq p^{min}$, then there exists an overdemanded set of goods. Further, if $p \nleq p^{min}$, then there exists a weakly underdemanded set of goods.*

*Proof*: The proof is in the appendix. ■

In every region of the price vector space with respect to $p^{min}$, Corollary 1 shows when an overdemanded set of goods or a weakly underdemanded set of goods always exists in that region. A similar result holds with respect to $p^{max}$.

COROLLARY **2** *If $p \ngeq p^{max}$, then there exists a weakly overdemanded set of goods. Further, if $p \nleq p^{max}$, then there exists an underdemanded set of goods.*

*Proof*: The proof is analogous to Corollary 1. ■

The results in Theorem 2 and Corollary 1 and Corollary 2 are illustrated in Figure 2 for the example in Figure 1. The labelling in various regions of the figure indicates whether (weakly) overdemanded sets of goods ((W)OD) and (weakly) underdemanded sets of goods ((W)UD) exist at all price vectors in these regions. By Theorem 1, there is no set of overdemanded or underdemanded goods in the lattice corresponding to the WE price vector region in Figure 2. The minimum and the maximum WE price vectors are characterized by Theorem 2. The interior of WE price vector space is chracterized later in Theorem 5 as the set of price vectors where every set of goods is both weakly overdemanded and weakly underdemanded. In any price vector inside the rectangle generated by drawing parallel lines to axes at the minimum and the maximum WE price vectors, we can find a weakly overdemanded and a weakly underdemanded set of goods. The other regions in Figure 2 are labelled

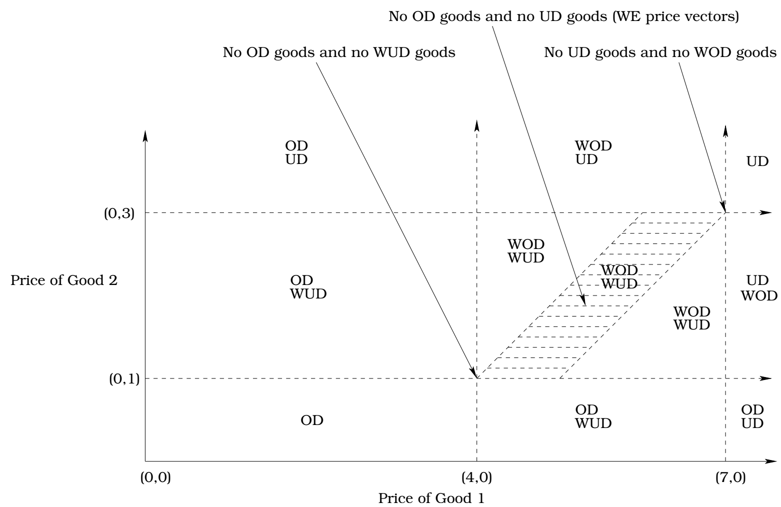


Figure 2: Various regions of the price vector space for the example in Figure 1

using Corollary 1 and Corollary 2. For example, for every price vector in the upper-right corner, an underdemanded set of goods exists, whereas for every price vector in the lower-left corner, an overdemanded set of goods exists. Notice that once every set of goods is weakly underdemanded, then no set of goods can be overdemanded. This happens, for example when all prices are set equal or above the highest valuation of the goods. Also, there exist regions (upper-left and lower-right corners in Figure 2) where sets of underdemanded and overdemanded goods co-exist.

We can say something more about various price vectors than what the results in Corollary 1 and Corollary 2 seem to indicate. If we decrease the prices of positive price goods at the minimum WE price vector by an equal amount such that no price goes below zero, then at the new price vector no weakly underdemanded goods exist. But, by Corollary 1, some set of goods is overdemanded. So, if $p^{min} \neq 0$, then there is some non-zero price vector $p \lneq p^{min}$ where no set of goods is weakly underdemanded but some set of goods is overdemanded. This argument illustrates that we can draw a piecewise linear path of prices from the minimum WE price vector to the zero price vector along which no set of goods is weakly underdemanded but some set of goods is overdemanded.

Similarly, if we increase the prices of positive price goods by an equal amount from the maximum WE price vector, no set of goods is weakly overdemanded at the new price vector, but some set of goods is underdemanded. So, the 45 degree straight line from the maximum WE price vector in the north-east direction is a set of (infinite) price vectors where no set of goods is weakly overdemanded but some set of goods is underdemanded.

# 4 Design of Iterative Auctions

In this section, we use the characterizations results of Theorem 2 to give a broad class of iterative auctions, which includes every known iterative auction for this setting. Thus, we unify various iterative auctions for the assignment (unit demand) setting under one broad class of auctions.

Iterative auctions, where prices monotonically increase (ascending auctions) or decrease (descending auctions) are practical and transparent methods to sell goods. The design of iterative auctions for our model has been studied earlier - ascending auctions can be found in Demange et al. (1986) and Sankaran (1994), whereas descending auctions can be found in Sotomayor (2002) and Mishra and Parkes (2008). These auctions terminate at a WE price vector - the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Parkes (2008) terminate at the minimum WE price vector, while the auction in Sotomayor (2002) terminates at the maximum WE price vector.[5] Moreover, the underlying price adjustment in these auctions is based on the ideas of overdemanded and underdemanded sets of goods. Interestingly, the papers on ascending auctions do not talk about underdemanded sets of goods and use the notion of overdemanded sets of goods only. Similarly, the papers on descending auctions do not talk about overdemanded sets of goods and use the notion of (weakly) underdemanded sets of goods only. The terminating conditions in these auctions are absence of overdemanded sets of goods for ascending auctions and absence of underdemanded sets of goods for descending auctions. Still, these auctions terminate at an extreme WE price vector. Our results can be used to explain why this is possible.

Consider the following class of ascending auctions:

- S0 Start the auction at a price vector $p$ where no set of goods is weakly underdemanded (by Corollary 1, $p \leq p^{min}$);
- S1 Collect demand sets of buyers and check if an overdemanded set of goods exist;
- S2 If no overdemanded set of goods exist, then stop (by Theorem 2, this is the minimum WE price vector);
- S3 Else increase prices of goods such that no set of goods is weakly underdemanded at the new price vector, and repeat from step S1.

The auctions in Demange et al. (1986) and Sankaran (1994) are such auctions, though they do not mention this explicitly. Both these auctions start from the zero price vector.[6] At

[5]Since minimum WE price vector corresponds to the VCG payments, the auctions in Demange et al. (1986), Sankaran (1994), and Mishra and Parkes (2008) have *truthful bidding* in an equilibrium, whereas buyers can *manipulate* the auction in Sotomayor (2002).

[6]To be precise, they use the reserve price of every good as the starting price, which is assumed to be zero in our model.

the zero price vector, no set of goods is weakly underdemanded. In step S3, Demange et al. (1986) increase prices by unity for goods in a minimal overdemanded set, whereas Sankaran (1994) increases prices by unity for goods in an overdemanded set, which he finds using a labeling algorithm of graph theory [7]. Both the price adjustments ensure that no set of goods is weakly underdemanded after the price increase (i.e., satisfy the condition in step S3), and we stay below the minimum WE price vector (by Corollary 1).

The descending auctions share an analogous feature. Consider the following class of descending auctions:

T0 Start the auction at a price vector $p$ where no set of goods is weakly overdemanded (by Corollary 2, $p \geq p^{max}$);

T1 Collect demand sets of buyers and check if an underdemanded set of goods exist;

T2 If no underdemanded set of goods exist, then stop (by Theorem 2, this is the maximum WE price vector);

T3 Else decrease prices of goods such that no set of goods is weakly overdemanded at the new price vector, and repeat from step T1.

The auction in Sotomayor (2002) starts from a very high price vector where every buyer demands only the dummy good. Hence no set of goods is weakly overdemanded. By decreasing prices by unity for goods in a minimal underdemanded set, no set of goods is weakly overdemanded after the price decrease, and the price in the auction stays above the maximum WE price vector.

This class of descending auctions can be modified to terminate at the minimum WE price vector. Such auctions have to start from a price vector where no set of goods is overdemanded (by Corollary 2 such a price vector is above the minimum WE price vector). These auctions should stop if no set of goods is weakly underdemanded, and price decrease should be such that no set of goods is overdemanded at the new price vector.

Thus, our characterization results unify the existing iterative auctions by bringing them under a broad class of auctions. We hope that this will be useful in identifying more iterative auctions from this class which are easier to implement in practice than the auctions known in the literature.

## 5 Potentials of Graphs and Walrasian Equilibrium Prices

The results in the previous section enable us to verify whether a price vector is the minimum or the maximum WE price vector given the demand set information of buyers. We pursue

[7] Demange et al. (1986); Sankaran (1994); Sotomayor (2002); Mishra and Parkes (2008) assume that valuations of buyers are integers.

this question now for any extreme point and interior point of the WE price vector lattice. However, before we can do so, we need more undestanding of the underlying mathematical structure of the WE price vector space. We do this in this section by interpreting the WE price vectors as *potential* functions of an appropriate directed graph. Such an interpretation helps us to prove several new results, and gives a graph theoretic interpretation to several known results. We begin by defining and proving some concepts related to graph theory.

## 5.1 Potentials of Strongly Connected Graphs

A **graph** is defined by a triple $G = (N, E, l)$, where $N = \{0, 1, \ldots, n\}$ is the set of $n + 1$ nodes, $E \subseteq \{(i, j) : i, j \in N, i \neq j\}$ is a set of ordered pairs of different nodes, called edges, and $l$ is a vector of weights on the edges in $E$ with $l_{ij} \in \mathbb{R}$ being the length of edge $(i, j) \in E$. As before denote $N_0 = N \setminus \{0\}$. A graph is **complete** if there is an edge between every pair of different nodes.

A **path** is a sequence of *distinct* nodes $(i^1, \ldots, i^k)$ such that $(i^j, i^{j+1}) \in E$ for all $1 \leq j \leq k - 1$. If $(i^1, \ldots, i^k)$ is a path, then we say that it is a path from $i^1$ to $i^k$. A graph is **strongly connected** if there is a path from every node in $N$ to every other node in $N$.

A **cycle** is a sequence of nodes $(i^1, \ldots, i^k, i^{k+1})$ such that $(i^1, \ldots, i^k)$ is a path, $(i^k, i^{k+1}) \in E$, and $i^1 = i^{k+1}$. The length of a path or a cycle $P = (i^1, \ldots, i^k, i^{k+1})$ is the sum of the edge lengths in the path or cycle, and is denoted as $l(P)$, i.e., $l(P) = l_{i^1i^2} + \ldots + l_{i^ki^{k+1}}$. When there is at least one path from node $i$ to node $j$, then a **shortest path** from node $i$ to node $j$ is a path from $i$ to $j$ having minimum length over all paths from $i$ to $j$. We denote the length of a shortest path from $i$ to $j$ as $s(i, j)$. For convenience, we define $s(i, i) = 0$ for all $i \in N$.

Definition 4 *A* **potential** *of a graph $G = (N, E, l)$ is a function $p : N \to \mathbb{R}$ such that $p(j) - p(i) \leq l_{ij}$ for all $(i, j) \in E$. For any $j \in N$, a $j-$***potential*** *of a graph $G$ is a potential $p$ of graph $G$ such that $p(j) = 0$.*

It is well known that a potential of graph $G$ exists if and only if $G$ has no cycles of negative length (Gallai, 1958). Moreover, the set of potentials of a graph form a lattice.

In case the graph is strongly connected, the lengths of the shortest paths from and to node 0 determine the components of the maximum and minimum 0-potential of the complete lattice set of 0-potentials, respectively.

Lemma 5 **(Duffin (1962))** *Suppose $G$ is a strongly connected graph with no cycle of negative length. Then the set of $0-$potentials of $G$ form a complete lattice. The maximum $0-$potential of $G$ is given by $p^{max}(j) = s(0, j)$ for all $j \in N$ and the minimum $0-$potential of $G$ is given by $p^{min}(j) = -s(j, 0)$ for all $j \in N$.*

*Proof*: We give a proof for completeness in the appendix. ■

Notice that the set of $0-$potentials of a given graph $G$ is a polytope, defined by the linear inequalities of the potentials and the equality that the potential of node 0 is equal to zero. Next, we characterize the extreme points of this polytope. For $j \in N$, define the potentials $\underline{p}^j$ and $\overline{p}^j$ as

$$\begin{aligned} \overline{p}^j(i) &= s(0,j) - s(i,j) \qquad \forall\ i \in N, \\ \underline{p}^j(i) &= s(j,i) - s(j,0) \qquad \forall\ i \in N. \end{aligned}$$

To see that $\overline{p}^j$, $j \in N$, is a potential, note that $p = \{-s(i,j)\}_{i \in N}$ is a $j$-potential. Scaling $p$ by $s(0,j)$ gives us another potential. Since value of $p(0) = -s(0,j)$, $\overline{p}^j$ is a 0-potential. Similarly, $\underline{p}^j$ is a 0-potential. These observations lead to the following lemma.

LEMMA **6** *Suppose $G$ is a strongly connected graph with no cycles of negative length. Then, for every $j \in N$, $\underline{p}^j$ and $\overline{p}^j$ are $0-$potentials of graph $G$.*

Notice that $\overline{p}^0 = p^{min}$ and $\underline{p}^0 = p^{max}$. Also, $\overline{p}^j(j) = p^{max}(j)$ and $\underline{p}^j(j) = p^{min}(j)$ for any $j \in N$.

For $\emptyset \neq S \subseteq N_0$, define the potential $\overline{p}^S$ as $\overline{p}^S(i) := \max_{j \in S} \overline{p}^j(i)$ for all $i \in N$ and the potential $\underline{p}^S$ as $\underline{p}^S(i) := \min_{j \in S} \underline{p}^j(i)$ for all $i \in N$. Since $\overline{p}^j$ is a 0-potential for every $j \in S$, it follows from the lattice structure of 0-potentials, that $\overline{p}^S$ is a 0-potential. Similarly, $\underline{p}^S$ is a 0-potential.

By definition of $\overline{p}^j(\cdot)$ and $\underline{p}^j(\cdot)$, $j \in N$, and using the fact that $\overline{p}^i(i) = p^{max}(i) = s(0,i)$ and $\underline{p}^i(i) = p^{min}(i) = -s(i,0)$ for all $i \in N$, we can rewrite $\overline{p}^S(\cdot)$ and $\underline{p}^S(\cdot)$ for every $\emptyset \neq S \subseteq N_0$ as

$$\overline{p}^S(i) = \begin{cases} s(0,i) & \text{if } i \in S \\ \max_{j \in S} \left[ s(0,j) - s(i,j) \right] & \text{otherwise} \end{cases}$$

and

$$\underline{p}^S(i) = \begin{cases} -s(i,0) & \text{if } i \in S \\ \min_{j \in S} \left[ s(j,i) - s(j,0) \right] & \text{otherwise.} \end{cases}$$

Next, we show that these potentials are precisely the extreme points of the lattice defined by all $0-$potential functions. Define $\mathbb{P}^e(G) := \{\overline{p}^S : \emptyset \neq S \subseteq N_0\} \cup \{\underline{p}^S : \emptyset \neq S \subseteq N_0\}$. The next theorem says that for every $S$, $\emptyset \neq S \subseteq N_0$, both vectors $\overline{p}^S$ and $\underline{p}^S$ are extreme points of the set of $0-$potentials of $G$ and, conversely, that every extreme point of the the set of $0-$potentials of $G$ is equal to $\overline{p}^S$ or $\underline{p}^S$ for some $S$, $\emptyset \neq S \subseteq N_0$. This leads to the main result of this section.

THEOREM **3** *Suppose $G$ is a strongly connected graph with no cycles of negative length. Then $\mathbb{P}^e(G)$ is the set of extreme points of the the set of $0-$potentials of $G$.*

*Proof*: The proof is in the appendix. ■

## 5.2 Potentials as Walrasian Equilibrium Prices

We now interpret the Walrasian equilibrium prices as potentials in an appropriate directed graph. Corresponding to an efficient allocation $\mu$ we describe a graph $G^{\mu}$. The graph $G^{\mu}$, called the **allocation graph** corresponding to $\mu$, has the set of goods $N$ as its set of nodes, and is complete. Let $N^{\mu}$ be the set of goods unassigned in $\mu$ and $M^{\mu}$ be the set of buyers assigned to the dummy good in $\mu$. Let $\mu^j$ be the buyer allocated to good $j \in N_0 \setminus N^{\mu}$ in allocation $\mu$. As before, $\mu_i$ denotes the good allocated to buyer $i \in M$ in allocation $\mu$. Note that if $p$ is a WE price vector, we must have $v_{\mu^k k} - p(k) \geq v_{\mu^k j} - p(j)$ and so $p(k) - p(j) \leq v_{\mu^k k} - v_{\mu^k j}$. If a good $k$ is not allocated to any buyer in $\mu$, then at a WE price vector $p$ we must have $p(k) = 0$. Hence, $p(k) \leq p(j)$ for all $j \in N$. Note that this implies $p(k) = 0$ if $p(0) = 0$, and $p(k) - p(j) \leq 0$. If buyer $i$ is assigned to good 0 in $\mu$, then the WE constraint says that $v_{i0} - p(0) = -p(0) \geq v_{ij} - p(j)$. Hence, $p(0) - p(j) \leq -v_{ij}$. Since more than one buyer can be allocated to the dummy good, we can write, $p(0) - p(j) \leq \min_{i:\mu_i=0} -v_{ij}$. This gives us an intuition on what the edge lengths of the graph $G^{\mu}$ must be.

For $j,\ k \in N$, we define the length from node $j$ to node $k$, $l_{jk}$, for three possible different cases.

1. The value of $l_{jk}$ is set equal to zero if $k \in N^{\mu}$.
2. The value of $l_{jk}$ is set equal to $v_{\mu^k k} - v_{\mu^k j}$ if $k \in N_0 \setminus N^{\mu}$.
3. The value of $l_{j0}$ is set equal to $\min_{i:\mu_i=0} -v_{ij}$ if $M^{\mu} \neq \emptyset$.

We now state the main result of this section.

Theorem 4 *If $p$ is a Walrasian equilibrium price vector, then $p$ is a $0-$potential of $G^{\mu}$ for any efficient allocation $\mu$, and if $p$ is a $0-$potential of $G^{\mu}$ for some efficient allocation $\mu$, then $(p, \mu)$ is a Walrasian equilibrium.*

*Proof*: The proof is in the appendix. ■

This result shows that the WE price vectors are 0-potentials of the allocation graph. This immediately explains why the set of WE price vectors is a complete lattice. We now use this result to characterize the interior and extreme points of the WE price vector lattice.

## 5.3 Interior Walrasian Equilibrium Prices

Results in Section 3 indicate that it is possible to identify any Walrasian equilibrium price vector and the minimum and the maximum Walrasian equilibrium price vector by using the demand set information at these price vectors. But the demand set submitted in these equilibrium prices may contain information which is *useless*. For instance, in the example in Figure 1, $p^{min} = (2, 2)$ and $D_1(p^{min}) = \{1\}$, $D_2(p^{min}) = \{2\}$, and $D_3(p^{min}) = \{0, 1, 2\}$. Note that the (efficient) allocation allocates buyer 1 to good 1, buyer 2 to good 2, and buyer 3 to the dummy good. Hence, by including goods 1 and 2 in his demand set, buyer 3 is submitting information, which is not necessary to verify a Walrasian equilibrium. The natural question is whether there exist Walrasian equilibrium price vectors where the demand sets of buyers are *minimally informative* in this way.

The answer to this question lies in the characterization of the set of interior Walrasian equilibrium price vectors. In this section, we characterize the interior points of the Walrasian equilibrium price vector space. Notice that the interior may be empty in some instances. For instance, suppose there are two *identical goods*, i.e., values for these two goods are equal for every buyer. In any WE price vector, prices of these two goods must be equal. This reduces the dimension of the WE price vector space, making the interior empty. We find necessary and sufficient conditions under which the interior is non-empty.

DEFINITION **5** *A Walrasian equilibrium price vector $p$ is an* **interior Walrasian equilibrium price vector** *if it is an interior point of the set of Walrasian equilibrium price vectors in $\mathbb{R}^n$.* [8]

THEOREM **5** *A price vector $p$ is an interior Walrasian equilibrium price vector if and only if every non-dummy good has positive price and is demanded by a unique buyer and every buyer demands exactly one good, i.e., $N^+(p) = N_0$ and $\#U(p, \{j\}) = \#O(p, \{j\}) = 1$ for all $j \in N_0$.*

*Proof*: The proof can be found in the appendix. ■

Another equivalent way to state this result is that the set of interior WE price vectors is fully characterized by the property that *every* set of goods is both weakly overdemanded but not overdemanded and weakly underdemanded but not underdemanded. Theorem 5 says that at interior WE price vectors the demand set of every buyer only consists of the good he is allocated in a WE. Thus, the interior WE price vectors elicit minimal information from buyers. However, such price vectors need not exist. We now identify conditions under which the interior of the Walrasian equilibrium price vector space is non-empty.

THEOREM **6** *An interior Walrasian equilibrium price vector exists if and only if there is a unique efficient allocation and $n \leq m$.*

*Proof*: The proof can be found in the appendix. ■

[8] Note that we draw the set of Walrasian equilibrium price vectors in $\mathbb{R}^n$, and not in $\mathbb{R}^{n+1}$, since the price of the dummy good is always zero.

The condition $n \leq m$ is quite natural in many settings. But existence of a unique efficient allocation is difficult to guarantee. For example, if two different buyers have exactly same valuations, then we violate this condition. Hence, it is quite possible we have empty interior in many settings. In that case, Theorem 6 can be viewed as a negative result - in many settings buyers need to report *useless* information in their demand sets to verify a Walrasian equilibrium.

### 5.4 Extreme Walrasian Equilibrium Prices

In this subsection we characterize the extreme points of the set Walrasian equilibrium price vectors. The central result that we use is Theorem 3. Our characterization of extreme Walrasian equilibrium price vectors is a careful interpretation of this result in terms of potentials of the allocation graph.

We first extend the definition of $K(p)$ and $L(p)$, defined in Section 3. At price vector $p$ and set $S$, $\emptyset \neq S \subseteq N_0$, define

$$K(p, S) = \{j \in S : T \text{ is not weakly overdemanded at } p \text{ for any } T \text{ satisfying } j \in T \subseteq S\}.$$

The set $K(p, S)$ is the subset of goods in $S$ that are not contained in some weakly overdemanded subset of $S$ at $p$. Every good in the set $S \setminus K(p, S)$ is contained in some weakly overdemanded subset of goods of $S$ at $p$.

Similarly, at price vector $p$ and set $S$, $\emptyset \neq S \subseteq N_0$, define

$$L(p, S) = \{j \in S : T \text{ is not weakly underdemanded at } p \text{ for any } T \text{ satisfying } j \in T \subseteq S\}.$$

Every good in $L(p, S)$ either has price zero or has the property that any subset of $S^+(p)$ that contains $j$ is not weakly underdemanded. Every good in the set $S \setminus L(p, S)$ is contained in some weakly underdemanded subset of goods of $S$ at $p$.

We denote the set $N_0 \setminus K(p, N_0)$ as $S^o(p)$. The set $S^o(p)$ contains all goods that are part of some weakly overdemanded set of goods at $p$. Similarly, we denote the set $N_0 \setminus L(p, N_0)$ as $S^u(p)$. The set $S^u(p)$ contains all goods that are part of some weakly underdemanded set of goods at $p$. Using these notions, we state our main result of this section.

Theorem 7 *A Walrasian equilibrium price vector is an extreme point if and only if* $L(p, N_0) \cup K(p, N_0) \neq \emptyset$ *and* $L(p, S^o(p)) = S^o(p)$ *if* $K(p, N_0) \neq \emptyset$ *and* $K(p, S^u(p)) = S^u(p)$ *if* $L(p, N_0) \neq \emptyset$.

*Proof*: The proof is in the appendix. ■

The characterization above says that at an extreme Walrasian equilibrium price vector no subset of the goods that are part of some weakly overdemanded set of goods can be weakly underdemanded and that simultaneously no subset of the goods that are part of some weakly underdemanded set of goods can be weakly overdemanded. Besides its mathematical elegance, Theorem 7 shows that the characterization in Theorem 2 is extendable to any extreme point of WE price space. Using this characterization and the interior WE price characterization, we can also conclude if a WE price vector is on a *face* of the WE price space. Thus, we have characterized the entire WE price space.

## 6 Conclusions

We characterized the extreme and interior points of the set Walrasian equilibrium price vectors for the assignment model. We also characterized the minimum and the maximum Walrasian equilibrium price vectors. Our characterizations indirectly characterize all Walrasian equilibrium price vectors that lie on any face of the Walrasian equilibrium price vector space. All characterizations involve conditions on the demand sets of buyers only. A future research direction is to extend these characterizations to a model where a buyer can be assigned more than one good with combinatorial values.

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# Appendix

## Proof of Theorem 1

*Proof*: Suppose $p$ is a WE price vector. By condition (**WE-2**), there exists a feasible allocation in which every good in $N^+(p)$ can be assigned to a unique demander of that good. Hence no set of goods is underdemanded. If some set of goods, say, $S \subseteq N_0$, is overdemanded, then condition (**WE-1**) will fail for some buyer in $O(S, p)$ in every feasible allocation, which is impossible since $p$ is a WE price vector. Hence, no set of goods can be overdemanded.

Suppose now that no set of goods is overdemanded and no set of goods is underdemanded at price vector $p$. By Lemma 1 there is a non-empty set of feasible allocations $\Gamma^*$ that allocates every buyer a good from his demand set. Choose an allocation $\mu \in \Gamma^*$ for which the number of goods from $N^+(p)$ that is allocated in $\mu$ is maximal over all the allocations in $\Gamma^*$. Let us call such an allocation a maximal allocation in $\Gamma^*$. Let $T^0 = \{j \in N^+(p) : \mu_i \neq j \ \forall\ i \in M\}$. If $T^0 = \emptyset$, then by definition $(p, \mu)$ is a WE. We will show that $T^0$ is empty. Assume for contradiction that $T^0$ is not empty.

We first show that for every buyer $i \in M$, if $\mu_i \notin N^+(p)$ then $T^0 \cap D_i(p) = \emptyset$. Assume for contradiction that for some $i \in M$ with $\mu_i \notin N^+(p)$ there exists $j \in T^0 \cap D_i(p)$. In that case, we can construct a new allocation $\mu'$ in which $\mu'_i = j$ and $\mu'_k = \mu_k$ for all $k \neq i$. Allocation $\mu'$ is in $\Gamma^*$ and assigns one good more from $N^+(p)$ than $\mu$ does. This is a contradiction since $\mu$ is a maximal allocation in $\Gamma^*$. As a result of this, the demanders of $T^0$ are assigned to goods in $N^+(p) \setminus T^0$. Let $X^0 = U(T^0, p)$. So, $X^0 \subseteq \{i \in M : \mu_i \in N^+(p) \setminus T^0\}$. Now, for any $k \geq 0$, consider a sequence $(T^0, X^0, T^1, X^1, \ldots, T^k, X^k)$, where for every $1 \leq q \leq k$, $T^q$ is the set of goods assigned to buyers in $X^{q-1}$ in $\mu$ and $X^q = U(\cup_{r=0}^{q} T^r, p) \setminus U(\cup_{r=0}^{q-1} T^r, p)$. Note that by definition $T^q \cap T^r = \emptyset$ for every $q \neq r$.

We show that if $T^q \neq \emptyset$ and $T^q \subseteq N^+(p)$ for all $0 \leq q \leq k$, then there exists $T^{k+1} \neq \emptyset$ such that $T^{k+1} \subseteq N^+(p)$ and $T^{k+1} \cap T^q = \emptyset$ for all $0 \leq q \leq k$. By definition of $X^q$, $0 \leq q \leq k$, and $T^q$, $1 \leq q \leq k$,

$$\begin{aligned} \#U(\cup_{q=0}^{k} T^q, p) &= \#U(\cup_{q=0}^{k-1} T^q, p) + \#X^k \\ &= \sum_{q=0}^{k} \#X^q \\ &= \sum_{q=1}^{k} \#T^q + \#X^k. \end{aligned} \tag{1}$$

Since $T^0, \ldots, T^k$ are disjoint and $\cup_{q=0}^{k} T^q \subseteq N^+(p)$ is not underdemanded, we have

$$\#U(\cup_{q=0}^{k} T^q, p) \geq \sum_{q=0}^{k} \#T^q. \tag{2}$$

Using (1) and (2), we get $\#X^k \geq \#T^0$. Since $T^0$ is non-empty, $X^k$ is non-empty. Define $T^{k+1}$ as the set of goods assigned to buyers in $X^k$ in $\mu$. Clearly, $T^{k+1}$ is non-empty and $T^{k+1} \cap T^q = \emptyset$ for every $0 \leq q \leq k$. To show that $T^{k+1} \subseteq N^+(p)$, assume for contradiction that there exists a buyer $i_k \in X^k$ such that $\mu_{i_k} \notin N^+(p)$. By definition of $X^k$, $i_k$ should demand some good $j_k \in T^k$. Now consider the sequence $(i_k, j_k, i_{k-1}, j_{k-1}, \ldots, i_0, j_0)$, where for every $0 \leq q \leq k-1$, $i_{q-1}$ is the buyer assigned to good $j_q$ in $\mu$ (note that $i_{q-1} \in X^{q-1}$ by definition) and $j_{q-1}$ is a good demanded by $i_{q-1}$ from $T^{q-1}$ (such a good exists by the definition of $X^{q-1}$ and $T^{q-1}$). Now, construct an allocation $\mu'$ with $\mu'_{i_q} = j_q$ for all $0 \leq q \leq k$ and $\mu'_i = \mu_i$ for any $i \notin \{i_0, \ldots, i_k\}$. Clearly, $\mu' \in \Gamma^*$. By assigning $i_k$ to $j_k$, $\mu'$ assigns one good more from $N^+(p)$ than $\mu$ does, contradicting the fact that $\mu$ is a maximal allocation in $\Gamma^*$. Hence $T^{k+1} \subseteq N^+(p)$. This process can be repeated infinitely many times starting from $T^0$. So $(T^0, T^1, \ldots)$ is an infinite sequence such that $T^q \cap T^r = \emptyset$ for every $q \neq r$, $T^q \neq \emptyset$ for all $q$, and $T^q \subseteq N^+(p)$ for all $q$. This is a contradiction since $N^+(p)$ is finite. So, $T^0 = \emptyset$, and therefore $(p, \mu)$ is a WE. ■

Proof of Lemma 3

*Proof*: If $p(j) = 0$, then by definition $j \in L(p)$. Suppose $p(j) = p^{min}(j)$ for some $j \in N^+(p)$. Then $p^{min}(j) > 0$ and for any $T \subseteq N^+(p)$ containing $j$ all goods in $T$ are assigned in a WE at $p$. Let $T'$ be the set of buyers assigned to those goods in $T$ in a WE at $p$. Since $p$ is a WE price vector, by Theorem 1 $T$ is not underdemanded, i.e., $\#U(p, T) \geq \#T = \#T'$. Assume for contradiction that $T$ is weakly underdemanded at $p$, i.e., $\#U(p, T) = \#T = \#T'$. Then, prices of goods in $T$ can be lowered from $p$ by a sufficiently small amount to get another WE price vector, contradicting the fact that $p(j) = p^{min}(j)$. Hence, $T$ is not weakly underdemanded at $p$, and therefore $j \in L(p)$.

Suppose $j \in L(p)$. If $p(j) = 0$, then $p(j) = p^{min}(j)$. Assume for contradiction $p(j) > 0$ and $p(j) > p^{min}(j)$. Let $X = \{k \in N_0 : p(k) > p^{min}(k)\}$. Notice that $j \in X$ and $X \subseteq N^+(p)$. By definition, $X$ is not weakly underdemanded at $p$. Comparing $p$ and $p^{min}$, all prices of goods in the set $X$ decrease, from $p$ to $p^{min}$, whereas prices of other goods remain the same. Hence, buyers in $U(p, X)$ will become exclusive demanders of $X$ at $p^{min}$. Since $\#U(X, p) > \#X$, we get $\#O(X, p^{min}) \geq \#U(X, p) > \#X$. Hence $X$ is overdemanded at $p^{min}$, a contradiction by Theorem 1. So, for every $j \in L(p)$, $p(j) = p^{min}(j)$. ■

Proof of Corollary 1

*Proof*: Suppose $p \ngeq p^{min}$. Let $S = \{j \in N : p(j) < p^{min}(j)\}$. Since $p \ngeq p^{min}$, $S \neq \emptyset$. Further, because $p^{min}(j) > p(j) \geq 0$ for all $j \in S$, $S \subseteq N^+(p^{min})$. Since prices of goods in $S$ decrease from $p^{min}$ to $p$ while prices of goods in $N \setminus S$ do not decrease, $U(S, p^{min}) \subseteq O(S, p)$. So, $\#O(S, p) \geq \#U(S, p^{min}) > \#S$, where the last inequality follows from Theorem 2 ($S$ is not weakly underdemanded at $p^{min}$). Hence $S$ is overdemanded at $p$.

Now, suppose $p \nleq p^{min}$. Define $S' = \{j \in N : p(j) > p^{min}(j)\}$. Because $p \nleq p^{min}$, $S' \neq \emptyset$. Further, since $p(j) > p^{min}(j) \geq 0$ for all $j \in S'$, $S' \subseteq N^+(p)$. Since prices of goods in $S'$ decrease from $p$ to $p^{min}$ while prices of goods in $N \setminus S'$ do not decrease, $U(S', p) \subseteq O(S', p^{min})$. So, $\#U(S', p) \leq \#O(S', p^{min}) \leq \#S'$, where the last inequality follows from Theorem 2 ($S'$ is not overdemanded at $p^{min}$). Hence $S'$ is weakly underdemanded at $p$. ■

PROOF OF LEMMA 5

*Proof*: First we show that the vectors $p^{max}$ and $p^{min}$ are $0-$potentials. Consider any edge $(i, j) \in E$ and the shortest path from 0 to $j$ and from 0 to $i$. If the shortest path from 0 to $i$ does not include node $j$, then $s(0, j) \leq s(0, i) + l_{ij}$ and therefore $s(0, j) - s(0, i) \leq l_{ij}$. If the shortest path from 0 to $i$ includes node $j$, then $s(0, j) \leq s(0, j) + s(j, i) + l_{ij} = s(0, i) + l_{ij}$, where the inequality comes from the assumption that $G$ has no negative cycle and the equality comes from the fact that the shortest path from 0 to $i$ includes node $j$. Hence, $p^{max}$ is a $0-$potential. A similar argument shows that $p^{min}$ is a $0-$potential.

Let $p$ be any $0-$potential of $G$. Notice that a $0-$potential of $G$ exists since $G$ has no cycle of negative length. Consider the shortest path from 0 to $j$ and let it be $(0, i^1, \ldots, i^k, j)$. We can write the following set of inequalities for every edge in this path:

$$\begin{aligned} p(i^1) - p(0) &\leq l_{0i^1} \\ p(i^2) - p(i^1) &\leq l_{i^1 i^2} \\ \ldots &\leq \ldots \\ p(j) - p(i^k) &\leq l_{i^k j}. \end{aligned}$$

Adding up all inequalities we get $p(j) - p(0) \leq s(0, j)$. Since $p(0) = 0$, we get $p(j) \leq s(0, j)$. A similar argument by using the shortest path from $j$ to 0 shows that $p(j) \geq -s(j, 0)$. This shows that $p^{max}$ is the maximum $0-$potential and $p^{min}$ is the minimum $0-$potential. Since the set of potentials form a lattice, the set of $0-$potentials form a complete lattice. ■

PROOF OF THEOREM 3

*Proof*: Let $\mathbb{P}^0(G)$ be the set of $0-$potentials of $G$. Consider any $\emptyset \neq S \subseteq N_0$. Due to the lattice structure of $\mathbb{P}^0(G)$ both vectors $\underline{p}^S$ and $\overline{p}^S$ are $0-$potentials of graph $G$.

Consider the linear programming problem

$$\min \theta^- \sum_{i \in S} p(i) - \sum_{i \in N_0 \setminus S} p(i)$$
$$\text{s.t.} \tag{P1}$$
$$p \in \mathbb{P}^0(G),$$

for some $\theta^- > 0$. By Lemma 5, for every $p \in \mathbb{P}^0(G)$ it holds that $p(i) \geq p^{min}(i) = -s(i, 0) \geq 0$ for all $i \in N$. Hence, for large enough $\theta^-$, at any optimal solution of (**P1**) we have $p(i) = -s(i, 0)$ for all $i \in S$. For $i \in N_0 \setminus S$, take any $j \in S$. Let a shortest path from $j$ to $i$ in $G$ be $(j, j^1, \ldots, j^k, i)$. We can write for any $p \in \mathbb{P}^0(G)$,

$$p(i) - p(j) \leq l_{jj^1} + l_{j^1 j^2} + \ldots + l_{j^k i} = s(j, i)$$

and so, for large enough $\theta^-$, at an optimal solution it holds that

$$p(i) \leq s(j, i) - s(j, 0).$$

Since this holds for all $j \in S$, we obtain that at the optimal solution for every $i \in N_0 \setminus S$ it holds that

$$p(i) \leq \min_{j \in S} \big[s(j, i) - s(j, 0)\big], \tag{3}$$

when $\theta^-$ is large enough. Hence, the maximum value of $p(i)$ for all $i \in N_0 \setminus S$ is $\min_{j \in S} \big[s(j, i) - s(j, 0)\big]$. Thus, $\underline{p}^S \in \mathbb{P}^0(G)$ is the unique optimal solution to (**P1**) for sufficiently large $\theta^-$. Hence, $\underline{p}^S$ is an extreme point of $\mathbb{P}^0(G)$.

Next, consider the linear programming problem

$$\begin{aligned} \max \sum_{i \in S} \theta^+ p(i) - \sum_{i \in N_0 \setminus S} p(i) \\ \text{s.t.} \\ p \in \mathbb{P}^0(G), \end{aligned} \tag{\textbf{P2}}$$

for some $\theta^+ > 0$. By Lemma 5, for every $p \in \mathbb{P}^0(G)$ we have $p(i) \leq p^{max}(i) = s(0, i)$ for all $i \in N$. Hence, for sufficiently large $\theta^+$, at the optimal solution of (**P2**) it holds that $p(i) = s(0, i)$ for all $i \in S$. For $i \in N_0 \setminus S$, consider any $j \in S$. We can write for any $p \in \mathbb{P}^0(G)$,

$$p(j) - p(i) \leq s(i, j)$$

and so, for large enough $\theta^+$, at an optimal solution it holds that

$$p(i) \geq s(0, j) - s(i, j).$$

Therefore, for all $i \in N_0 \setminus S$,

$$p(i) \geq \max_{j \in S} \big[s(0, j) - s(i, j)\big]. \tag{4}$$

Hence, for large enough $\theta^+$, the minimum value of $p(i)$ for all $i \notin S$ is $\max_{j \in S} \big[s(0, j) - s(i, j)\big]$. Thus, $\overline{p}^S \in \mathbb{P}^0(G)$ is the unique optimal solution to (**P2**) for sufficiently large $\theta^+$. Hence, $\overline{p}^S$ is an extreme point of $\mathbb{P}^0(G)$.

It remains to be proved that the elements of $\mathbb{P}^e(G)$ are the only extreme points of $\mathbb{P}^0(G)$. Assume for contradiction that there exists an extreme point $p \notin \mathbb{P}^e(G)$. Let $X := \{j \in N_0 : p(j) = p^{max}(j) \text{ or } p(j) = p^{min}(j)\}$. We argue that $X \neq \emptyset$. Assume for contradiction $X = \emptyset$. Then, none of the constraints in $\mathbb{P}^0(G)$ involving $p(0)$ can be tight. This is because $p(j) - p(0) = l_{0j}$ implies $p(j) = p^{max}(j)$ and $p(0) - p(j) = l_{j0}$ implies $p(j) = p^{min}(j)$. Hence, there exists $p', p'' \in \mathbb{P}^0(G)$ such that $p'(j) = p(j) + \epsilon$ and $p''(j) = p(j) - \epsilon$ for all $j \neq 0$ for sufficiently small $\epsilon > 0$. Hence, $p(j) = \frac{p'(j)+p''(j)}{2}$ for all $j \in N$, contradicting the fact that $p$ is an extreme point. So, $X \neq \emptyset$.

Define $S := \{j \in X : p(j) = p^{max}(j)\}$. Suppose $S$ is non-empty. Then for all $i \in N_0 \setminus S$, we have $p(i) < p^{max}(i)$. By our argument earlier, $p(i) \geq \max_{j \in S}\big[s(0,j) - s(i,j)\big]$ for all $i \in N_0 \setminus S$. Let $T = \{i \in N_0 \setminus S : p(i) > \max_{j\in S}\big[s(0,j) - s(i,j)\big]\}$. Since $p \notin \mathbb{P}^e(G)$, $T$ is non-empty. Also, by definition of $T$, $p(i) < p^{max}(i)$ for all $i \in T$. Hence, for any $i \in T$ and any $j \notin T$, the two constraints between $i$ and $j$ in $\mathbb{P}^0(G)$ are not tight. Hence, we can construct two $0-$potentials $p'$ and $p''$ as follows: $p'(i) = p''(i) = p(i)$ if $i \notin T$ and $p'(i) = p(i) + \epsilon$ and $p''(i) = p(i) - \epsilon$ for all $i \in T$ for sufficiently small $\epsilon > 0$. Clearly, $p(i) = \frac{p'(i)+p''(i)}{2}$ for all $i \in N$, contradicting the fact that $p$ is an extreme point. A similar argument works if $X \setminus S$ is non-empty. Hence, every extreme point of $\mathbb{P}^0(G)$ is in $\mathbb{P}^e(G)$. ■

Proof of Theorem 4

*Proof*: Suppose $p$ is a WE price vector. Consider any efficient allocation $\mu$ and the allocation graph $G^\mu$ corresponding to $\mu$. So, $(p,\mu)$ is a WE. Take any edge $(j,k)$ of $G^\mu$. Now, consider the next three possible cases.

1. If $k \in N^\mu$, then $p(k) = 0$. Hence $p(k) - p(j) = -p(j) \leq 0 = l_{jk}$.
2. If $k \in N_0 \setminus N^\mu$, then $v_{\mu^k k} - p(k) \geq v_{\mu^k j} - p(j)$ (since $(p,\mu)$ is a WE). This implies that $p(k) - p(j) \leq v_{\mu^k k} - v_{\mu^k j} = l_{jk}$.
3. If $k = 0$ and $M^\mu \neq \emptyset$, then consider any buyer $i$ allocated to 0 in $\mu$. Then, $v_{i0} - p(0) \geq v_{ij} - p(j)$. Hence, $p(0) - p(j) \leq -v_{ij}$. Since this is true for all $i$ such that $\mu_i = 0$, we can write $p(0) - p(j) \leq \min_{i:\mu_i=0} -v_{ij} = l_{j0}$.

This shows that $p$ is a potential of $G^\mu$. Since $p(0) = 0$ in a WE price vector, we get that $p$ is a $0-$potential of $G^\mu$.

Now, suppose that $p$ is a $0-$potential of $G^\mu$ for some efficient allocation $\mu$. By definition, $p(0) = 0$. For every $j \in N$, $p(0) - p(j) \leq l_{j0} \leq 0$. Using $p(0) = 0$, we get $p(j) \geq 0$. Hence, $p$ is a price vector. Consider any $j \in N^\mu$. By definition of a $0-$potential, we can write $p(j) - p(0) \leq l_{0j} = 0$. Using $p(0) = 0$, we get $p(j) \leq 0$. Hence, $p(j) = 0$ for all $j \in N^\mu$.

Now, consider any $i \in M$. If $i \in M^\mu$, then $\mu_i = 0$. By definition of potential, $p(\mu_i) - p(j) = p(0) - p(j) \leq l_{j0} \leq v_{i0} - v_{ij}$ for all $j \in N$. Hence, $v_{i0} - p(0) \geq v_{ij} - p(j)$ for all $j \in N$.

So, $0 \in D_i(p)$. If $i \notin M^\mu$, then by definition of potential, $p(\mu_i) - p(j) \leq l_{j\mu_i} = v_{i\mu_i} - v_{ij}$ for all $j \in N$. This gives $\mu_i \in D_i(p)$. Therefore, $p$ is a WE price vector. ■

PROOF OF THEOREM 5

*Proof*: For the proof of the theorem, we use the following claim.

CLAIM **1** *Let $(p, \mu)$ be a Walrasian equilibrium. $p$ is an interior Walrasian equilibrium price vector if and only if $D_i(p) = \{\mu_i\}$ for all $i \in M$ and $N^+(p) = N_0$.*

*Proof*: Since every WE price vector is a $0-$potential (Theorem 4), the set of WE price vectors is a polytope in $\mathbb{R}^n$ defined by $p(k) - p(j) \leq l_{jk}$ for all $j, k \in N$ with $p(0) = 0$. An interior point of this polytope is a point $p^*$ satisfying $p^*(k) - p^*(j) < l_{jk}$ for all $j, k \in N$ with $p^*(0) = 0$.

Suppose $p$ is an interior WE price vector. Clearly $N^+(p) = N_0$, since otherwise some goods will have zero prices only. Now, consider the allocation graph $G^\mu$. Since $p$ is an interior WE price vector, it is a $0-$potential of $G^\mu$ (by Theorem 4), and for every $j, k \in N$, $j \neq k$, we have $p(k) - p(j) < l_{jk}$. Now consider any buyer $i \in M$. Let buyer $i$ be assigned to good $j \in N$. By definition, $j \in N \setminus N^\mu$. Hence, $p(j) - p(k) < l_{kj} = v_{ij} - v_{ik}$ for all $k \in N \setminus \{j\}$. This gives, $v_{ij} - p(j) > v_{ik} - p(k)$ for all $k \in N \setminus \{j\}$. Hence, $D_i(p) = \{j\}$.

Now, suppose $D_i(p) = \{\mu_i\}$ for all $i \in M$ and $N^+(p) = N_0$. Since $N^+(p) = N_0$, we get that every good in $N_0$ is assigned to some buyer in $\mu$. Consider any $j \in N_0$. Let $j$ be assigned to $i$ in $\mu$. Since $D_i(p) = \{j\}$, we get $v_{ij} - p(j) > v_{ik} - p(k)$ for all $k \in N \setminus \{j\}$. So, $p(j) - p(k) < v_{ij} - v_{ik}$ for all $k \in N \setminus \{j\}$. If $j \neq 0$, then we get $v_{ij} - v_{ik} = l_{kj}$ and $p(j) - p(k) < l_{kj}$. If $j = 0$, then, consider $i'$ such that $-v_{i'k} \leq v_{ik}$ for all $i$ with $\mu_0 = i$. By assumption $D_{i'}(p) = \{0\}$. Hence, we can write $0 - p(j) > v_{i'k} - p(k)$ for all $k \in N$. This gives, $p(j) - p(k) < -v_{i'k} = \min_{i:\mu_i=0} -v_{ik} = l_{kj}$. This shows that $p(j) - p(k) < l_{kj}$ for all $j, k \in N$, $j \neq k$. Hence, $p$ is an interior WE price vector. ■

Suppose $p$ is an interior WE price vector. Let $\mu$ be an efficient allocation. Then $(p, \mu)$ is a WE. By Claim 1, $N^+(p) = N_0$ and $D_i(p) = \{\mu_i\}$ for every $i \in M$. Since $N^+(p) = N_0$, every $j \in N_0$ is allocated in $\mu$. Hence, $U(\{j\}, p) = O(\{j\}, p) = \{\mu^j\}$. Since $N^+(p) = N_0$, we can equivalently say that $\{j\}$ is weakly overdemanded and weakly underdemanded at $p$ for all $j \in N_0$.

Suppose $\#U(\{j\}, p) = \#O(\{j\}, p) = 1$ for every $j \in N_0$ and $N^+(p) = N_0$. Therefore, every good in $N^+(p) = N_0$ is demanded by a unique buyer. Hence, these goods can be assigned to those corresponding unique buyers, and the remaining buyers can be assigned to the dummy good (notice that these buyers must be demanding the dummy good). Let this allocation be $\mu$. So, $(p, \mu)$ is a Walrasian equilibrium and $D_i(p) = \{\mu_i\}$ for all $i \in M$. Using Claim 1, $p$ is an interior WE price vector. ■

Proof of Theorem 6

*Proof*: Suppose an interior WE price vector $p$ exists. Assume for contradiction that $\mu$ and $\mu' \neq \mu$ are two efficient allocations. Then, $(p, \mu)$ and $(p, \mu')$ are two Walrasian equilibria. Since $p$ is an interior WE price vector, $N^+(p) = N_0$, and hence, every good in $N_0$ is assigned in $\mu$ and $\mu'$. Since $\mu \neq \mu'$, for some good $j \in N_0$, $\mu_i = \mu'_{i'} = j$ where $i \neq i'$. This means $\{i, i'\} \subseteq U(\{j\}, p)$, which implies that $\#D_i(p) \geq 2$ and $\#D_{i'}(p) \geq 2$. This is a contradiction by Theorem 5.

Also, by definition $N^+(p) = N_0$ for every interior price vector $p$. By definition of WE, no good from $N_0$ is unassigned. Hence, $n \leq m$.

Suppose there is a unique efficient allocation $\mu$ and $n \leq m$. Let $(p, \mu)$ be any WE. First, we prove the following claim.

Claim **2** *Every good in $N_0$ is assigned in $\mu$.*

*Proof*: Suppose some good $j \in N_0$ is not assigned in $\mu$. Hence, $p(j) = 0$. Since $n \leq m$, some buyer $i \in M$ is assigned the dummy good. But $v_{ij} - p(j) = v_{ij} \geq 0$. This means $j \in D_i(p)$. Therefore, assigning $i$ to $j$ gives another allocation that is efficient, which is a contradiction since $\mu$ is the unique efficient allocation. Hence, every good in $N_0$ is assigned in $\mu$. ■

Consider the minimum WE price vector $p^{min}$. Let $M^+$ be the set of buyers assigned to goods from $N_0$ in $\mu$.

Claim **3** *For every buyer $i \in M^+$, $v_{i\mu_i} - p^{min}(\mu_i) > 0$.*

*Proof*: Assume for contradiction that for some buyer $i \in M^+$, $v_{i\mu_i} - p^{min}(\mu_i) = 0$. Since every set of goods $S \subseteq N_0$ is not overdemanded and every $S \subseteq N^+(p^{min})$ is not weakly underdemanded at $p^{min}$ (Theorem 1), by removing buyer $i$ from the economy, it is still not overdemanded and not underdemanded at $p^{min}$. This means, we can find a WE allocation $\mu'$ without buyer $i$. By assigning buyer $i$ to the dummy good, which is in his demand set at $p^{min}$, we get an efficient allocation different from $\mu$. This is a contradiction. ■

Now, construct a price vector $\bar{p}$ by increasing the prices of goods in $N_0$ by sufficiently small amount from $p^{min}$. Every good in $N_0$ is assigned by Claim 2 to buyers in $M^+$. By increasing prices by sufficiently small amount from $p^{min}$, and using Claim 3, every buyer in $M^+$ continues to demand $\mu_i$ at $\bar{p}$, and buyers in $M \setminus M^+$ demand only the dummy good. Hence, $\bar{p}$ is a WE price vector, and $N^+(\bar{p}) = N_0$. Further, for sufficiently small price increase from $p^{min}$, we can have for every buyer $i \in M^+$, $v_{i\mu_i} - \bar{p}(\mu_i) > 0$ by Claim 3. Now, consider the following claim.

Claim 4 *Consider a WE $(p, \mu)$ such that $O(N_0, p) = M^+$ and $N^+(p) = N_0$. Consider a set of goods $S \subseteq N_0$ with $\#S \geq 2$. Define $T = \{i \in M : \mu_i \in S\}$. Then, there exists some buyer $i \in T$ such that $D_i(p) = \{\mu_i\}$.*

*Proof*: Suppose $\#D_i(p) \geq 2$ for all $i \in T$. Consider any good $j^0 \in S$. We construct a sequence of pairs of buyers and goods. Initially set $K = \{j^0\}$ and $L = \{i^0\}$, where $\mu_{i^0} = j^0$. Consider some $j^1 \in D_{i^0}(p) \setminus \{j^0\}$ (such a $j^1$ exists since $\#D_{i^0}(p) \geq 2$). Set $K := K \cup \{j^1\}$ and $L := L \cup \{i^1\}$, where $\mu_{i^1} = j^1$. If some $j^2 \in D_{i^1}(p) \setminus \{j^1\}$ also belongs to $K$, then stop. Else, update $K := K \cup \{j^2\}$ for some $j^2 \in D_{i^1}(p) \setminus \{j^1\}$ and $L := L \cup \{i^2\}$, where $\mu_{i^2} = j^2$. We repeat this process. At any stage of the process, we have $K = \{j^0, j^1, \ldots, j^{k-1}\}$ and $L = \{i^0, i^1, \ldots, i^{k-1}\}$. If some $j^k \in D_{i^{k-1}}(p) \setminus \{j^{k-1}\}$ belongs to $K$, where $j^{k-1} = \mu_{i^{k-1}}$, then we stop. Otherwise, we set $K := K \cup \{j^k\}$ for some $j^k \in D_{i^{k-1}}(p) \setminus \{j^{k-1}\}$ and $L := L \cup \{i^k\}$, where $\mu_{i^k} = j^k$. The process is finite, since when we reach $L = T$, we have $K = S$.

Now, at the end of the process, let the final buyer to be inserted to $L$ be $i^k$. Let $i^k$ demand $j^{k'} \neq \mu_{i^k}$ from $K$ at $p$. By the definition of the sequence above, there exists an assignment $\mu'$ where $\mu'_{i^k} = j^{k'}$ and $\mu'_{i^l} = j^{l+1}$ with $\mu'_{i^l} \in D_{i^l}(p)$ for all $k' \leq l < k$, and $\mu'_i = \mu_i$ otherwise. Clearly, $(p, \mu')$ is a WE. Hence, $\mu'$ is an efficient allocation, contradicting the fact the $\mu$ is unique. ■

The proof is done by repeatedly applying Claim 4. First, set $p = \bar{p}$, $S = N^+(p) = N_0$, and $T = M^+$ in Claim 4. Then, we get a buyer $i \in M^+$ such that $D_i(p) = \{\mu_i\}$. Since $v_{i\mu_i} - p(\mu_i) > 0$, we can increase the price of $\mu_i$ by sufficiently small amount to get a new WE price vector $p'$ such that $D_i(p') = \{\mu_i\}$ and $U(\{\mu_i\}, p') = \{i\}$. Now, set $p = p'$, $S = N^+(p') \setminus \{\mu_i\}$, and $T = M^+ \setminus \{i\}$, and apply Claim 4 again. After repeating this procedure for all the buyers, we get a WE price vector $\hat{p}$ where for every buyer $i \in M$, $D_i(\hat{p}) = \{\mu_i\}$. By Theorem 5, $\hat{p}$ is an interior WE price vector. ■

Proof of Theorem 7

*Proof*: We begin the proof with two claims.

Claim 5 *For any WE price vector $p$ satisfying $K(p, N_0) \neq \emptyset$, $L(p, S^o(p)) = S^o(p)$ if and only if, for all $j \in S^o(p)$, $p(j) = \max_{k \in K(p,N_0)} \big[s(0,k) - s(j,k)\big]$, where shortest paths $s(\cdot,\cdot)$ are computed in an allocation graph corresponding to any efficient allocation.*

*Proof*: Suppose $L(p, S^o(p)) = S^o(p)$ and $K(p, N_0) \neq \emptyset$. By Lemma 4, $p(j) = p^{max}(j)$ for all $j \in K(p, N_0)$. By Theorem 3, $p(j) \geq \max_{k \in K(p,N_0)} \big[s(0,k) - s(j,k)\big] \geq 0$ for all $j \in N_0 \setminus K(p, N_0)$. Let

$$X = \{j \in N_0 \setminus K(p, N_0) : p(j) > \max_{k \in K(p,N_0)} \big[s(0,k) - s(j,k)\big]\}.$$

Assume for contradiction that $X$ is non-empty. Clearly $X \subseteq N^+(p)$. Since $L(p, N_0 \setminus K(p, N_0)) = N_0 \setminus K(p, N_0)$, $X$ is not weakly underdemanded at $p$. Hence, $\#U(p, X) > \#X$. Consider the price vector $p'$ where prices of all goods except goods in $X$ remain the same as in $p$, but for all $j \in X$, $p'(j) = \max_{k \in K(p,N_0)} \big[s(0, k) - s(j, k)\big] < p(j)$. Hence, buyers in $U(p, X)$ become exclusive demanders of $X$ at $p'$. So, $\#O(p', X) \geq \#U(p, X) > \#X$. Hence, $X$ is overdemanded at $p'$. Since $p'$ is a WE price vector by Theorem 3, we get a contradiction by Theorem 1.

Suppose for all $j \in S^o(p)$, we have $p(j) = \max_{k \in K(p,N_0)}[s(0, k) - s(j, k)]$. Assume for contradiction that there exists $j \in N_0 \setminus K(p, N_0)$ such that $j \notin L(p, N_0 \setminus K(p, N_0))$. Clearly, $p(j) > 0$. This means that for some $T \subseteq N^+(p) \setminus K(p, N_0)$ containing $j$ the set $T$ is weakly underdemanded. Consider a price vector $p'$ where prices of goods in $T$ are lowered by sufficiently small amount, whereas prices of other goods remain the same. Since $T$ is weakly underdemanded, $p'$ is a WE price vector. By (4), $p'(j) \geq \max_{k \in K(p,N_0)}[s(0, k) - s(j, k)]$ for all $j \in N_0 \setminus K(p, N_0)$. This gives us a contradiction. ■

Claim **6** *For any WE price vector $p$ satisfying $L(p, N_0) \neq \emptyset$, $K(p, S^u(p)) = S^u(p)$ if and only if, for all $j \in S^u(p)$, $p(j) = \min_{k \in L(p,N_0)} \big[s(k, j) - s(k, 0)\big]$, where shortest paths $s(\cdot, \cdot)$ are computed in an allocation graph corresponding to any efficient allocation.*

*Proof*: The proof is similar to the proof of Claim 5. ■

Suppose $p$ is an extreme WE price vector. Then, by Theorem 3 and Lemmas 3 and 4, $L(p, N_0) \cup K(p, N_0) \neq \emptyset$. If $K(p, N_0) \neq \emptyset$, then by Theorem 3 and Claim 6, we have that $L(p, S^o(p)) = S^o(p)$. Similarly, if $L(p, N_0) \neq \emptyset$, then by Theorem 3 and Claim 6, we have that $K(p, S^u(p)) = S^u(p)$. The converse statement also follows from Theorem 3 and Claims 5 and 6. ■