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Abstract

We consider the problem of choosing a partition of a set of objects by a set of agents. The private information of each agent is a strict ordering over the set of partitions of the objects. A social choice function chooses a partition given the reported preferences of the agents. We impose a natural restriction on the allowable set of strict orderings over the set of partitions, which we call an *intermediate* domain. Our main result is a complete characterization of strategy-proof and tops-only social choice functions in the intermediate domain. We also show that a social choice function is strategy-proof and unanimous if and only if it is a *meet* social choice function.

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1 Introduction

The general mechanism design problem is concerned with choosing an alternative among a set of alternatives, when each agent has a preference ordering over the alternatives, which is his private information. The seminal work of Gibbard (1973) and Satterthwaite (1975) showed that if the preferences of agents over alternatives is unrestricted and the range of the social choice function has at least three alternatives, then the only strategy-proof social choice function is a dictatorship. A large body of literature has since focused on relaxing the underlying assumptions in the Gibbard-Satterthwaite theorem. One way to escape this impossibility result is to impose domain restrictions. Indeed, many real life problems have inherent domain restrictions.

We study one such model. In our model, a set of agents are faced with a set of objects. The agents have to collectively choose a partition of the set of objects. Each agent has a strict preference ordering over the set of partitions (of the set of objects), which is his private information. A social choice function asks for the preference orderings of the agents, and based on the reported preference orderings, chooses a partition. We describe some settings where such a model can be applied.

- Creation of a network. There is a set of cities, and the government wants to create a network by connecting the cities (using highways, high-speed cables, high-speed rails etc.). Connections are transitive - if city a is connected to b and b is connected to c, then a is also connected to c. Such networks will lead to a partition of the cities. Various firms will use the network. Firms have preferences over networks. These preferences may arise because each firm may have manufacturing and distribution centers at different cities. A center at a particular city will want to be connected by high-speed infrastructure by a fixed set of cities - it may not want to be connected to all cities because there may be costs of connections which will offset any utility of connection. The government collects the preferences of agents over networks, and chooses a network to build.

- Creating Political Districts from Geographical Districts. Consider a state which has a set of geographical districts. The state wants to create a set of political districts (constituencies) by partitioning the geographical districts. Different political parties may have different preferences over the partitions of geographical district. The state asks each political party to reveal its preference ordering of the partitions, and then chooses a collective partition of the set of geographical districts.

We impose the standard notion of strategy-proofness on the social choice functions - it must be a dominant strategy for every agent to report his true preference ordering over partitions. If all possible preference orderings are allowed, then, under a mild range condition or unanimity, the Gibbard-Satterthwaite theorem will say that the only strategy-proof social choice function is a dictatorship.

We consider a restricted domain of preference orderings. Note that a partition must determine, for every pair of objects i and j, whether i and j should be together or separate. Call two partitions similar in i and j if they treat i and j similarly, i.e., either both of them put i and j together or both of them put i and j separately. Now, consider an agent who has A as the top partition in a preference ordering, and consider two other partitions B and C. Suppose whenever A and C are similar for any pair of objects, A and B are also similar for that pair of objects. This implies that overall, B is more similar to A than C is to A. Our domain restriction says that in such a case this agent must rank B over C, whenever A is his top ranked partition. We allow for all preference orderings which is consistent with such restriction, and call such a domain an *intermediate domain*, and investigate the consequence of strategy-proofness in this domain.

1.1 Our Contribution

We give a complete characterization of strategy-proof and tops-only social choice functions in the intermediate domain. Tops-onlyness property stipulates that at two preference profiles if the tops of the agents are the same, then the chosen partitions must also be the same. Because the number of partitions is quite large, tops-only property significantly reduces the communication requirement of each agent to the mechanism designer.

We prove our main result by proving another result, which is interesting in its own right. This result uses another mild property called *Pareto*$^+$, which requires that if all the agents want to put a pair of objects together in their top-ranked partition, then the social choice function must put them together. We show that if the number of objects is at least three, then a social choice function is strategy-proof, tops-only, and satisfies Pareto$^+$ if and only if it is a *meet** social choice function. A meet* social choice function identifies a subset of agents (may be empty) as oligarchs, and for every pair of objects, they are put together if and only if the top-ranked partition of each oligarch puts them together.

Our main result uses this characterization. It says that if a social choice function is strategy-proof and tops-only, then it can be *decomposed*. Decomposability roughly says that there is exists a canonical partition such that a pair of objects belonging to different bundles of this partition are never put together. Further, the social choice function can be viewed as union of a set of strategy-proof social choice functions, each defined for a bundle of the canonical partition. We show that if a social choice function is strategy-proof and tops-only, we can decompose it into a set of social choice functions, each of which is strategy-proof, tops-only, and satisfies Pareto$^+$. As a result, we can invoke our earlier characterization to get a complete characterization of strategy-proof and tops-only social choice functions.

We show that if we impose unanimity, then we can get rid of tops-onlyness property. In particular, we show that if the number of objects is at least three, then a social choice function is strategy-proof and satisfies unanimity if and only if it is a meet social choice function (a meet social choice function is a meet* social choice function where the the oligarchs are non-empty). Hence, unanimity and strategy-proofness imply tops-onlyness in our domain.

On the other hand, if we impose Pareto efficiency, then also we can get rid of tops-onlyness property (Pareto efficiency implies unanimity), but it reduces the class of strategy-proof social choice functions significantly. In particular, we get a Gibbard-Satterthwaite-like impossibility - if the number of objects is at least three, then the only strategy-proof and Pareto efficient social choice function is a dictatorship.

1.2 Past Literature

Since the seminal work of Gibbard and Satterthwaite, many interesting restricted domains have been investigated - for a survey, see Barbera (2010) and Moulin (1983). Some prominent examples of restricted domains that have been studied are: models with single-peaked preferences (Moulin, 1980) and multi-dimensional single-peaked domains (Barbera et al., 1993), matching models (Papai, 2000; Svensson, 1999), location on a network model (Schummer and Vohra, 2002), choosing a subset of objects from a set of objects in a separable environment (Barbera et al., 1991), and many more. Different restrictions bring out different possibilities, e.g., *median rules* and its generalizations are strategy-proof in various single-peaked domains (Moulin, 1980; Barbera et al., 1993).

As far as we know, there is no literature studying strategy-proof social choice functions in our model. A recent related paper is that of Duddy and Piggins (2010). They study strategy-proof social choice functions in a model where agents need to classify each object as 1 (good) or 0 (i.e., a partition into two bundles), a model first studied in Kasher and Rubinstein (1997) in the context of axiomatic aggregation (see also Miller (2008)) and studied in Barbera et al. (1991) under separable preferences. Barbera et al. (1991) find voting by committees rules to be the only onto and strategy-proof social choice functions for this model in separable domain. Under some mild technical conditions and a range condition, Duddy and Piggins (2010) show that the only onto and strategy-proof social choice function in their domain is a dictatorship.

The literature on separable preferences is related to ours. Models with separable domains work in a multi-dimensional environment. Suppose there are k dimensions of an alternative (in the choice of a subset of objects from a set of objects, each object is a dimension). There are A_j set of outcomes in dimension j, and the set of alternatives is $A_1 \times A_2 \times \ldots \times A_k$. The separable domain roughly says that preferences over alternatives is separable over each dimension. Separable domains where an alternative is a product of outcomes on each dimension is well-studied. Barbera et al. (1991) and Barbera et al. (2005) studied it in the context of choosing a subset of objects from a set of objects. While these two papers consider the case where each object can have two outcomes (chosen or not chosen), Svensson and Tortstensson (2008) consider the case where each object can have more than two outcomes - see also Reffgen and Svensson (2010). Barbera et al. (1993) studied a separable domain in the context of multi-dimensional single-peaked preferences. The main insights of these papers is that in separable domains where an alternative is a product set of outcomes in each dimension, a strategy-proof social choice function can be decomposed into social choice functions in each dimension. The most general version of this result is found in Le Breton and Sen (1999) - see also Le Breton and Weymark (1999) and Weymark (1999).

Why is our decomposability result not implied by these results? It is possible to imagine each pair of objects as a dimension in our model. The outcome for each pair of objects is either together or separate. But we *cannot* write the set of alternatives in our model (the set of partitions) as a product of possible outcomes for each pair of objects. This is because of the requirement that a partition is an *equivalence relation*, and must satisfy a transitivity property - if the pair of objects i and j is together, and the pair of objects j and k is together, then the outcome for the pair of objects i and k is fixed (i and k must be together). This is a crucial departure from the literature which studies separable domains on product outcome sets. Because of this difference, none of our results is implied by any of the results from the literature on separable domains. Moreover, our decomposability result is of completely different nature. We do not require any onto condition, usually a standard assumption in the separability literature (Le Breton and Sen, 1999). We get our decomposability result with strategy-proofness and tops-onlyness (which is weaker than ontoness). Further, we do not get decomposability on each pair of objects.

Our model has some resemblance to the coalition formation literature. However, the coalition formation literature usually does not consider externality between coalitions, i.e., focuses on *hedonic* coalition formation, where an agent only cares about the coalition he is in (Bogomolnaia and Jackson, 2002). In hedonic coalition formation games, Rodriguez-Alvarez (2004) shows that strategy-proofness, individual rationality, non-bossiness, and voter-sovereignty are incompatible. Rodriguez-Alvarez (2009) considers strategy-proof hedonic coalition formation in restricted domains of separable preferences. With individual rationality and non-bossiness, he characterizes a class of rules called the "single-lapping" rules. This rule is also central in characterizing *coalition form games* which give rise to unique core-stable coalitions (Papai, 2004). The coalition formation literature is too long to describe here, and the domain is very restricted. So, none of our results can be applied in the coalition formation setting.

There is a large body of literature studying Arrovian type aggregation in our model. This literature is inspired by Wilson (1978), who advocates Arrovian aggregation in abstract models such as those described in Rubinstein and Fishburn (1986). This literature considers *aggregators*, which are maps from a profile of partitions of agents to a partition. This literature does not consider a preference ordering over partitions for agents. Rather, each agent has a partition (an equivalence relation) of objects, and an aggregator chooses a collective partition. The main axioms used in that literature are binary independence and some form of unanimity (Mirkin, 1975; Leclerc, 1984; Barthélemy et al., 1986; Fishburn and Rubinstein, 1986; Barthélemy, 1988; Dimitrov et al., 2011). Broadly, this literature concludes that binary independence along with some form of unanimity gives us meet aggregators (when there are at least three objects) - see also Dimitrov et al. (2011); Chambers and Miller (2011) for a different characterization of meet aggregators. This literature does not focus on preferences of agents over partitions, and does not consider strategy-proof aggregation. Much like Gibbard-Satterthwaite theorem is proved using Arrow's theorem (see Reny (2001) for a unified proof of Arrow's theorem and Gibbard-Satterthwaite theorem), we use one of the results in the aggregation literature of this model to prove one of our results. Thus, our results provide a strategic foundation to this literature on aggregating partitions.

2 The Domain of Preferences

Let $N = \{1, \ldots, n\}$ be the set of agents and $M = \{1, \ldots, m\}$ be the set of objects. A **partition** A of objects in M is an equivalence relation, and can be represented by an $m \times m$-$\{0,1\}$ matrix satisfying (a) reflexivity: $A_{ii} = 1$ for all $i \in M$, (b) symmetry: $A_{ij} = A_{ji}$ for all $i, j \in M$, and (c) transitivity: $A_{ij} = A_{jk} = 1$ implies $A_{ik} = 1$ for all $i, j, k \in M$ [1]. The value of A_{ij} reflects whether objects i and j are together in partition A or not. In particular, $A_{ij} = 1$ indicates that objects i and j are together in partition A, whereas $A_{ij} = 0$ indicates that objects i and j are separate in partition A. A **bundle** of a partition A is a set of objects $S \subseteq M$ such that for all $i, j \in S$, $A_{ij} = 1$ and for all $i \in S$ and $j \notin S$, $A_{ij} = 0$ [2]. Hence, a partition can be written as a collection of bundles. Let $\mathbb{M}$ be the set of all partitions of M.

A preference ordering is a complete, transitive, and anti-symmetric binary relation over $\mathbb{M}$. Let $\mathcal{P}$ be the set of all strict orderings over $\mathbb{M}$. An agent $h \in N$ has a preference ordering $\succ_h$ over $\mathbb{M}$, where $\succ_h (k)$ denotes the k-th ranked partition according to $\succ_h$. We impose a natural restriction on the allowable set of preference ordering.

Definition 1 *A domain of preferences $\mathcal{D} \subseteq \mathcal{P}$ is* **intermediate** *if for every $\succ_h \in \mathcal{D}$ with $A = (\succ_h (1))$ and every $B, C \in \mathbb{M}$ such that*

$$\{\{i,j\} : i, j \in M, i \neq j, C_{ij} = A_{ij}\} \subsetneq \{\{i,j\} : i, j \in M, i \neq j, B_{ij} = A_{ij}\},$$

we have $B \succ_h C$.

In some sense, if for every $i, j \in M$, $C_{ij} = A_{ij}$ implies $B_{ij} = A_{ij}$, then B is more similar to A than C is to A. A strict ordering in $\succ_h$ belonging to $\mathcal{D}$ must satisfy the property that if a partition B is "more similar" to the top partition of $\succ_h$ than a partition C, then $B \succ_h C$. We assume that the preference ordering $\succ_h$ of every agent $h \in N$ must belong to the domain of intermediate preferences $\mathcal{D}$.

Our intermediate domain uses a familiar notion of *betweenness* for any relation. Its use can be traced back to Grandmont (1978). A partition is an equivalence relation. Using the terminology of Grandmont (1978), our domain restriction says that if a partition B is between partitions A and C, then if A is at the top, then B must be ranked above C.

[1] Here, A_{ij} specifies the value of the entry in the ith row and jth column.

[2] The conventional mathematical terminology for partition is "equivalence relation" and for bundle is "equivalence class". We use partition and bundle for convenience.

Now, we give an equivalent way of stating our domain restriction. For this, we define utility functions for every agent. Let $\mathcal{M} \equiv \{\{i,j\} : i,j \in M, i \neq j\}$. Define a utility function for agent h as $u^h : \{0,1\} \times \mathcal{M} \to \mathbb{R}$. Instead of writing $u^h(x, \{i,j\})$, where $x \in \{0,1\}$, we will write $u^h_{ij}(x)$ for simplicity. Here, $u^h_{ij}(0)$ and $u^h_{ij}(1)$ denote the utility gained by agent h from the pair of objects $\{i,j\}$ if i and j are put separately and together respectively in a partition. For any partition A, denote by $U^h(A)$ the sum $\sum_{\{i,j\}\in\mathcal{M}} u^h_{ij}(A_{ij})$. This is agent h's utility from partition A.

We say a utility function u^h is consistent with partition A if for all $\{i,j\} \in \mathcal{M}$ we have

$$u^h_{ij}(A_{ij}) > u^h_{ij}(1 - A_{ij}).$$

Clearly, if a utility function u^h *represents* a preference ordering $\succ_h$, it must be consistent with $\succ_h (1)$.

Definition **2** *A partition B* **dominates** *partition C at partition A if for all utility functions u^h consistent with A we have*

$$U^h(B) > U^h(C).$$

The following result establishes the connection between intermediate domain and such utility representation.

Proposition **1** *Let $\succ_h$ be any preference ordering. It belongs to the intermediate domain $\mathcal{D}$ if and only if for all partitions B and C such that B dominates C at $\succ_h (1)$, we have $B \succ_h C$.*

The proof of Proposition 1 is in the Appendix. Proposition 1 clarifies the exact nature of our domain restriction. In our domain, an agent must evaluate a partition by assigning utility numbers to every pair of object and each of the two possible states for each pair of object. The utility of an agent for a partition is then obtained by summing those utility numbers. In the creation of network example, this means that the utility of connecting city a and b is the same whether any other city is connected to them or not.

We give an example to clarify some of the nuances of the intermediate domain.

Example **1** *Suppose $M = \{a,b,c,d\}$. Consider a preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A$, where A refers to the partition with the following bundles: $\{a,b,c\}$ and $\{d\}$. Consider four more partitions and their corresponding bundles:*

- *B is the partition with bundles $\{a,b\}$, $\{c\}$, and $\{d\}$.*
- *C is the partition with bundles $\{a,b,d\}$ and $\{c\}$.*
- *D is the partition with bundles $\{a\}$, $\{b\}$, and $\{c,d\}$.*
- *E is the partition with bundles $\{a\}$ and $\{b,c,d\}$.*

Comparing B and C with A, we see that whenever C and A agree on a pair of objects, B and A also agree on the same pair. Hence, $B \succ_h C$.

However, for B and E, we cannot make such a comparison - a and b are together in A and B but separate in E, but b and c are together in A and E but separate in B. So, it is possible that $B \succ_h E$ or $E \succ_h B$. Similarly, D and E can be ranked either way when A is the top.

On the other hand, wherever A and D agree on a pair of objects, A and B also agree on that pair. Hence, $B \succ_h D$.

A **social choice function (SCF)** is a mapping $F : \mathcal{D}^n \rightarrow \mathbb{M}$, i.e., given the intermediate preference orderings of agents, it selects a partition. For a profile $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$, the output of F is denoted by $F(\succ_1, \ldots, \succ_n)$, and $F(\succ_1, \ldots, \succ_n)_{ij} \in \{0, 1\}$ denotes whether $i, j \in M$ belong to the same bundle or not in $F(\succ_1, \ldots, \succ_n)$. Often, we write the profile $(\succ_1, \ldots, \succ_n)$ as $\succ$ and the profile $(\succ'_1, \ldots, \succ'_n)$ as $\succ'$, and so on.

We impose the usual strategy-proofness requirement on an SCF - every agent must have a dominant strategy to submit his true preference ordering. An agent h **manipulates** an SCF F at $(\succ_h, \succ_{-h}) \in \mathcal{D}^n$ via $\succ'_h \in \mathcal{D}$ if $F(\succ'_h, \succ_{-h}) \succ_h F(\succ_h, \succ_{-h})$.

Definition 3 *An SCF F is* **strategy-proof** *if no agent $h \in N$, can manipulate at any preference profile $(\succ_h, \succ_{-h}) \in \mathcal{D}^n$ via any preference ordering $\succ'_h \in \mathcal{D}$.*

3 Strategy-proof, Tops-only, Pareto Properties

The number of partitions grows exponentially with the number of objects. Hence, the number of possible orderings grows even faster with the number of objects. Even in the intermediate domain, the number of allowable orderings increases at an exponential rate with the number of objects. In such a scenario, a natural restriction to impose is that agents only report their top ranked partition to the social choice function. In particular, the following well-known requirement seems plausible in our model.

Definition 4 *An SCF F is* **tops-only** *if for every pair of profiles $\succ, \succ' \in \mathcal{D}^n$ such that $\succ_h(1) = \succ'_h(1)$ for all $h \in N$, then $F(\succ) = F(\succ')$.*

Tops-onlyness is a well-studied axiom in social choice theory. It comes as a consequence of unanimity in various domains (Chatterji and Sen, 2011). We discuss this issue further for our model later.

If an SCF is tops-only, then it only cares about the top ranked partition of each agent. In such a case we can focus on aggregators instead of SCFs.

An **aggregator** v is a mapping $v : \mathbb{M}^n \rightarrow \mathbb{M}$. So for a profile of partitions $(A^1, \ldots, A^n)$, an aggregator gives a partition $v(A^1, \ldots, A^n)$.

An agent h **manipulates** an aggregator v at (A^h, A^{-h}) via $B^h \in \mathbb{M}$ if for some preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h(1) = A^h$, we have $v(B^h, A^{-h}) \succ_h v(A^h, A^{-h})$.

Definition 5 *An aggregator v is* **strategy-proof** *if no agent $h \in N$ can manipulate at any $(A^h, A^{-h}) \in \mathbb{M}^n$ via any B^h.*

A tops-only SCF induces an aggregator. Suppose F is a tops-only SCF. Then, define $v^F(A^1, \ldots, A^n) = F(\succ_1, \ldots, \succ_n)$, where $\succ \in \mathcal{D}^n$ is such that $\succ_h (1) = A^h$ for every $h \in N$ - note that for every $A^h \in \mathbb{M}$, there exists $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A^h$ (such domains are called **minimally rich**). Clearly, if F is strategy-proof, then v^F is strategy-proof.

3.1 An Implicit Characterization

We will now establish an implicit characterization of strategy-proof *aggregators*. This characterization will identify a simple property of an aggregator which is equivalent to strategy-proofness. This of course implies a characterization of strategy-proof SCFs which are tops-only.

Definition 6 *An aggregator v is responsive, if for every $h \in N$, for every A^{-h}, for every $A^h \in \mathbb{M}$, and every $i, j \in M$ we have $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$ implies that*

$$v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} \qquad \forall\ B^h \in \mathbb{M}.$$

Responsiveness requires that if an agent's preference for a pair of objects is not fulfilled for some partition (keeping profile of other agents fixed), then the outcome for that pair of objects do not change by changing the partition.

Proposition 2 *An aggregator is strategy-proof if and only if it is responsive.*

Proof: Suppose v is strategy-proof. Now, fix an agent $h \in N$, and fix the profile of other agents at A^{-h}. Let $A^h \in \mathbb{M}$ be such that $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$ for some $i, j \in M$. Consider another partition $B^h \in \mathbb{M}$. Let $v(A^h, A^{-h}) = B$, and $v(B^h, A^{-h}) = C$. Assume for contradiction $B_{ij} \neq C_{ij}$. By definition, $A^h_{ij} \neq B_{ij}$ and $A^h_{ij} = C_{ij}$. We claim that there is a preference ordering $\succ''_h$ such that $\succ''_h (1) = A^h$, and $C \succ''_h B$. If this was not true, then $B \succ_h C$ for all $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A^h$. This implies that if $C_{ij} = A^h_{ij}$, then $B_{ij} = A^h_{ij}$. This is a contradiction. But $C \succ''_h B$ implies that agent h will manipulate at (A^h, A^{-h}) via B^h. This is a contradiction to the fact that v is strategy-proof.

Suppose v is responsive. Fix an agent $h \in N$, and a profile A^{-h} of other agents. Consider partitions $A^h, B^h \in \mathbb{M}$. Let $v(A^h, A^{-h}) = A$ and $v(B^h, B^{-h}) = B$. If $A = B$, agent h cannot manipulate at (A^h, A^{-h}) via B^h. Else, $A \neq B$. If $A = A^h$, then agent h cannot manipulate at (A^h, A^{-h}) via B^h. So, assume $A \neq A^h$. Consider any $i, j \in M$, such that $A^h_{ij} \neq A_{ij}$. By responsiveness, $A_{ij} = B_{ij}$. This implies that whenever $B_{ij} = A^h_{ij}$ for some $i, j \in M$, $A_{ij} = A^h_{ij}$. Now, for any preference ordering $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A^h$, we must have $A \succ_h B$. Hence, agent h cannot manipulate at (A^h, A^{-h}) via B^h. This implies that v is strategy-proof. ∎

We now define an independence axiom for an aggregator. For every $i, j \in M$ and any profile of partitions $(A^1, \ldots, A^n)$, denote the n-dimensional vector $(A^1_{ij}, \ldots, A^n_{ij})$ as $\mathbf{A}_{ij}$.

Definition 7 *An aggregator v satisfies* **binary independence** *if for every $i, j \in M$ and for every pair of profiles $(A^1, \ldots, A^n)$ and $(B^1, \ldots, B^n)$ such that $\mathbf{A}_{ij} = \mathbf{B}_{ij}$, we have*

$$v(A^1, \ldots, A^n)_{ij} = v(B^1, \ldots, B^n)_{ij}.$$

Binary independence points at some kind of separability of aggregation. In particular, it says that whether a pair of objects remain separate or together must depend *only* on agents' preferences about that pair of objects. It is a widely studied axiom in Arrovian aggregation literature of this model (Mirkin, 1975; Fishburn and Rubinstein, 1986; Dimitrov et al., 2011). Below, we show that every strategy-proof aggregator satisfies binary independence.

Proposition 3 *If an aggregator is strategy-proof, then it satisfies binary independence.*

Proof: Let v be an aggregator which is strategy-proof. By our characterization in Proposition 2, v is responsive. Consider any $i, j \in M$. Let $(A^1, \ldots, A^n)$ and $(B^1, \ldots, B^n)$ be two profiles such that $\mathbf{A}_{ij} = \mathbf{B}_{ij}$. Consider the profile $(B^1, A^2, \ldots, A^n)$. Assume for contradiction that $v(A^1, \ldots, A^n)_{ij} \neq v(B^1, A^2, \ldots, A^n)_{ij}$. Since $A^1_{ij} = B^1_{ij}$, either $v(A^1, \ldots, A^n)_{ij} \neq A^1_{ij}$ or $v(B^1, A^2, \ldots, A^n)_{ij} \neq B^1_{ij}$. By responsiveness, if $v(A^1, \ldots, A^n)_{ij} \neq A^1_{ij}$, then $v(A^1, \ldots, A^n)_{ij} = v(B^1, A^2, \ldots, A^n)_{ij}$, and if $v(B^1, A^2, \ldots, A^n)_{ij} \neq B^1_{ij}$, then $v(A^1, \ldots, A^n)_{ij} = v(B^1, A^2, \ldots, A^n)_{ij}$. This is a contradiction.

We can repeat this argument by changing the preference of one agent at a time to reach the profile $(B^1, \ldots, B^n)$, and conclude $v(A^1, \ldots, A^n)_{ij} = v(B^1, \ldots, B^n)_{ij}$. ■

Although binary independence is implied by a strategy-proofness, it is not sufficient for strategy-proofness if $|M| \geq 3$. Consider the following aggregator v^*. Suppose $|M| \geq 3$. For any pair of objects $\{k, l\} \neq \{i, j\}$ and every profile $(A^1, \ldots, A^n)$,

$$v^*(A^1, \ldots, A^n)_{kl} = 0.$$

For every profile $(A^1, \ldots, A^n)$,

$$v^*(A^1, \ldots, A^n)_{ij} = 1 \text{ if and only if } A^1_{ij} = 1 \text{ and } A^h_{ij} = 0 \ \forall\ h \in N \setminus \{1\}.$$

Clearly, v^* satisfies binary independence. But it is not strategy-proof. To see this, consider agent 2, and fix the profile of other agents at (A^{-2}) such that $A^1_{ij} = 1$ and $A^h_{ij} = 0$ for all $h \in N \setminus \{1, 2\}$. Consider A^2 such that $A^2_{ij} = 0$. By definition $v^*(A^2, A^{-2})_{ij} = 1 \neq A^2_{ij}$. Now, consider B^2 such that $B^2_{ij} = 1$. By definition $v^*(B^2, A^{-2})_{ij} = 0 \neq v^*(A^2, A^{-2})_{ij}$. Hence, v^* is not responsive, and by Proposition 2, it is not strategy-proof.

3.2 A Rich Family of Aggregators

In this section, we define a rich class of aggregators, and the corresponding SCFs. Note that an aggregator v induces an SCF F^v as follows. For every profile of preference orderings $\succ \in \mathcal{D}^n$, define $F^v(\succ) = v(\succ_1(1), \ldots, \succ_n(1))$. Clearly, F^v is tops-only.

Definition 8 *An aggregator v is a* **meet*** *aggregator if there exists a set of agents (called oligarchs) $S \subseteq N$ such that for all $(A^1, \ldots, A^n)$, we have for all $i, j \in M$, $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$. If the set of oligarchs is the empty set, then we call the aggregator a* **trivial aggregator**. *An aggregator is a* **meet** *aggregator if it is a meet* aggregator but not a trivial aggregator. An aggregator is a* **dictatorship** *if it is a meet aggregator with a unique oligarch.*

An SCF F is a **meet*** *SCF if there exists a meet* aggregator v such that $F = F^v$. An SCF F is a* **trivial** *SCF if there exists a trivial aggregator v such that $F = F^v$. An SCF F is a* **meet** *SCF if it is a meet* SCF but not a trivial SCF. An SCF F is a* **dictatorship** *if there exists an aggregator v which is dictatorship, and $F = F^v$.*

Note that a trivial aggregator always gives the partition where all the objects in M are put in one bundle.

Not every meet* aggregator is strategy-proof when the domain of preferences is unrestricted. However, in intermediate domains, every meet* aggregator is strategy-proof.

Proposition 4 *If v is a meet* aggregator, then it is strategy-proof.*

Proof: Let v be a meet* aggregator with $S \subseteq N$ being the set of oligarchs.

Note that any agent $h \notin S$ cannot manipulate v at any profile via any partition. If $S = \emptyset$, we are done. Else, consider an agent $h \in S$, and assume for contradiction that h can manipulate v at (A^h, A^{-h}) via B^h. This means, $v(B_h, A^{-h}) \succ_h v(A^h, A^{-h})$ for some $\succ_h \in \mathcal{D}$ with $\succ_h(1) = A^h$. Let $v(A^h, A^{-h}) = B$, and $v(B^h, A^{-h}) = C$. By definition, $C \succ_h B$.

Consider any $i, j \in M$. By definition of meet aggregators, if $A^h_{ij} \neq B_{ij}$, then $A^h_{ij} = 1$ and $B_{ij} = 0$ - this is because, since $h \in S$, if $A^h_{ij} = 0$ then $B_{ij} = 0$. In that case, there is some agent $h' \in S$ and $h' \neq h$ such that $A^{h'}_{ij} = 0$. Hence, if agent h changes his report to B^h, $v(B^h, A^{-h})_{ij} = C_{ij} = 0$. Hence, for every $i, j \in M$, $A^h_{ij} \neq B_{ij}$ implies $A^h_{ij} \neq C_{ij}$. Since $\succ_h \in \mathcal{D}$, $B \succ_h C$. This is a contradiction. ■

As we will show later, the entire set of strategy-proof aggregators is much larger than the meet* family of aggregators. Our first result is a characterization of strategy-proof aggregators in the presence of the following weak requirement.

Definition 9 *An aggregator v satisfies* **Pareto$^+$** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^h_{ij} = 1$ for all $h \in N$, we have $v(A^1, \ldots, A^n)_{ij} = 1$.*

A social choice function F satisfies **Pareto$^+$** *if for every $i, j \in M$ and for every preference profile $\succ \in \mathcal{D}^n$ with $(\succ_h(1))_{ij} = 1$ for all $h \in N$, we have $F(\succ)_{ij} = 1$.*

Pareto$^+$ says that if each agent puts objects i and j together, then the aggregator must put i and j together.

THEOREM 1 *Suppose $|M| \geq 3$. A social choice function is strategy-proof, tops-only, and satisfies Pareto$^+$ if and only if it is a meet* social choice function.*

Proof: A meet* social choice function is tops-only. By Proposition 4, every meet* aggregator is strategy-proof, and hence, every meet* social choice function is also strategy-proof. Clearly, a meet* social choice function also satisfies Pareto$^+$.

For the converse, let F be a strategy-proof, tops-only social choice function satisfying Pareto$^+$. Since F is tops-only, v^F is well-defined. Further v^F is strategy-proof and satisfies Pareto$^+$. By Proposition 3, v^F must satisfy binary independence. Finally, Dimitrov et al. (2011) show that if an aggregator satisfies binary independence and Pareto$^+$, then it must be a meet* aggregator. Hence, v^F is a meet* aggregator, and F is a meet* social choice function. ■

Another way to state Theorem 1 is that an aggregator is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet* aggregator.

We remark that strategy-proofness and Pareto$^+$ property of a social choice function does not imply tops-onlyness. The following example illustrates that.

EXAMPLE 2 *Let $\hat{A}$ be the partition where each bundle is a singleton (i.e., all the objects are put separately). Consider the social choice function which chooses agent 1's top ranked partition from the set $\mathbb{M} \setminus \{\hat{A}\}$ at every preference profile. This social choice function is clearly strategy-proof and satisfies Pareto$^+$, but it is not tops-only.*

3.3 The Two Object Case

If $|M| = 2$, then we have many more aggregators which are strategy-proof and satisfy Pareto$^+$. Suppose $M = \{i, j\}$. Consider the following family of aggregators. Let $\mathbb{S} = \{S_1, \ldots, S_k\}$, where S_p with $p \in \{1, \ldots, k\}$ is a subset of agents (may be empty), called an oligarchy. A set of oligarchies $\mathbb{S}$ is **non-nested** if for every $S_p, S_q \in \mathbb{S}$, S_p is not a subset of S_q. Note that if a non-nested set of oligarchies $\mathbb{S}$ contains $\emptyset$, then it is the only element of $\mathbb{S}$.

DEFINITION 10 *Suppose $M = \{i, j\}$. An aggregator v is a* **meet*-join aggregator** *if there exists a set of non-nested oligarchies $\mathbb{S}$ such that for every $(A^1, \ldots, A^n) \in \mathbb{M}^n$, we have*

$$v(A^1, \ldots, A^n)_{ij} = 1 \text{ if and only if } A^h_{ij} = 1 \; \forall \; h \in S_p \text{ for some } S_p \in \mathbb{S}.$$

A social choice function F is a **meet*-join social choice function** *if there is a meet*-join aggregator v such that $F = F^v$.*

Note that a meet*-join aggregator is a meet aggregator if $\mathbb{S}$ is a singleton.

Proposition 5 *Suppose $|M| = 2$. An aggregator is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet*-join aggregator. Further, a social choice function is strategy-proof and satisfies Pareto$^+$ if and only if it is a meet*-join social choice function.*

Note that when $|M| = 2$, every social choice function is tops-only. Also, when $|M| = 2$ in our model, we are in the standard Gibbard-Satterthwaite model with two alternatives. The characterization of strategy-proof social choice functions is well-known in that case - see, for example, Moulin (1983) and Barbera et al. (1991). Applying Pareto$^+$, we get Proposition 5 immediately. So, we skip the proof of Proposition 5 here, but provide it in the appendix for completeness.

4 Strategy-proof and Tops-Only SCFs

The Pareto$^+$ property used in Theorem 1 may not be completely appealing in our model. Consider the example of building a network. Even if all firms agreed to connect a pair of cities a and b, it may be infeasible for the government to connect them because of their distance or budget constraints.

In this section, we drop the Pareto$^+$ requirement of a social choice function. We provide a complete characterization of strategy-proof aggregators. In other words, we provide a complete characterization of strategy-proof and tops-only social choice functions. Our result comes as a consequence of a decomposability result we are able to prove in our model. To define decomposability, we require some notation. For every subset of objects $X \subseteq M$, let $\mathbb{X}$ be the set of partitions of objects in X. For any partition A^h of agent h, we can look at the restriction of A^h to some subset of objects X, and that restriction is denoted as $A^{h,X}$.

Definition 11 *An aggregator $v : \mathbb{M}^n \to \mathbb{M}$ is* **decomposable** *if there exists a partition $\bar{A}$ of M with bundles $\bar{A}_1, \ldots, \bar{A}_k$ and k strategy-proof aggregators $v_1, \ldots, v_k$, where $v_p : \bar{\mathbb{A}}_p^n \to \bar{\mathbb{A}}_p$ for all $p \in \{1, \ldots, k\}$, and the following two conditions hold for every profile $(A^1, \ldots, A^n)$:*

- *$v(A^1, \ldots, A^n)_{ij} = v_p(A^{1,\bar{A}_p}, \ldots, A^{n,\bar{A}_p})_{ij}$ if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$.*
- *$v(A^1, \ldots, A^n)_{ij} = 0$ if $i \in \bar{A}_p$ and $j \in \bar{A}_q$ for some $p \neq q$ and $p, q \in \{1, \ldots, k\}$.*

In such a case, we say v can be decomposed into $v_1, \ldots, v_k$ via partition $\bar{A}$ with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$.

Every strategy-proof aggregator is decomposable - it can be decomposed into itself via the complete partition with the unique bundle M. However, we can say something non-trivial about decomposing any strategy-proof aggregator.

Proposition 6 *If v is a strategy-proof aggregator, then it can be decomposed into strategy-proof aggregators $v_1, \ldots, v_k$ via some partition $\bar{A}$ with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$ such that for all $p \in \{1, \ldots, k\}$, each v_p satisfies Pareto$^+$.*

Proof: Since v is strategy-proof, it is responsive due to Proposition 2, and satisfies binary independence by Proposition 3. Consider a profile of partitions, where the partition of each agent is the complete partition, where a complete partition is a partition where all the objects are put together, i.e., there is a single bundle containing all the objects. Let the outcome of the aggregator at this profile be $\bar{A}$ with bundles $(\bar{A}_1, \ldots, \bar{A}_k)$.

Consider the restriction of v to $\bar{A}_p$ for all $p \in \{1, \ldots, k\}$, and denote it by v_p. In particular, define for every $(X^1, \ldots, X^n)$, where $X^h \in \bar{\mathbb{A}}_p$ for all $h \in N$, and for every $i, j \in \bar{A}_p$

$$v_p(X^1, \ldots, X^n)_{ij} = v(A^1, \ldots, A^n)_{ij}.$$

Since v satisfies binary independence, each v_p is well-defined. Further, since v is responsive and satisfies binary independence, each v_p is also responsive. By Proposition 2, each v_p is strategy-proof.

Next, we will show that for every $i \in \bar{A}_p$ and $j \in \bar{A}_q$ where $p \neq q$ and $p, q \in \{1, \ldots, k\}$, $v(A^1, \ldots, A^n)_{ij} = 0$ for all profiles $(A^1, \ldots, A^n)$. Fix an $i \in \bar{A}_p$ and $j \in \bar{A}_q$ where $p \neq q$ and $p, q \in \{1, \ldots, k\}$, and a profile $(A^1, \ldots, A^n)$. By definition, $\bar{A}_{ij} = 0$ [3]. Assume for contradiction $v(A^1, \ldots, A^n)_{ij} = 1$.

Construct another profile $(B^1, \ldots, B^n)$ such that $B^h_{ij} = A^h_{ij}$ and $B^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. By binary independence, $v(B^1, \ldots, B^n)_{ij} = 1$. Let $S = \{h \in N : B^h_{ij} = 0\}$. If $S \neq \emptyset$, then choose $h \in S$, and consider C^h such that $C^h_{ij} = 1$ and $C^h_{st} = B^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. By responsiveness, $v(C^h, B^{-h})_{ij} = v(B^h, B^{-h})_{ij} = 1$. Continuing in this manner by choosing a new agent from S in every iteration, we will get to a profile $(C^1, \ldots, C^n)$, where $C^h_{ij} = 1$ and $C^h_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$, and $v(C^1, \ldots, C^n)_{ij} = 1$. Now, consider the profile $(D^1, \ldots, D^n)$, where D^h is the complete partition for every $h \in N$. By binary independence $v(D^1, \ldots, D^n)_{ij} = 1$. But, by definition, $v(D^1, \ldots, D^n) = \bar{A}$, and $\bar{A}_{ij} = 0$. This is a contradiction.

Finally, we show that each v_p for $p \in \{1, \ldots, k\}$ satisfies Pareto$^+$. Assume for contradiction some v_p does not satisfy Pareto$^+$. Then, by the definition of v_p, for some preference profile $(A^1, \ldots, A^n)$ such that $A^h_{ij} = 1$ for all $h \in N$ for some $i, j \in \bar{A}_p$, we have $v(A^1, \ldots, A^n)_{ij} = 0$. Now, consider the preference profile $(D^1, \ldots, D^n)$, where D^h is the complete partition for every $h \in N$. By binary independence, $v(D^1, \ldots, D^n)_{ij} = 0$ (since v is strategy-proof, it satisfies binary independence by Proposition 3). This is a contradiction by the definition of $\bar{A}$. ∎

Proposition 6 says that every strategy-proof aggregator can be decomposed into aggregators, which satisfy Pareto$^+$. This is non-trivial since we did not impose Pareto$^+$ for the main aggregator. As a consequence of this, we can say precisely how a strategy-proof aggregator must look like.

[3] We let $\bar{A}_p$ (single superscript) to denote a bundle in $\bar{A}$, but $\bar{A}_{ij}$ (double superscript) to denote the value of the i-th row and j-th column entry corresponding to the $0-1$ matrix of partition $\bar{A}$. We apologize for this notational clumsiness.

We are now ready to define a general family of aggregators, which includes the meet* family.

Definition 12 *An aggregator v is a* **decomposed meet* aggregator** *if there exists a partition $\bar{A}$ with bundles $\bar{A}_1, \ldots, \bar{A}_k$, and for every $\bar{A}_p$ with $p \in \{1, \ldots, k\}$, we have*

- *a set of oligarch $S_p \subseteq N$ if $|\bar{A}_p| > 2$,*
- *and a set of non-nested oligarchies $\mathbb{S}_p$ if $|\bar{A}_p| = 2$,*

such that

- *if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$ with $|\bar{A}_p| > 2$, then $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S_p$ (meet*),*
- *if $i, j \in \bar{A}_p$ for some $p \in \{1, \ldots, k\}$ with $|\bar{A}_p| = 2$, then $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$ for some $S \in \mathbb{S}_p$,*
- *and, $v(A^1, \ldots, A^n)_{ij} = 0$, if $i \in \bar{A}_p$ and $j \in \bar{A}_q$, where $p \neq q$.*

An SCF F is a decomposed meet* SCF if there exists a decomposed meet* aggregator v such that $F = F^v$.

Different choices of $\bar{A}$ result in interesting aggregators. If we choose, $\bar{A}$ to be the complete partition, then the resulting decomposed meet* aggregator is a meet* aggregator if $|M| > 3$ and meet*-join aggregator if $|M| = 2$. Choosing any $\bar{A}$, and choosing $\emptyset$ as oligarchs in each bundle of $\bar{A}$ gives $\bar{A}$ as the output in every profile. Intuitively, $\bar{A}$ reflects the bias of the mechanism designer towards a particular partition. Such bias may be inherent in some applications, e.g., in the political district formation example discussed earlier, the state may have an inherent bias to put two far-off geographical districts separate.

We now state the main result of the paper.

Theorem 2 *A social choice function is strategy-proof and tops-only if and only if it is a decomposed meet* social choice function.*

Proof: Let F be a strategy-proof and tops-only social choice function. Define the aggregator v^F as follows. For every $(A^1, \ldots, A^n) \in \mathbb{M}^n$, let $v^F(A^1, \ldots, A^n) = F(\succ_1, \ldots, \succ_n)$ such that $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$ and $\succ_h (1) = A^h$ for all $h \in N$. Since F is tops-only, v^F is well-defined. Further, v^F is strategy-proof. By Proposition 6, v^F can be decomposed into $v_1, \ldots, v_k$ via partition $\bar{A}$ with bundles $\bar{A}_1, \ldots, \bar{A}_k$. For each $p \in \{1, \ldots, k\}$, v_p is strategy-proof and satisfies Pareto$^+$. Using Theorem 1 and Proposition 5, we conclude that v^F is a decomposed meet* aggregator. Hence, F is a decomposed meet* social choice function.

Suppose F is a decomposed meet* social choice function. By definition, F only uses information in the top-ranked partition of each agent. So, it is tops-only. Let v^F be the decomposed meet* aggregator induced by F. Consider the partition $\bar{A}$ corresponding to this decomposed meet* aggregator, let $(\bar{A}_1, \ldots, \bar{A}_k)$ be the bundles in this partition. Denote the restriction of v^F to $\bar{A}_p$ for every $p \in \{1, \ldots, k\}$ as v_p. By definition, each v_p is well-defined. Further, each v_p is either a meet* aggregator or a meet*-join aggregator. By Theorem 1 and Proposition 5, each v_p is strategy-proof. Strategy-proofness of each v_p implies strategy-proofness of v^F (by definition of v^F). ■

Another way to state Theorem 2 is that an aggregator is strategy-proof if and only if it is a decomposed meet* aggregator. Note that the only decomposed meet* aggregators which satisfy Pareto$^+$ are meet* aggregators when $|M| \geq 3$ and meet*-join aggregators when $|M| = 2$.

The tops-onlyness property in Theorem 2 is essential for the characterization. For instance, consider the aggregator in Example 2. This is an aggregator which is not tops-only, but strategy-proof. Hence, it is not a decomposed meet* aggregator.

5 Unanimity and Tops-Only Property

The tops-only property says that the only relevant information in the preference orderings of agents are their tops. It is a useful tool, and often the biggest obstacle, in establishing characterization results in social choice theory (Chatterji and Sen, 2011; Weymark, 2008). Tops-only property is critical in our characterizations of Theorems 1 and 2. At the same time, it may not be entirely appealing. However, we have already seen that tops-only property is not implied by Pareto$^+$. In this section, we intend to introduce two new appealing properties for social choice functions, and show their connection to tops-onlyness.

The first property is Pareto$^-$. It is an analogue of Pareto$^+$ property. It can be found, for example, in Fishburn and Rubinstein (1986).

Definition 13 *An aggregator v satisfies* **Pareto$^-$** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^h_{ij} = 0$ for all $h \in N$, we have $v(A^1, \ldots, A^n)_{ij} = 0$.*

A social choice function F satisfies **Pareto$^-$** *if for every $i, j \in M$ and for every preference profile $\succ \in \mathcal{D}^n$ with $(\succ_h (1))_{ij} = 0$ for all $h \in N$, we have $F(\succ)_{ij} = 0$.*

Unlike Pareto$^+$, Pareto$^-$ has different consequences when applied to strategy-proof and tops-only SCFs. For example, using Theorem 2, we can characterize the class of SCFs which are tops-only, strategy-proof, and satisfy Pareto$^-$ - these are decomposed meet SCFs (i.e., no bundle in the partition $\bar{A}$ will have an empty set as an oligarch). Notice that this class is not *symmetric or dual* to the class of SCFs we get in Theorem 1, where we had imposed Pareto$^+$ in addition to strategy-proofness and tops-onlyness.

We now define unanimity. It says that whenever agents have the same top-ranked partition, the social choice function must choose that partition. It is an appealing property, and used extensively in social choice theory.

Definition 14 *An aggregator v satisfies* **unanimity** *if for every $i, j \in M$ and for every profile $(A^1, \ldots, A^n)$ with $A^1 = \ldots = A^n = A$, we have $v(A^1, \ldots, A^n) = A$.*

A social choice function F satisfies **unanimity** *if for every profile $(\succ_1, \ldots, \succ_n) \in \mathcal{D}^n$ such that $\succ_1 (1) = \ldots = \succ_n (1) = A$, we have $F(\succ_1, \ldots, \succ_n) = A$.*

The main result of this section is the following.

Theorem 3 *Suppose $|M| \geq 3$. Then, the following statements are equivalent.*

1. *A social choice function is a meet social choice function.*

2. *A social choice function is strategy-proof and satisfies unanimity.*

3. *A social choice function is strategy-proof and satisfies Pareto$^+$ and Pareto$^-$.*

As discussed earlier, when $|M| \geq 3$, Fishburn and Rubinstein (1986) characterized meet aggregators using binary independence, Pareto$^+$, and Pareto$^-$. Similarly, Mirkin (1975) characterized meet aggregators using binary independence and unanimity. Theorem 3 is the strategic counterpart of these results.

The main hurdle in proving Theorem 3 is to show that unanimity implies the tops-only property for strategy-proof social choice function. To establish this property for our domain, we use a general result in Chatterji and Sen (2011). Chatterji and Sen (2011) identify a general sufficient condition on domains such that every strategy-proof function which satisfies unanimity in that domain is tops-only. We show that this sufficient condition is satisfied in our domain.

To be able to use their result, we need to first explore some structure of our domain. First, we define the notion of *betweenness*. For any pair of distinct partitions A and B, let

$$\beta(A, B) = \{C : \ \forall \ \succ_h \in \mathcal{D} \text{ with } \succ_h (1) = A, C \notin \{A, B\}, C \succ_h B\}.$$

So, $\beta(A, B)$ contains all partitions which will lie between A and B, whenever A is the top ranked partition. An alternate way to define $\beta(A, B)$ is the following. To remind, $\mathcal{M} := \{\{i, j\} : i, j \in M, i \neq j\}$, i.e., all pairs of objects such that the objects are distinct. For any pair of partitions, let $L(A, B) = \{\{i, j\} \in \mathcal{M} : A_{ij} = B_{ij}\}$.

Lemma 1 *Consider a pair of partitions A, B. A partition $C \in \beta(A, B)$ if and only if for every $\{i, j\} \in \mathcal{M}$, $A_{ij} = B_{ij}$ implies $A_{ij} = C_{ij}$ (i.e., $L(A, B) \subseteq L(A, C)$).*

Proof: This follows from the definition of $\beta(A, B)$. ■

The following lemma says the partitions which lie between A and B when A is the top, are also the partitions which lie between A and B when B is the top.

Lemma 2 *For any pair of partitions A and B, $\beta(A, B) = \beta(B, A)$.*

Proof: Suppose $C \in \beta(A, B)$. This means for every $\{i, j\} \in \mathcal{M}$, if $B_{ij} = A_{ij}$ then $C_{ij} = A_{ij}$. But this also implies that if $B_{ij} = A_{ij}$ then $C_{ij} = B_{ij}$. Hence, $C \in \beta(B, A)$. A symmetric argument establishes $C \in \beta(B, A)$ implies $C \in \beta(A, B)$. ■

Given a preference ordering $\succ_h$ and a partition B, let

$$\alpha(B, \succ_h) = \{A \in \mathbb{M} : A \succ_h B\}.$$

So, $\alpha(B, \succ_h)$ are all the partitions which are above B in preference ordering $\succ_h$. The following lemma says that if we take a pair of partitions A and B, we can find a preference ordering $\succ_h$ with A being the top-ranked partition such that the partitions between A and B in $\succ_h$ are exactly the partitions in $\beta(A, B)$.

Lemma 3 **(Squeezing)** *For every pair of partitions A and B, there exists a preference ordering $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A$ and $\alpha(B, \succ_h) = \{A\} \cup \beta(A, B)$.*

Proof: Fix a pair of partitions A and B. Define $T(A) = \{\succ_h \in \mathcal{D} : \succ_h (1) = A\}$. For every $\succ_h \in T(A)$, define $S(\succ_h) = \alpha(B, \succ_h) \setminus (\{A\} \cup \beta(A, B))$. Choose $\succ'_h \in T(A)$ such that $|S(\succ'_h)| \leq |S(\succ_h)|$ for all $\succ_h \in T(A)$. If $|S(\succ'_h)| = 0$, we are done. Assume for contradiction $|S(\succ'_h)| = r > 0$.

Let C be a partition above B in $\succ'_h$ such that $C \notin \beta(A, B)$, i.e., $C \in \alpha(B, \succ'_h) \setminus (\{A\} \cup \beta(A, B))$, and for all $D \neq C$ and $D \in \alpha(B, \succ'_h) \setminus (\{A\} \cup \beta(A, B))$ we have $D \succ'_h C$. In other words, C is the lowest ranked partition above B which does not lie in $\beta(A, B)$. Such a C exists since $|S| = r > 0$.

We construct another profile $\succ''_h$ by moving C just below B and keeping all the other partitions at the same position. We claim that $\succ''_h \in \mathcal{D}$. Assume for contradiction that $\succ''_h \notin \mathcal{D}$. Then, there must exist two partitions $X, Y \in \mathbb{M} \setminus \{A\}$ such that $X \succ''_h Y$ but $L(A, X) \subseteq L(A, Y)$. Since $\succ'_h \in \mathcal{D}$, we must have $Y = C$ and $X \in \alpha(B, \succ'_h)$ but $X \notin \alpha(C, \succ'_h)$, i.e., X must be a partition between B and C in $\succ'_h$, and Y must be C. Then, by definition of C, $X \in \beta(A, B)$. Since $C \notin \beta(A, B)$, there is a preference ordering $\hat{\succ}_h \in \mathcal{D}$ with $\hat{\succ}_h(1) = A$ and $X \hat{\succ}_h C$. This is a contradiction to the fact $L(A, X) \subseteq L(A, C)$.

Hence, there exists a preference ordering $\succ''_h \in T(A)$ such that $|S(\succ''_h)| = r - 1$. This is a contradiction. ■

We are now ready to state the tops-only result. Before stating the result, we state one notation, which we use in the proof. For every $\succ_h \in \mathcal{D}$ and for every $A \in \mathbb{M}$, define $\omega(A, \succ_h) = \{B \in \mathbb{M} : A \succ_h B\}$, i.e., all the partitions in $\succ_h$ that are worse than A.

Proposition **7** *If a social choice function is strategy-proof and satisfies unanimity, then it is tops-only.*

Proof: We are going to use a result due to Chatterji and Sen (2011). They show that the following two conditions are sufficient for tops-onlyness if a social choice function is strategy-proof and satisfies unanimity.

- A domain $\mathcal{Z} \subseteq \mathcal{P}$ is **minimally rich** if for every partition $A \in \mathbb{M}$, there exists $\succ_h \in \mathcal{Z}$ such that $\succ_h (1) = A$.

- Let $A \in \mathbb{M}$ and $\succ_h \in \mathcal{P}$ such that $A \neq \succ_h (1) = B$. We say A is **satisfactory** in domain $\mathcal{Z} \subseteq \mathcal{P}$ for $\succ_h \in \mathcal{Z}$ if for all $C \in \{B\} \cup \beta(A, B)$ there exists a $\succ'_h \in \mathcal{Z}$ such that $\succ'_h (1) = A$ and $C \succ'_h D$ for all $D \in \omega(A, \succ_h)$. In other words, A is satisfactory for $\succ_h$ if for every C in $\{B\} \cup \beta(A, B)$, there is a preference ordering where A is the top and C is better than all the alternatives worse than A in $\succ_h$.

  A domain $\mathcal{Z} \subseteq \mathcal{P}$ satisfies **Property** T^* if for all $\succ_h \in \mathcal{Z}$ and $A \in \mathbb{M} \setminus \{\succ_h (1)\}$, A is satisfactory for $\succ_h$.

The intermediate domain $\mathcal{D}$ is clearly minimally rich. We show that our intermediate domain $\mathcal{D}$ satisfies Property T^*. Pick an arbitrary $\succ_h \in \mathcal{D}$, and let $\succ_h (1) = B$. Pick $A \in \mathbb{M} \setminus \{B\}$. We need to show that A is satisfactory for $\succ_h$. Pick a $C \in \{B\} \cup \beta(A, B)$. By the squeezing lemma (Lemma 3), and using the fact $\beta(A, B) = \beta(B, A)$ (Lemma 2), there is a preference ordering $\succ'_h$ such that $\succ'_h (1) = A$ and $\alpha(B, \succ'_h) = \{A\} \cup \beta(B, A)$. Clearly, for all $D \in \omega(A, \succ_h)$, we have $B \succ'_h D$. Since $C \in \{B\} \cup \beta(B, A)$, we get that $C \succ'_h D$. Hence, the intermediate domain $\mathcal{D}$ satisfies Property T^*.

Finally, Chatterji and Sen (2011) show that if F is strategy-proof and $\mathcal{D}$ is minimally rich and satisfies Property T^*, then F is tops-only [4]. Hence, F is tops-only. ■

We can now state the proof of Theorem 3.

Proof of Theorem 3

Proof: (1) $\Rightarrow$ (2), (3): Clearly, any meet social choice function is strategy-proof (Theorem 1), and satisfies unanimity, Pareto$^+$, and Pareto$^-$.

(2) $\Rightarrow$ (3): Let F be a strategy-proof social choice function satisfying unanimity. By Proposition 7, F is tops-only. Hence, the aggregator v^F is well-defined. Further, v^F is strategy-proof, and satisfies unanimity. By Proposition 3, v^F satisfies binary independence. Unanimity and binary independence implies that v^F must also satisfy Pareto$^+$ and Pareto$^-$.

(3) $\Rightarrow$ (1): Let F be a strategy-proof social choice function satisfying Pareto$^+$ and Pareto$^-$. Then, it must satisfy unanimity. By Proposition 7, F is tops-only. Hence, the aggregator v^F is well-defined. Further, v^F is strategy-proof, and satisfies Pareto$^+$ and Pareto$^-$. By Theorem 1, v^F must be a meet aggregator, and F must be a meet social choice function. ■

There are other ways to prove Theorem 3 once we establish that tops-only property holds.

[4] Chatterji and Sen (2011) consider a general model, a la Gibbard (1973) and Satterthwaite (1975), and provide sufficient conditions on domains for a strategy-proof social choice functions to be tops-only.

For instance, Mirkin (1975) has shown that for $|M| \geq 3$, an aggregator satisfying unanimity and binary independence must be a meet aggregator. This establishes the equivalence between (1) and (2). Similarly, Fishburn and Rubinstein (1986) have shown that for $|M| \geq 3$, an aggregator satisfying Pareto$^+$, Pareto$^-$, and binary independence must be a meet aggregator. Since Pareto$^+$ and Pareto$^-$ imply unanimity, this establishes the equivalence between (1) and (3).

The analogue of Theorem 3 for $|M| = 2$ can be derived too using Theorem 2. Note that when $|M| = 2$, every social choice function is tops-only. Using unanimity and Theorem 2, we conclude that every strategy-proof social choice function satisfying unanimity must be a *meet-join* social choice function, where a meet-join SCF is a meet*-join SCF where the empty set is not part of the set of oligarchies.

6 Pareto Efficiency and Dictatorship

It is well known that unanimity is equivalent to a social choice function being onto. Unanimity is also equivalent to the following definition Pareto efficiency in many domains, including the unrestricted domain in Gibbard (1973) and Satterthwaite (1975) and the single-peaked domain in Moulin (1980).

Definition **15** *A social choice function F is* **Pareto efficient** *if for every profile $\succ \in \mathcal{D}^n$, there exists no partition $A \in \mathbb{M}$ such that $A \succ_h F(\succ)$ for all $h \in N$.*

A meet SCF need not be Pareto efficient as the following example illustrates.

Example **3** *Let $M = \{a, b, c\}$ and $N = \{1, 2\}$. We refer to the partition where all the objects are put together as the complete partition and the partition where all the objects are kept separately as the empty partition. Consider the meet SCF where the set of oligarchs are $\{1, 2\}$. Consider a profile $(\succ_1, \succ_2)$ such that*

- *$\succ_1 (1)$ is the partition with bundles $\{a, b\}$ and $\{c\}$,*
- *and $\succ_2 (1)$ is the partition with bundles $\{a, c\}$ and $\{b\}$.*

By definition, $F(\succ)$ is the empty partition. Let A be the complete partition. Note that A and $\succ_1 (1)$ agree that a and b should be together but A and $F(\succ)$ do not agree on that. Similarly, A and $\succ_2 (1)$ agree that a and c should be together but A and $F(\succ)$ do not agree on that. Hence, we can assume, without loss of generality, that $\succ_1$ is such that $A \succ_1 F(\succ)$ and $\succ_2$ is such that $A \succ_2 F(\succ)$. So, F is not Pareto efficient.

The intuition of Example 3 carries over more generally.

Theorem **4** *Suppose $|M| \geq 3$. A social choice function is strategy-proof and Pareto efficient if and only if it is a dictatorship.*

Proof: Clearly, a dictatorship is Pareto efficient and strategy-proof. Suppose F is a strategy-proof and Pareto efficient SCF. Since Pareto efficiency implies unanimity, by Theorem 3, F is a meet social choice function, and v^F is a meet aggregator. Let $S \subseteq N$ be the set of oligarchs of v^F. We complete the proof by showing $|S| = 1$. Suppose $|S| > 1$. Fix three objects $i, j, k \in M$. Consider the following partitions for each agent. For some $h \in S$, let the top-ranked partition of agent h be A^h, and $A^h_{ik} = 1$ and $A^h_{st} = 0$ for all $\{s, t\} \neq \{i, k\}$. For every $h' \in S \setminus \{h\}$, let the top-ranked partition of agent h' be $A^{h'}$, and $A^{h'}_{ij} = 1$ and $A^{h'}_{st} = 0$ for all $\{s, t\} \neq \{i, j\}$. For every $h'' \in N \setminus S$, let the top-ranked partition of agent h'' be $A^{h''} = A$, where A is the complete partition.

Now, by definition $v^F(A^1, \ldots, A^n) = B$, where B is the empty partition. Now, note that $A^h_{ik} = A_{ik} = 1$ but $A^h_{ik} \neq B_{ik}$. Also, $A^{h'}_{ij} = A_{ij}$ but $A^{h'}_{ij} \neq B_{ij}$ for all $h' \in S \setminus \{h\}$. Hence, there exists a preference ordering $\succ_h \in \mathcal{D}$ such that $\succ_h (1) = A^h$ and $A \succ_h B$. Also, for every $h' \in S \setminus \{h\}$, there exists a preference ordering $\succ'_{h'} \in \mathcal{D}$ such that $\succ'_{h'} (1) = A^{h'}$ and $A \succ_{h'} B$. By definition, for every $h'' \in N \setminus S$, $A \succ_{h''} B$ for any preference ordering $\succ_{h''} \in \mathcal{D}$. This implies that F is not Pareto efficient, which is a contradiction. Hence, $|S| = 1$, and F is a dictatorship. ■

Theorem 4 shows that we are back to Gibbard-Satterthwaite type impossibility if we impose Pareto efficiency in addition to strategy-proofness. This is in sharp contrast to various possibilities we have seen in the presence of unanimity or tops-onlyness.

7 Conclusion

This paper adds a new restricted domain to the literature on strategic social choice theory initiated by the papers of Gibbard (1973) and Satterthwaite (1975). We provide characterizations of strategy-proof social choice functions (a) under tops-onlyness property, (b) under unanimity, and (c) under Pareto efficiency. Our model and the restricted domain we consider are natural, and has some plausible applications (as discussed in Section 1).

It will be interesting to consider even more restriction of preferences in our model. For instance, one natural notion of comparing two partitions with respect to a reference partition is using "distance" between them. If A is the top-ranked partition in a preference ordering, then a partition B may be preferred over partition C if and only if the distance between A and B is less than the distance between A and C. One can verify that this is a smaller domain than our intermediate domain. It will be interesting to find a characterization of strategy-proof social choice functions in this domain.

Rubinstein and Fishburn (1986) consider an abstract model of algebraic aggregation. The current paper, specially Theorem 3, gives a strategic foundation to their result on partitions (Fishburn and Rubinstein, 1986). It will be interesting to extend this result to the abstract model of algebraic aggregation.

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Appendix: Missing Proofs

Proof of Proposition 1

Proof: Suppose $\succ_h \in \mathcal{D}$ with $\succ_h (1) = A$. Pick a pair of partitions B and C such that B dominates C at A. To show that $B \succ_h C$, it is sufficient to show that for every $\{i,j\} \in \mathcal{M}$, $C_{ij} = A_{ij}$ implies $B_{ij} = A_{ij}$. Assume for contradiction that for some $\{i,j\} \in \mathcal{M}$, we have $C_{ij} = A_{ij}$ but $B_{ij} \neq A_{ij}$. In that case, choose a utility function u^h consistent with A such that $u^h_{ij}(A_{ij})$ is a very large positive number, and $u^h_{kl}(x)$, where $\{k,l\} \neq \{i,j\}$ and $x \in \{0,1\}$, is either 1 or 0. Note that such a u^h consistent with A exists. Further, $U^h(C) > U^h(B)$ since $u^h_{ij}(C_{ij})$ is a very large positive number. But since B dominates C at A, we must have $U^h(B) > U^h(C)$. This is a contradiction. Hence, $B \succ_h C$.

For the converse, for all partitions B and C such that B dominates C at $\succ_h (1)$, we have $B \succ_h C$. Let $\succ_h (1) = A$. To show $\succ_h$ belongs to the intermediate domain, take any B and C with

$$\{\{i,j\} \in \mathcal{M} : C_{ij} = A_{ij}\} \subsetneq \{\{i,j\} \in \mathcal{M} : B_{ij} = A_{ij}\}. \tag{1}$$

Take any utility function u^h consistent with A. Now,

$$\begin{aligned} U^h(B) &= \sum_{\{i,j\} \in \mathcal{M}} u^h_{ij}(B_{ij}) \\ &= \sum_{\{i,j\} \in \mathcal{M}: B_{ij} = C_{ij}} u^h_{ij}(B_{ij}) + \sum_{\{i,j\} \in \mathcal{M}: B_{ij} \neq C_{ij}, B_{ij} = A_{ij}} u^h_{ij}(B_{ij}) + \sum_{\{i,j\} \in \mathcal{M}: B_{ij} \neq C_{ij}, C_{ij} = A_{ij}} u^h_{ij}(B_{ij}). \end{aligned}$$

By the relation in 1, $\{\{i,j\} \in \mathcal{M} : B_{ij} \neq C_{ij}, C_{ij} = A_{ij}\}$ is empty. Hence, we can write

$$\begin{aligned} U^h(B) &= \sum_{\{i,j\} \in \mathcal{M}: B_{ij} = C_{ij}} u^h_{ij}(B_{ij}) + \sum_{\{i,j\} \in \mathcal{M}: B_{ij} \neq C_{ij}, B_{ij} = A_{ij}} u^h_{ij}(B_{ij}) \\ &> \sum_{\{i,j\} \in \mathcal{M}: B_{ij} = C_{ij}} u^h_{ij}(C_{ij}) + \sum_{\{i,j\} \in \mathcal{M}: B_{ij} \neq C_{ij}, B_{ij} = A_{ij}} u^h_{ij}(C_{ij}) \\ &= U^h(C), \end{aligned}$$

where the strict inequality followed from the fact u^h is consistent with A and the relation in 1. This shows that $U^h(B) > U^h(C)$, and hence, B dominates C at $\succ_h (1)$. This implies that $B \succ_h C$. ■

Proof of Proposition 5

Proof: With $|M| = 2$, every social choice function is tops-only. So, without loss of generality, we focus on aggregators instead of social choice functions. Consider a meet*-join aggregator v. Consider agent $h \in N$, and fix the profile of other agents at A^{-h}. Let A^h be a partition such that $A^h_{ij} \neq v(A^h, A^{-h})_{ij}$. Consider another partition B^h. If agent h is not an oligarch (i.e., $h \notin S_p$ for some $S_p \in \mathbb{S}$), then $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij}$. If h is an oligarch, then there are two cases to consider.

- $A^h_{ij} = 0$ and $v(A^h, A^{-h})_{ij} = 1$ implies that there is some oligarchy $S_p \in \mathbb{S}$ such that $h \notin S_p$, and $A^{h'}_{ij} = 1$ for all $h' \in S_p$. In that case, $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} = 1$.

- $A^h_{ij} = 1$ and $v(A^h, A^{-h})_{ij} = 0$ implies that in every oligarch $S_p \in \mathbb{S}$ there is some agent $h' \in S_p \setminus \{h\}$ such that $A^{h'}_{ij} = 0$. In that case, $v(A^h, A^{-h})_{ij} = v(B^h, A^{-h})_{ij} = 0$.

This shows that v is responsive. Hence, it is strategy-proof by Proposition 2.

Suppose v is strategy-proof and satisfies Pareto$^+$. Then, call a set of agents $S \subseteq N$ decisive if for every $(A^1, \ldots, A^n)$, $v(A^1, \ldots, A^n)_{ij} = 1$ if and only if $A^h_{ij} = 1$ for all $h \in S$. A decisive set exists because v satisfies Pareto$^+$. Also, for any $S \subsetneq T \subseteq N$, if S is decisive, then T is also decisive. This follows from responsiveness of v (since v is strategy-proof). Let $\mathbb{S} = \{S_1, \ldots, S_k\}$ such that each $S_p \in \mathbb{S}$ is *minimally* decisive. Since each $S_p \in \mathbb{S}$ is minimally decisive, for any $S_q, S_r \in \mathbb{S}$, S_q cannot be a subset of S_r. Hence, by definition $\mathbb{S}$ is a set of oligarchs, and v is a meet*-join aggregator.

Hence, a social choice function is a meet*-join aggregator if and only if it is strategy-proof and satisfies Pareto$^+$. ■