General Equilibrium, Wariness and Bubbles*

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Abstract: We say that a consumer is wary if she overlooks gains but not losses in remote sets of dates or states. We formulate this by requiring preferences to be upper but not lower Mackey semi-continuous and Bewley's result on existence of Arrow-Debreu equilibrium whose prices are not necessarily countably additive holds. We relate wariness to some concepts studied in decision theory like lack of myopia and ambiguity aversion. Wary infinite lived agents are not impatient, have optimality conditions, in the form of weaker transversality conditions, that allow them to be creditors at infinity and bubbles occur for positive net supply assets completing the markets. In a two date economy, with infinite states, wary agents are not myopic and bubbles occur, as asset prices do not have to equal the series of returns weighted by state prices. A large class of efficient allocations can only be implemented with asset bubbles. Pessimistic attitudes lead agents to overvalue assets or durable goods with hedging properties, like gold.

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1 Introduction

Occurrence and bursting of bubbles in the prices of assets in positive net supply are important phenomena, but the theory of general equilibrium has not managed to accommodate these events satisfactorily. Contrary to the well known examples of bubbles in long lived assets traded by overlapping generations, there was no robust case when both agents and assets are either infinite lived or short lived. In this paper, we show that wary consumers have precautionary demands for assets and the resulting equilibrium prices may include a bubble, even under positive net supply. We say that a consumer is wary when she neglects gains but not losses in consumption that happen in a remote set of dates or states. This notion can be formulated by modeling consumption bundles as bounded sequences (on a countable infinite set of dates or states) and assuming that preferences are upper but not lower Mackey semi-continuous.

Wariness encompasses some important concepts discussed in decision theory. It is related to lack of impatience, as infinite lived wary agents are only semi-impatient, in the sense of overlooking what they earn but not what they lose at far away dates. Similarly, in a finite horizon economy with a countable infinite set of states, wary consumers are only semi-myopic, tending to ignore increases in consumption but not decreases in remote events.

Examples of wary preferences can be obtained in a context of ambiguity aversion, for instance, when agents maximize the minimum expected utility over probabilities that dominate some convex capacity, as in Schmeidler [32] [1]. In this context, wariness at some consumption plan is equivalent to the discontinuity of the capacity at the full set (by a result in Epstein and Wang [14]). This is the case for the ϵ-contamination capacity, which generates a utility that deviates from separable utility by adding a term dealing specifically with the infimum of the utilities (at all states or dates), as in our main examples. The discontinuity at the full set is as if some state (or date) is missing and the asset bubble can be intuitively thought of as the asset's payoff at that missing node.

A multiple discount factors model was suggested by Gilboa [15] and explored by Marinacci [26] to characterize complete patience. This model can portrait a con-

[1] Similar maxmin problems have been studied in the recent macroeconomics literature on robust decision rules under model misspecification (see Hansen and Sargent [18, 19]).

sumer being unsure about his own preferences in a distant future. The multiplicity of future preferences has motivated some authors to model the incompleteness of preferences (see Dubra, Maccheroni and Ok [12]), but our approach allows us to get interesting insights while preserving completeness.

Mackey discontinuous preferences have been studied before in the general equilibrium literature and, initially, in the context of a contingent claims economy with a single budget constraint. Bewley's [5] stronger result, on the existence of Arrow-Debreu (AD for short) equilibrium with countably additive prices, does not hold. However, the weaker version states that equilibrium prices are bounded finitely additive set functions, which can be decomposed into a countably additive functional and a pure charge[2]. We identify conditions that prevent AD prices from being countably additive and give examples that are related to earlier examples by Araujo [1], Barrios [4], Sawyer [30] and Werner [36])[3]. We believe that Gilles and LeRoy [17] were the first to have suggested the relevance of non-countably additive AD prices for the study of bubbles. These authors refered to such prices as bubbles, although it was not clear what was the precise relation between these AD prices and the bubbles of financial assets.

We implement AD equilibria with non-countably additive prices in two different contexts: an infinite horizon deterministic or an infinite states two date economies. In both, the AD budget equation is replaced by countably many budget constraints and the asset used to transfer wealth has a bubble.

In the infinite horizon case, the transversality condition, necessary for individual optimality, becomes more flexible. It may be optimal to be a creditor at infinity if that covers some desired asymptotic excess of consumption over resources. Transversality conditions are now compatible with bubbles in the prices of long lived assets in positive net supply, even for deflators yielding finite present value of wealth. Implementation is achieved by imposing (as it is usual in the literature) a portfolio constraint that mimics the transversality condition (or a related constraint bounding debt date by date, but in a more flexible way than as been done before). This constraint prevents, in the limit, a non-financed excess of

[2]See Appendix A.2 on the definition of pure charge and the decomposition result.

[3]Radner [28] and Kurz and Majumdar [24] had already discussed and illustrated this in the context of linear production problems.

consumption over endowments (that is, precludes asymptotic bankruptcy).

Other types of portfolio constraints could either fail to implement or be too restrictive, introducing a non-necessary friction. Huang and Werner [21] gave an example of an AD equilibrium with a price pure charge that could be implemented with an asset bubble, under a portfolio constraint unrelated to the transversality condition and requiring the asset position to be constant after some date. Kocherlakota [23] introduced constraints independently of what agents' preferences are, so that the lower bound on asset positions, exogenous but no longer non-positive, prevents agents from selling the bubble on the initial holdings. If the standard borrowing constraints, which guaranteed existence of equilibrium under impatience, were used instead, positive net supply assets would be free of bubbles, in complete markets, unless the present value of wealth is infinite (as in the example by Bewley [6]), by Theorem 3.1 in Santos and Woodford [29] (see also Magill and Quinzii [25])[4]. Moreover, the presence of wary consumers could prevent efficient allocations from being implemented under standard constraints.

Our analysis of infinite horizon economies led us to important results. We show that asset prices must be found by using as deflator the countably additive part of the AD price. This is the only possible choice (under interiority and some differentiability conditions), since the deflator ratios must be the marginal rates of intertemporal substitution, which coincide with the ratios of elements in the AD countably additive component. Thus, the present value of bounded endowments is finite. We show also that there are AD equilibria whose sequential implementation using a positive net supply asset, paying dividends, requires a price bubble, as long as the constraint satisfies some minimal requirement[5]. We give also an example where one agent is impatient but the other one is not. The optimal consumption of the former is not uniformly bounded away from zero (as she sells gradually the endowments to the latter, who places a higher value on distant consumption) and this allows for AD prices with pure charges, implementable with asset price bubbles. Bubbles in infinite horizon economies are addressed in Section 4.

[4]Even under incomplete markets such bubbles could only occur, for finite present values of wealth, if uniform impatience failed (and Ponzi schemes were avoided in another manner, say by imposing no-short sales).

[5]When the constraint implies the inclusion of the sequential choice set in the AD budget set or allows for the admissibility of a scalar multiple of the equilibrium portfolio plan, for a scalar close enough to one (see Theorems 1 and 2, respectively, in Subsection 4.3).

Bubbles in short-lived assets have not been treated successfully before. In the literature, the fundamental value has been defined as the series of deflated returns on an infinite set of dates and the excess of price over it was the limiting deflated price, as time went to infinity, which would be zero in the case of a short-lived asset. We will now define a bubble of a short-lived asset as the difference between the price and the infinite series of returns on states of nature, weighted by state prices. When consumers are wary, the non-arbitrage functional may fail to be countably additive and prices of short-lived assets have two components. One (the fundamental value) is the series of returns weighted by the state prices computed using the countably additive part of this functional. The other (the bubble) is determined by the value that the pure charge part of this functional takes at the sequence of returns. This observation holds also under incomplete markets. There are not yet very general results on existence of equilibrium for infinite states two dates incomplete markets[6]. However, in the single good and countable states case, these difficulties are avoided by applying Bewley's [5] existence theorem to the subspace spanned by the assets. There is also an interesting two goods case that we address, when there is a durable good serving at the same time as a precautionary vehicle to hedge against undesirable persistent endowment shocks. This durable good, that might be though of as a commodity-money, has a speculative price (determined by the pure charge of the non-arbitrage functional, evaluated at the sequence of relative commodity prices), even without having to appeal to infinite horizon features. Bubbles in two date economies are addressed in Section 5.

2 Guiding Examples

Consider a deterministic infinite horizon economy with a single commodity and two agents (indexed by $i = 1, 2$) whose preferences depart from the standard time separable utilities as agents are particularly worried about the worst lifetime outcome of each consumption plan $x = (x_t)_t \in \ell^\infty_+$ (the set of bounded and nonnegative sequences, equipped with the supremum norm):

$$U^i(x) = \sum_{t\geq 1} \kappa^{t-1} u^i\left(x_t\right) + \beta \inf_{t\geq 1} u^i(x_t)$$

with $\kappa \in (0, 1)$, $\beta \in [0, \infty)$.

[6]Even under state-separability of utility, existence is only guaranteed under the assumption of positivity of endowments accrued by asset returns.

This precautionary behavior is equivalent to a maxmin attitude: the consumer looks for a plan that maximizes the worst discounted time separable utility, within a class of discount factors (not necessarily of the exponential form) having κ^{t-1} as lower bound at each date t. It is as if the consumer were unsure about the discount factor that should be used and, therefore, uses the severest one. More precisely (as shown in Appendix B.1),

$$\sum_{t\geq 1} \kappa^{t-1} u^i(x_t) + \beta \inf_{t\geq 1} u^i(x_t) = \inf_{(\vartheta_t)_{t\geq 1}\in\mathcal{A}} \sum_t \vartheta_t u^i(x_t)$$

where $\mathcal{A}$ is the set of all real sequences $(\vartheta_t)_{t\geq 1}$ such that $\sum_{t\geq 1}\vartheta_t = \frac{1}{1-\kappa}+\beta$ and $\vartheta_t \geq \kappa^{t-1}$ for every date. We will explore the connection with the ambiguity literature in Section 3.2.2.

Examples 1 and 2 compute AD equilibrium and address the sequential implementation using an infinite-lived asset paying $R_t = 1$, for $t > 1$, with initial holdings $z_0^i > 0$. Agents trade an amount $z_t \in I\!R$ of the asset at a price q_t. AD endowments W^i and sequential endowments ω^i are related by $\omega_t^i = W_t^i - R_t z_0^i$, where z_0^i must be such that $\omega^i \geq 0$[7]. We will see that the asset has a price bubble even though the present value of endowments is finite (for the non-arbitrage kernel deflator given by the countably additive part of the AD price). Sequential budget constraints are

$$x_t + q_t z_t \leq \omega_t^i + (q_t + R_t) z_{t-1} \quad \forall\, t \geq 1. \tag{1}$$

Portfolio constraints must be added to avoid Ponzi schemes.

Example 1: Bubble with Converging Net Trades

To start with a simple example we pick an economy where the AD equilibrium consists in exchanging the endowment sequences, that is, $x^i = W^j$ for $i \neq j$. This happens just for specific endowment and preferences parameters. Our second example illustrates a more general situation. Given $h \in (0, 3/4)$, let $u^1(r) = \sqrt{r}$, $W^1 = (\frac{t+8}{t} - h)_{t\geq 1}$ and $u^2(r) = \ln(r+h)$, $W^2 = \frac{1}{4}([\frac{t+8}{t}]^2)_{t\geq 1}$ and take κ to be the same for the two agents. We claim that the common supporting-price π at $x^i = W^j$, for $i = 1, 2$, is given by

[7]Notice that the endowments ω^i of the sequential economy accrued by the returns from asset initial holdings are the resources that each agent has when she does not trade in the asset market and, therefore, these resources should coincide with the AD endowments W^i.

$$\pi x = \sum_{t\geq 1} \frac{t}{t+8}\kappa^{t-1}\, x_t \,+\, \beta\,\mathrm{LIM}(x) \tag{2}$$

where $\mathrm{LIM} : \ell^\infty \to I\!R$ is a generalized limit (that is, any linear and norm continuous functional on ℓ^∞ taking at x a value on $[\liminf x, \limsup x]$ and, therefore, coinciding with $\lim x$ when this limit exists). This follows by Proposition 4 (in Subsection 3.3). If, in addition, $\pi(x^i) = \pi(W^i)$ holds, then (π, x^1, x^2) is an AD equilibrium. This is the case for some (κ, β) (for other combinations agents should not be exchanging endowments). In particular, for $\kappa = 1/2$ we can find the compatible $\beta > 0$ (see details in Appendix C.2).

We implement (x^1, x^2) sequentially. Take $(z_0^1, z_0^2) = (1/8, 1/8)$. Then, $\omega_1^i = W_1^i$ and, for $t > 1$, $\omega_t^i = W_t^i - R_t z_0^i = W_t^i - 1/8$. Given $(p_t)_{t\geq 1} = (\frac{t}{t+8}(\frac{1}{2})^{t-1})_{t\geq 1}$, define asset prices by $p_1 q_1 = \pi(R) = \sum_{t>1} p_t + \beta$ and $q_t = \frac{p_{t-1}}{p_t} q_{t-1} - 1$ for $t > 1$. The countably additive component $(p_t)_t$ of AD price π is the non-arbitrage deflator (and this is the only possible choice, up to a scalar multiple, as established in Proposition 7, in Section 4.2). Thus, β/p_1 is the asset *bubble*, at $t = 1$, as $\frac{1}{p_1}\sum_{t>1} p_t$ is the fundamental value.

The portfolio z^i that implements the above consumption allocation is given by $z_1^i = \frac{1}{q_1}(W_1^i - x_1^i) + \frac{1}{8}$ and, for $t > 1$, $z_t^i = z_{t-1}^i + \frac{1}{q_t}(\omega_t^i - x_t^i + z_{t-1}^i)$.

Now, we claim that $(q, (x^i, z^i)_i)$ is an equilibrium for the sequential economy under some portfolio constraints that will mimic the necessary *transversality condition*, which requires (see Remark 4 in Subsection 4.1.2):

$$\lim_t p_t q_t z_t^i = \beta \lim_t (x_t^i - \omega_t^i) \tag{3}$$

Contrary to the standard case, agents can have an asymptotic (present value) long position, as they try to avoid a bad outcome at distant dates. Following the usual approach, we impose on every portfolio a borrowing constraint that mimics (3):

$$\lim_t p_t q_t z_t \geq \beta \limsup_t (x(z)_t - \omega_t^i) \tag{4}$$

where $(x(z), z)$ are such that (1) holds with equality at each date t. To see that z^i is optimal under (1) and (4), it is enough to show that for any plan $(x(z), z)$ satisfying (1) and (4), $x(z)$ must belong to the AD budget set. Multiplying both sides of (1) by p_t and summing over dates gives $\sum_{t\geq 1} p_t(x(z)_t - \omega_t^i) = p_1 q_1 z_0^i - \lim_t p_t q_t z_t$ (as non-arbitrage equations hold at each t by construction). Hence

$\pi(x(z) - W^i) = \sum_t p_t(x(z)_t - \omega^i_t) + \beta\text{LIM}(x(z) - \omega^i) - z^i_0\pi(\mathbb{1}) = -\lim_t p_t q_t z_t + \beta\text{LIM}(x(z) - \omega^i) \leq 0$, by (4). Moreover, the equality holds at x^i and, therefore, x^i is optimal. Besides, commodity and asset markets clear, thus, $(q, (x^1, z^1), (x^2, z^2))$ is a sequential equilibrium with a bubble.

Example 2: Bubble with Non-converging net trades

Suppose $u^i(r) = \sqrt{r}$ and there are endowment shocks that agents try to get rid of. Let $W^1 = (\frac{t+1}{t} + \varphi_t)_{t\geq 1}$ and $W^2 = (\frac{t+1}{t} - \varphi_t)_{t\geq 1}$, where φ_t is 1/2 when t is even and $-1/4$ when t is odd. Let us find an AD equilibrium. Again by Proposition 4, given $a > 0$, the plan $x^a = a\,(W^1 + W^2) = 2\,a\,(\frac{t+1}{t})_{t\geq 1}$ is optimal under the budget constraint $\pi^a x \leq \pi^a W^i$ when prices π^a are given by

$$\pi^a x \equiv \sum_t \kappa^{t-1} u'(x^a_t)\, x_t + \beta u'(\inf x^a)\text{LIM}(x) = \frac{1}{\sqrt{2^3 a}} \left[\sum_t \kappa^{t-1}\sqrt{\frac{t}{t+1}}\, x_t + \beta\text{LIM}(x)\right]$$

and $\pi^a x^a = \pi^a W^i$. The homogeneity of the budget constraint allows us to rewrite prices π as

$$\pi x \equiv \sum_t \kappa^{t-1}\sqrt{\frac{t}{t+1}}\, x_t + \beta\text{LIM}(x)$$

and, picking out $a(i) = \pi(W^i)/\pi(W^1 + W^2)$, for the plans $x^i = x^{a(i)}$, the condition $\pi x^i = \pi W^i$ holds for $i = 1, 2$.

An example for the functional LIM is a Banach limit B, i.e., a generalized limit that satisfies the additional requirement $B((x_t)_t) = \lim_n \frac{1}{n}\sum_{t=1}^n x_t$ when this limit exists. Let LIM $\equiv B$ on the AD prices formula[8]. Since $x^1 + x^2 = W^1 + W^2$, we have that (π, x^1, x^2) is an AD equilibrium.

In the sequential implementation we can choose $z^i_0 \in (0, 1/2)$, as $W^i_t > 1/2,\ \forall\, t$. Taking as deflator $(p_t) = (\kappa^{t-1}\sqrt{\frac{t}{t+1}})$, asset prices are given by $p_1 q_1 = \sum_{t\geq 1} p_t + \lim_t p_t q_t$ and $p_t q_t = p_{t+1}(q_{t+1} + 1)\ \forall t$, so the bubble is positive at $t = 1$. The transversality condition, when net trades are not converging, is

$$\lim_t \kappa^{t-1} u'(x^i_t) q_t z^i_t \in [\beta \liminf_t (q_t(z^i_{t-1} - z^i_t) + z^i_{t-1}), \beta \limsup_t (q_t(z^i_{t-1} - z^i_t) + z^i_{t-1})]$$

which becomes the usual one when $\beta = 0$.

As we did before, we impose the following portfolio constraint, which mimics the above transversality condition on every feasible portfolio plan: $\lim p_t q_t z_t \geq \nu(x(z) - \omega^i)$, where ν is the pure charge component of the AD price (that is, the non-countably additive part given by $\beta B(.)$). The remaining argument is analogous to the one done in the previous example.

[8]Notice that the choice of the generalized limit will determine the coefficient $a(i)$ and, therefore, the real equilibrium allocation.

3 Wariness and Arrow-Debreu Equilibrium

3.1 Wariness

Let us introduce our basic hypothesis on preferences.

Assumption A1: Preferences $\succsim$ on ℓ^∞_+ are complete, transitive, monotonous [9] and norm continuous on ℓ^∞_+ [9].

The attitude of consumers with respect to gains or losses in remote sets of dates or states can be described using the *Mackey topology*, the finest topology on ℓ^∞ for which the dual is ℓ^1. Let us refer to the Mackey upper and lower semi-continuities as Mackey usc and lsc, respectively.

Assumption A2: Preferences $\succsim$ are Mackey usc.

The counterparty is not assumed as we want to allow for wariness, the willingness to neglect gains but not losses in distant sets of dates or in events with very small likehood. More precisely,

Definition: A consumer whose preferences satisfy A1-A2 is *wary* at a point $x \in \ell^\infty_+$ when $\exists\, y \succ x$ such that $\forall\, n$, $x \succsim (y_1, ..., y_n, 0, 0, ...)$. If this condition holds at every point, she is said to be wary.

We have the following characterization (proven in Appendix A.1).

Proposition 1: Wariness

Under A1, preferences $\succsim$ are not Mackey lsc at x if and only if $\exists\, y \succ x$ such that, $\forall\, n$, $x \succsim (y_1, ..., y_n, 0, 0, ...)$.

That is, we can not find n large enough for which we do not have a reversal of the preference ordering due to losses beyond n. Analogously, the upper semi-continuity could characterized by the non-reversal of the ordering for gains occuring beyond some n large enough (see Lemma 4 in Appendix A.1).

3.2 Relationship with Several Decision Theory Concepts

We show that Mackey continuity of preferences turns out to be equivalent, when consumption bundles are bounded nonnegative sequences, to the axiom of monotone continuity proposed by Arrow [3], building on Villegas [35] work on

[9] The preferences $\succsim$ are monotonous when $x > x'$ implies $x \succ x'$. Notation and some basic concepts of the space ℓ^∞ and of the theory of charges can be found in Appendix A.

bets. Let us formalize this axiom, in our framework. Let $\mathbb{1}$ be the sequence whose terms are all equal to one; and, for each $n \in \mathbb{N}$, E_n be the set of natural numbers bigger than n. Given $x \in \ell_+^\infty$ and $A \subset \mathbb{N}$, let x_A be the sequence whose s-th term is equal to x_s if $s \in A$ and is equal to zero otherwise.

MONOTONE CONTINUITY:
If $y \succ x$, given $m \in I\!R_+$ and a decreasing sequence $(A_n)_n$ of subsets of $\mathbb{N}$ with empty intersection, there is n_0 such that (i) $y_{A^c_{n_0}} + m\mathbb{1}_{A_{n_0}} \succ x$ and (ii) $y \succ x_{A^c_{n_0}} + m\mathbb{1}_{A_{n_0}}$.

That is, the preference ordering should not change when one of the bundles is modified only on a far enough tail of an infinite subset of $\mathbb{N}$ (as this tail is a distant enough term of a vanishing sequence of events or date sets).

Our Proposition 1 already shows that lack of Mackey lsc at x implies that (i) fails (for the sequence $A_n = E_n$ and $m = 0$). We can say more:

PROPOSITION 2: *Under A1, preferences $\succsim$ are Mackey continuous if and only if satisfy monotone continuity.*

Sufficiency follows from Proposition 1 and Lemma 4, whereas necessity follows from Lemma 3 (both lemmas are in Appendix A.1) [10]. This characterization is crucial as it shows that the willingness to pay for consumption at arbitrarily remote states or dates is zero if and only if preferences are Mackey continuous.

As we will be assuming Mackey usc for existence purposes, a pure charge in a supporting price occurs at $x \ggg 0$ only if the consumer is wary at x (as will become clearer in subsection 3.3). Such pure charges will induce bubbles in the prices of assets implementing efficient allocations.

3.2.1 Impatience and Myopia

The concept of myopia has been proposed and applied both to the infinite states case and the infinite horizon case, refered too as impatience in the latter. Preferences are said to be strongly myopic if $x \succ y$ implies, for any $z \in \ell_+^\infty$, $x \succ y + z_{E_n}$ for n large enough (recall that z_{E_n} is the tail of z after n). That is, if x is better than y, no matter how large the terms of a sequence z might be, there

[10] Bewley [5] had already remarked a version of the necessity with $A_n = E_n$.

is always some n such that adding just the tail of z beyond n to y will not make it become better than x. It was shown by Brown and Lewis [8] that the Mackey topology is the strongest topology for which any continuous preference is strongly myopic. Actually, by Lemma 4 in Appendix A.1, strong myopia is equivalent to Mackey usc, provided preferences are monotonous and norm continuous.

However, strong myopia captures only one form of myopia that we can refer to as *upper semi-myopia* (or upper semi-impatience in a dynamic context). Wary consumers never satisfy *lower semi-myopia*, defined by $x - z_{E_n} \succ y$, for any z such that $x - z \in \ell^\infty_+$, when $x \succ y$ and n is sufficient large. Lower semi-myopia is equivalent to Mackey lsc, under monotonicity and norm continuity.

It is interesting to note that the necessary and sufficient condition for Mackey lsc that we found in Proposition 1 is exactly the same as Assumption (IV) in the note by Prescott and Lucas [27] on pricing in infinite dimensional spaces: for $y \succ x$, there is n_0, such that, $\forall n \geq n_0$, $(y_1, ..., y_n, 0, 0, ...) \succ x$ [11]. This assumption implies another notion of lower semi-impatience, proposed by Chateauneuf and Rébillé [9]), which requires, for any x, $(x + \epsilon \mathbb{1})_{E_n^c} \succ x$ when n is large enough.

3.2.2 Ambiguity Aversion and Imprecise Future Preferences

Ambiguity refers to a situation where agents have a collection of beliefs (that is, probability distributions). Gilboa and Schmeidler [16] and Schmeidler [32] modeled ambiguity aversion by considering an utility functional which is the minimum of the expected utilities over this collection of beliefs. More precisely, preferences are described by an utility function U:

$$U(x) = \min_{\eta \in C} \int_{\mathbb{N}} u \circ x \, d\eta, \tag{5}$$

where $u : I\!R_+ \to I\!R$ and C is a convex and weak* closed subset of ba [12] (the refered papers discussed axiomatically this representation). The minimal integral over beliefs represents a precautionary or pessimistic behavior. The minimization solution η^* puts more weight on sets where u attains its lowest values. In our applications we assume that u is increasing, concave and C^1 on $(0, +\infty)$

[11] For this reason, the results by Bewley [5] and Prescott and Lucas [27] on countable additivity of equilibrium prices are equivalent when ℓ^∞_+ is the consumption set.

[12] See Dunford and Schwartz [13], ch. III.2, for the definition of integral with respect to a charge η.

Let us look at a less general case by specifying the set C. Denote by $2^{\mathbb{N}}$ the collection of all subsets of $\mathbb{N}$. We say that a set function $\nu : 2^{\mathbb{N}} \to I\!R$ is a *capacity* if $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity ν is *convex* when $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ $\forall A, B \subset \mathbb{N}$. We normalize $\nu(\mathbb{N}) = 1$. The core of a capacity ν is $\text{core}(\nu) = \{\eta \in ba : \eta \geq \nu, \ \eta(\mathbb{N}) = 1\}$. When C is the core of a convex capacity ν[13], more can be said about the absence of Mackey lsc. A capacity ν is said to be *continuous at certainty* if, for any sequence $(A_n) \subset 2^{\mathbb{N}}$ such that each $A_n \subset A_{n+1} \subset \mathbb{N}$ and $\cup_n A_n = \mathbb{N}$, we have $\lim \nu(A_n) = \nu(\mathbb{N})$. Now, U is Mackey lsc if and only if the capacity is continuous at certainty (by Theorem 2.1 in Epstein and Wang [14]). The discontinuity at certainty can be interpreted as if there were a missing state.

Remark 1: (i) For preferences represented by utilities of the form (5) where C is the core of a convex capacity, Mackey lsc becomes equivalent to the impatience notion proposed by Chateauneuf and Rébillé [9] (as the authors noticed).
(ii) An example of upper semi-myopic preferences that are not lower semi-myopic is given by taking a utility given by (5) generated by a convex capacity which is not continuous at certainty[14].

We give now a well known example of a convex capacity. Given a probability measure μ, let ν_ϵ be the convex capacity obtained through a linear distortion of μ with coefficient $(1-\epsilon) \in (0, 1]$, i.e., taking $\nu_\epsilon(A) = (1-\epsilon)\,\mu(A)$ for $A \subsetneq \mathbb{N}$ and $\nu_\epsilon(\mathbb{N}) = 1$. This is called the ϵ-*contamination capacity* with respect to μ and allow us to rewrite (5) as[15]

$$U(x) = (1-\epsilon) \int_{\mathbb{N}} u \circ x \, d\mu + \epsilon \inf u \circ x. \tag{6}$$

Actually, in this case, the minimum over normalized dominating charges coincides with the infimum over dominating probability measures (see Lemma 9 in Appendix B). That is, $U(x) = \inf_{\eta \in ca \cap core(\nu_\epsilon)} \int_{\mathbb{N}} u \circ x \, d\eta$. Clearly, ν_ϵ is discontinuous at certainty and, therefore, this utility represents wary preferences at some point[16].

[13] In this case, the utility function is a Choquet integral (see Schmeidler [32]).

[14] In fact, (5) for any convex capacity represents upper semi-myopic preferences but the discontinuity at certainty precludes the Mackey lsc and, therefore, preferences are not lower semi-myopic.

[15] This notion appears in statistical works since the fifties (see, Hodges and Lehmann [20]). The fact that the Choquet integral coincides with the right hand side of (6) can be seen, for instance, in Dow and Werlang [10].

[16] Wariness holds actually at every $x \ggg 0$ as lack of Mackey lsc can be seen by letting

In a deterministic setting, agents may be unsure about the discount factor and, by analogy, we say that (5) or (6) represent ambiguous discounting. Gilboa [15] had remarked that the use of the Choquet integral as a representation of deterministic preferences is appropriate when the agent dislikes wobbles in consumption and is concerned with the worst outcome. Under this interpretation, for (ζ, β) proportional to $((1-\epsilon)\mu, \epsilon)$, the utility can be rewritten (up to a scalar multiple) as

$$U(x) = \sum_{t=1}^{\infty} \zeta_t u\left(x_t\right) + \beta \inf_{t \geq 1} u(x_t) \tag{7}$$

By (5), this utility function can be reinterpreted as the minimal separable utility when the discount factors have a lower bound given by ζ_t [17]. Consumers end up maximizing the worst discounted utility, over a set of possible discount factors.

Imprecise discounting may be a particular case of a more general situation where the agent does not know what she will be later on (having what is called a *divided self*) or where a random element affects future preferences. Uncertainty about future tastes has interesting connections with recent work on the incompleteness of preferences (see Dubra, Maccheroni and Ok [12]) and has also been addressed in terms of preferences for flexibility (see Kreps [22]). The approach just described addresses this uncertainty while preserving the completeness hypothesis and the determinacy of current choices. Moreover, even if each agent had standard discounted utilities, a policy-maker ignoring individual discount factors but worried about not leaving some agent extremely unhappy, might want to use a representative consumer model with the above imprecise discounting feature.

3.3 Arrow-Debreu Equilibrium

Suppose that there is a finite number I of consumers. Each consumer i is characterized by preferences $\succsim^i$ on ℓ^∞_+ and endowments $W^i \in \ell^\infty_+$.

Assumption A3: Preferences $\succsim$ on ℓ^∞_+ are represented by the restriction to ℓ^∞_+ of $U : \ell^\infty \to I\!R \cup \{-\infty\}$ which is a concave function, that has finite values on its

$z^n = (1+\varepsilon)\, x_{E_n^c}$. Notice that $z^n \to (1+\varepsilon)\, x$ in the Mackey topology (by Lemma 3 in Appendix), but, when $\varepsilon > 0$ is small enough, we have for all n that $U(z^n) < U(x)$ (as $\inf_t u(z^n_t) = u(0) < \inf_s u(x_s)$). So the lower contour set of x is not Mackey closed.

[17] If $u(y_{t_0}) = \inf_t u(y_t)$ and this is the only date for which this holds, t_0 utility is discounted by the rate $(\zeta_{t_0} + \beta)^{1/(t_0-1)}$. The degree of impatience for each date changes according to the relative level of consumption.

effective domain D and takes the value $-\infty$ outside[18].

Assuming that preferences $\succsim^i$ satisfy A3, we denote by U^i its the utility function given by this assumption. For prices π given by a positive linear functional on ℓ^∞, we define the AD budget set of consumer i as the set $B_{AD}(\pi, W^i) = \{x \in \ell^\infty_+ : \pi(x - W^i) \leq 0\}$. An *AD equilibrium* is a couple $(\pi, (x^i)_{i=1}^I)$ such that, $\forall i$, x^i maximizes U^i on $B_{AD}(\pi, W^i)$ and $\sum_{i=1}^I (x^i - W^i) = 0$.

By Bewley's [5] existence theorem, if preferences are convex, satisfy assumptions A1-A2 and $W^i \ggg 0$ $\forall i$, then there exist equilibrium prices $\pi \in \mathrm{ba}_{++}$ and, under the additional assumption of Mackey lsc, $\pi \in \ell^1_{++}$. We gave examples where AD equilibrium prices are not countably additive (Section 2), but we postponed some details that we now justify. The following lemma (proven in Appendix C) shows that, under A1, A2 and A3, a pure charge in a supporting price occurs at $x \ggg 0$ only if the consumer is wary at x.

Lemma 1: *Let U be an increasing concave utility function on ℓ^∞_+. If U is Mackey continuous, then $\partial U(x) \subset \ell^1$ for $x \ggg 0$.*

For U given by (7), a sufficient condition to get $\partial U(x)$ not contained in ℓ^1 is that the infimum of x be a cluster point of x, as the next two propositions (that are consequences of Lemma 10 in Appendix C.1) show. Given $x \in \ell^\infty_+$, let $\underline{x} \equiv \inf x$.

Proposition 3: *If U is given by (7) with $u : \mathbb{R}_+ \to \mathbb{R}$ concave, increasing and of class $C^1(0, \infty)$, then, given $x \ggg 0$,*

a) If $\underline{x}$ is not a cluster point of x, $\partial U(x) \subset \ell^1$;

b) If $\underline{x}$ is attained for infinite indices t, $\partial U(x) \cap \ell^1 \neq \emptyset$ but $\partial U(x) \not\subset \ell^1$

c) If $\underline{x}$ is not attained, $\partial U(x) \cap \ell^1 = \emptyset$.

Proposition 4: *Under the same hypotheses of Proposition 3, if $\underline{x}$ is a cluster point never attained of the sequence x, then x is maximal for U in $B_{AD}(\pi, W)$ when (i) W is such that $\pi W = \pi x$ and (ii) $\pi \in ba$ is given by*

$$\pi y = \sum_{t \geq 1} \zeta_t \, u'(x_t) \, y_t \; + \; \beta \, u'(\underline{x}) \, \mathrm{LIM}(y)$$

[18]Under A3, U is finite and norm continuous at x if and only if U is bounded from below in some norm neighborhood of x. Even when $D = \ell^\infty_+$, $A1$ and $A3$ are not in contradiction, as A1 is stated for preferences and the restriction of U to ℓ^∞_+ is continuous. Monotonicity implies that $\ell^\infty_+ \subset D$.

where LIM *is a generalized limit such that* $\mathrm{LIM}(x) = \underline{x}$.

(When x converges to $\underline{x}$, any generalized limit LIM fulfills the requirement)

REMARK 2: In Proposition 4, $\beta u'(\underline{x})$ LIM is the pure charge component of the supporting price [19]. Gilles and LeRoy [17] suggested that the pure charge component of an AD equilibrium price could be interpreted as a bubble, but did not relate it to bubbles in prices of the assets that serve to complete the markets. We will establish this precise relation in the next section.

4 Sequential Equilibrium and Bubbles

Along this section, we assume that preferences of each agent i satisfy A1 and A3 and that U^i is the representation of $\succsim^i$ given by the latter.

In a sequential economy agents can transfer income across different dates [20] using an infinite lived asset that pays $R_t \geq 0$ units of the consumption good at each date $t > 1$ ($R_1 = 0$) and may be in positive net supply. We assume that $R = (R_t)_{t\geq 0} \in \ell^\infty_+$. Initial asset holdings are denoted by $z^i_0 \in I\!R_+$ and sequential commodity endowments by $\omega^i = (\omega^i_t)_{t\geq 1} \in \ell^\infty_+$. Let the commodity be the numeraire at each date and denote by $q_t \geq 0$ the asset price at date t. The sequential problem of agent i consists in choosing a consumption plan $x = (x_t)_{t\geq 1} \in \ell^\infty_+$ and a portfolio plan $z = (z_t)_{t\geq 1} \in I\!R^\infty$, in order to maximize U^i subject to the sequential budget constraints (1).

This problem does not have a solution, under monotonicity of preferences, as the agent can do Ponzi schemes. To prevent this, portfolio constraints should be introduced. Examples of such constraints, in the previous literature, were of the form $q_t z_t \geq -M_t$, where $M \in \ell^\infty_+$. Given a portfolio constraint P, the *sequential P-constrained problem* consists in maximizing U^i subject to $B_P(q, \omega^i, z^i_0)$, where this set denotes the set of consumption plans $x \in \ell^\infty_+$ for which there is a P-feasible portfolio z such that $x_t \leq \omega^i_t + q_t(z_{t-1} - z_t) + R_t z_{t-1}$.

An *equilibrium for the sequential economy* with portfolio constraint P consists in a vector of plans $(x^i, z^i)_i$ and asset prices q such that (i) x^i maximizes, for each

[19] In Appendix C.2, we provide an example of an AD equilibrium with positive pure charging prices, for preferences not given by (7), assuming $x \ggg 0$ but no other assumption on $\underline{x}$.

[20] As we are interested in efficient bubbles, we focus on deterministic economies, but we could have examined also sequentially complete markets.

i, U^i on $B_P(q, \omega^i, z_0^i)$, (ii) $\sum_i (x_t^i - \omega_t^i) = \sum_i R_t z_0^i$ and $\sum_i (z_t^i - z_0^i) = 0$ for all $t \geq 1$ and (iii) $x^i = x^i(z^i)$ where $x^i(z)_t \equiv \omega_t^i + q_t(z_{t-1} - z_t) + R_t z_{t-1}$.

The following property of an optimal plan x^* will be assumed in most results (but not in general statements found in the Appendix).

ASSUMPTION A4: U^i is *differentiable along any canonical direction* at the optimal solution x^* for the sequential P-constrained problem, that is,

$$\forall t, \quad \exists \; \delta U^i(x^*; e_t) \equiv \mu_t^i.$$

Under this assumption, all supergradient of U^i at x^* have the same countably additive component μ^i. If U^i given by (7), A4 holds when the infimum of consumption is not attained (as in Examples 1 and 2, with $\mu_t^i = \kappa^{t-1}(u^i)'(x_t^i)$).

For these preferences, if decisions were made only tomorrow, would the choices for next dates remain the same? Consistency holds when the infimum of utilities in not attained (as it is the case for the endowments in the guiding examples) and would hold for any plan x if we would define the utility for the problem starting at t with a coefficient β_t, dependent on the values of x up to t. However, our results hold for more general Mackey discontinuous preferences.

4.1 Necessary Conditions for Optimality

To study the optimal solutions for the sequential P-constrained problem[21], we need to define the concept of admissibility. Given a portfolio constraint P, the sequence v is said to be a *right-admissible direction* at $x \in B_P(q, \omega, z_0)$ when $\exists \; \varepsilon' > 0$ such that $x + \tau v \in B_P(q, \omega, z_0) \quad \forall \tau \in [0, \varepsilon')$. It is said to be *left-admissible* when $\tau \in (-\varepsilon'', 0]$ instead, for some $\varepsilon'' > 0$, and is said to be *admissible* if it is right and left-admissible. If z^* is optimal and v is an admissible direction (at $x(z^*)$), then the directional derivative $\delta U(x(z^*); v)$, when it exists, is zero.

4.1.1 Euler Conditions

Consider the direction $v(t) = -q_t e_t + (q_{t+1} + R_{t+1}) e_{t+1}$. We suppose that portfolio constraints are such increases in asset positions at a particular date are always allowed at $x \gg 0$, that is, *the right-admissibility of $v(t)$ at x always holds.*

[21] As we will be studying the problem of an individual agent we omit, in this subsection, to simplify the notation, the indices refering to the specific agent. For instance, if A4 holds for x^*, μ_t denotes $\delta U(x^*; e_t)$.

Proposition 5: *Let* $x^* \gg 0$ *be an optimal solution to the sequential P-constrained problem satisfying A4. Then*

$$q_t\mu_t \geq (q_{t+1} + R_{t+1})\mu_{t+1}. \tag{8}$$

(The proof can be found in Appendix D)

Remark 3: (i) Under left-admissibility (which may fail when decreases in asset positions violate the constraint), the opposite inequality would hold too.
(ii) If $x^* \ggg 0$, even without A4, we can always say that $\exists T \in \partial U(x^*) : q_t T e_t \geq (q_{t+1} + R_{t+1}) T e_{t+1}$[22]. For other $v(t')$, the supergradient might be different.

4.1.2 Transversality Conditions

Transversality conditions should not be confused with constraints that have been imposed by several authors to guarantee the existence of a solution to sequential problems. The former are properties that the optimal solution must exhibit. The latter restrict the choice set by requiring portfolio plans to mimic that property. We need to study some directions for changes in the portfolio plan.

Given a portfolio z and $n \in \mathbb{N}$, let $y(n)$ be defined by $y(n)_t = 0$ if $t < n$, $y(n)_n = -q_n z_n$ and $y(n)_t = q_t(z_{t-1} - z_t) + R_t z_{t-1} = x(z)_t - \omega_t$ if $t > n$. We will always assume the left-admissibility of $y(n)$, which means that constraint P allows, at any $t > n$, for the replacement of $(z_t)_{t \geq n}$ by $((1+h)z_t)_{t \geq n}$ with $h < 0$ arbitrarily close to 0 (the absolute value of the portfolio is decreased at date t). Sometimes we will impose the counterpart:

Assumption A5: For constraint P, *direction* $y(n)$ *is right-admissible at* $x(z)$.

A5 holds for constraints $q_t z_t \geq -M_t$ if $(q_t z_t)_t \ggg -M$ and always for constraints that mimic transversality conditions (classical or the ones we present). In this case, the absolute value of the portfolio can be increased at date t. Anyway, we shall always assume that constraints should always be such that, for every n, i*f* $y(n)$ *is right-admissible, then* $y(n+1)$ *is also right-admissible, for any plan* z.

Proposition 6: Transversality Conditions
Let $x^* \ggg 0$ *be an optimal solution to the sequential P-constrained problem satisfying A4 and let* z^* *be such that* $x^* = x(z^*)$. *Then,*

[22] Under left-admissibility, the opposite inequality holds for some $T' \in \partial U(x^*)$ and, as $\partial U(x^*)$ is convex, Euler equation holds for some $T'' \in \partial U(x^*)$.

(i) $\exists$ *pure charge* $\nu : \nu(x^* - \omega) \geq \limsup \mu_n q_n z_n^*$ *and* $\mu + \nu \in \partial U(x^*)$;

(ii) Under A5, $\exists$ *pure charge* $\eta : \eta(x^* - \omega) \leq \liminf \mu_n q_n z_n^*$ *and* $\mu + \eta \in \partial U(x^*)$;

(This follows from Lemma 11 in Appendix D.1)

Remark 4: For U given by (7), when $\inf x^*$ is never attained, we can compute the directional derivative $\delta U(x^*; \mathbb{1}_{E_n}) = \sum_{t>n} \zeta_t u'(x_t^*) + \beta u'(\inf x^*)$, whose limit, as $t \to \infty$, is $\beta u'(\inf x^*)$. As A4 holds in this case, we get, using Lemma 8 (in Appendix A.3), $\lim \mu_n q_n z_n^* \in \beta u'(\inf x^*)[\liminf(x^* - \omega), \limsup(x^* - \omega)]$.

4.2 Sequential Implementation and Room for Efficient Bubbles

We say that an AD equilibrium $(\pi, (x^i)_i)$, for the endowments $(W^i)_i$, *is implemented sequentially,* under given portfolio constraints, with an asset with returns $R \in \ell_+^\infty$ if we can find initial holdings $z_0^i > 0$, verifying $\omega^i \equiv W^i - (R_t z_0^i) \geq 0$, prices q and portfolios $(z^i)_i$ so that $(q, (x^i, z^i)_i)$ is an equilibrium of this sequential economy. The implementation depends on the choice of a deflator:

Definition: Given asset returns R and prices q, a sequence $\lambda = (\lambda_t)_t \gg 0$ is a *non-arbitrage deflator* if $\lambda_t q_t = \lambda_{t+1}(q_{t+1} + R_{t+1})$ for every $t \geq 1$[23].

At any date t, $\lambda_t q_t = \sum_{s>t} \lambda_s R_s + \lim_s \lambda_s q_s$, where both the series and the limit are finite, since $R \geq 0$ and q_t is finite[24]. For this deflator, the asset *fundamental value* and the asset *bubble* at t are, respectively, $\frac{1}{\lambda_t} \sum_{s>t} \lambda_s R_s$ and $\frac{1}{\lambda_t} \lim_s \lambda_s q_s$.

Note that if $x^i \gg 0$ satisfies A4 and (8) with equality at each t, (μ_t^i) is a non-arbitrage deflator. If, in addition, $x^i \ggg 0$, we can use the ℓ^1 component p of the AD price as a deflator, as the first order condition of the AD problem requires[25]

$$\exists \rho^i > 0 : \rho^i \pi \in \partial U^i(x^i). \tag{9}$$

Can we use a deflator which is not, up to a scalar multiple, equal to the countably additive part of the AD price? Could we distribute the pure charge across all dates to get a different deflator? This could not be done when utilities are of the form (7) and the infimum is not attained (as in Examples 1 and 2)). More generally:

[23] Given R and q, the sequence λ is uniquely determined up to the choice of some term, say the initial term λ_1. Conversely, given λ and R, q is uniquely determined up to one term.

[24] Notice that if $R \ggg 0$, as in the guiding examples, λ must be in ℓ^1.

[25] See Zeidler [37], p.391, Theorem 47.C.

Proposition 7: ℓ^1 AD Deflators

Let $(\pi, (x^i)_i)$ be an AD equilibrium such that, for each i, $x^i \ggg 0$ satisfies A4. For portfolio constraints that allow an agent with a long position at t to reduce it by a small enough amount, if the asset is in positive net supply or $(x^i_t)_i \neq (W^i_t)_i\ \forall t$, then the countably additive component p of π is, up to a scalar multiple, the only possible choice of a deflator to implement sequentially $(\pi, (x^i)_i)$.

Proof: We can not implement $(\pi, (x^i)_i)$ with positive shadow prices for the portfolio constraints, otherwise the uniquely defined marginal rates of intertemporal substitution would be different across agents (as at each date, some agent must be purchasing the asset and does not have the constraint binding at that date), contradicting efficiency. Then, for any non-arbitrage deflator λ, $\lambda_t/\lambda_{t+1} = (q_{t+1} + R_{t+1})/q_t = \delta U^i(x^i; e_t)/\delta U^i(x^i; e_{t+1}) = p_t/p_{t+1}$. *Q.E.D.*

When discussing implementation we use the following (proven in Appendix D):

Lemma 2: *Let $\pi = p + \nu$ be such that ν is a pure charge, $p \in \ell^1$ is a deflator for (R, q) with $q \gg 0$. Given $x \in \ell^\infty_+$, take z to be such that (1) holds. Then, for $\omega^i_t = W^i_t - R_t z^i_0$, we have $x \in B_{AD}(\pi, W^i)$ if and only if*

$$\nu(x^i(z) - \omega^i) - \lim_t p_t q_t z_t \leq z_0(\nu(R) - \lim p_t q_t), \tag{10}$$

Actually, (10) holds with equality when $\pi x = \pi W^i$. In this case, $\exists \lim_t p_t q_t z_t$.

Could we use standard portfolio constraints to implement sequentially efficient allocations? Given a deflator λ, take three types of constraints that have been extensively used to avoid Ponzi schemes under impatience assumptions:

(a) $\lim \lambda_t q_t z_t = 0$ (transversality constraint);
(b) $\lambda_t z_t \geq -M_t$ with $M \in \ell^\infty_+$ (bounded short-sales);
(c) $\lambda_t q_t z_t \geq -\sum_{s>t} \lambda_s \omega^i_s$ (debt dependent on future ability to repay).

It is well known that these types of constraints rule out bubbles for positive net supply assets, in the deterministic case (actually in the complete markets case), when the present value of wealth is finite. Besides, in equilibrium, even with zero net supply, (b) or (c) imply $\lim \lambda_t q_t z^i_t = 0$ when $\lambda \in \ell^1$ (see Appendix D.3)[26].

[26] $\|\lambda\|_1 < +\infty$ is sufficient for the present value of wealth to be finite and also necessary when $\sum_i(\omega^i + (R_t z^i_0)_t) \ggg 0$.

Proposition 8: Standard Constraints Do Not Implement AD

Let $(\pi, (x^i)_i)$ *be an AD equilibrium such that, for each* i, $x^i \ggg 0$ *satisfies A4 and* π *has a pure charge that is non-zero valued at* $(x^{i_0} - W^{i_0})$ *of some agent* i_0. *Under portfolio constraints of type* (a) *or* (b) *or* (c), *it is impossible to implement sequentially* $(\pi, (x^i)_i)$ *when* $\sum_i z_0^i > 0$ *or* $(x_t^i)_i \neq (W_t^i)_i\ \forall t$.

Proof: Suppose $(\pi, (x^i)_i)$ can be implemented as $(q, (x^i, z^i)_i)$. By Proposition 7, p is, up to a scalar multiple, the unique deflator. Let i be such that $\nu(x^i - W^i) > 0$. Since $\nu(x^i - W^i) = \nu(x^i - \omega^i) - z_0^i \nu(R)$ and, in equilibrium, $\lim_t p_t q_t z_t^i = 0$, Lemma 2 implies $\nu(x^i - W^i) \leq -z_0^i \lim_t p_t q_t \leq 0$, a contradiction. *Q.E.D.*

In Examples 1 and 2, $\nu(x^i - W^i) = \beta LIM(x^i - W^i)$ which is positive for one agent and negative for the other, in the AD equilibrium. At least for constraints of type (a) (even in the weaker form $\lim \lambda_t q_t z_t \geq 0$), if $(x^i)_i \neq (W^i)_i$ and a consumer i has $U^i(x^i) \cap \ell^1 = \emptyset$, it can be shown that the result in Proposition 8 holds without having to assume A4 or that the AD price pure charge ν has $\nu(x^i - W^i) \neq 0$ for some i (which does not hold when $(x_t^i - W_t^i) \to 0$). Do positive net supply assets have price bubbles when other portfolio constraints are used instead?

The first important observation is that, under wariness, *transversality conditions do not prevent bubbles*. When, for each i, $x^i \ggg 0$ satisfies A4, by (i) in Proposition 6 and condition (9), $\sum_i \lim p_t q_t z_t^i \leq \sum_i \alpha^i (\limsup (x^i - W^i) + z_0^i \limsup R)$, where $\alpha^i \geq 0$ is given by Lemma 8 in Appendix A.3. If $\alpha^i > 0\ \forall i$, even when $x^i - W^i$ converges, by normalization of utilities (to have $\alpha^i = 1$, adjusting ρ^i), we get $\lim_t p_t q_t \leq \limsup R$, which does not rule out bubbles under a finite present value of wealth (as $p \in \ell^1$ and $W^i \in \ell^\infty$).

On the contrary, if utilities U^i were Mackey continuous, then, at norm interior optimal bundles, the supergradients had to belong to ℓ^1 (see Lemma 1) and, if Euler condition (8) holds with equality at each t (assuming A4), the transversality conditions became $\lim p_t q_t z_t^i = 0$, implying $(\sum_i z_0^i) \lim p_t q_t = 0$, that is, bubbles of assets in positive net supply would be ruled out (under the deflator given by Proposition 7, which yields a finite present value of wealth). Intuitively, impatient agents would like to sell the bubble. However, in the absence of initial holdings and under appropriate portfolio constraints, agents may be prevented from doing it. We say that the agent *wants to sell the bubble*, at date t and when consuming x, if there is some $h > 0$ such that $U^i(x + h(q_t e_t - \sum_{\tau>t} R_\tau e_\tau)) > U^i(x)$. We have the following result (proven in Appendix D.4) is in the spirit of Tirole's [33] argument that bubbles can not survive an infinite horizon arbitrage.

Proposition 9: Impatient Agents Sell the Bubble

Suppose U^i (given by A3) is increasing, Mackey continuous and such that A4 holds at $x \ggg 0$. If there were a bubble in the asset price and x satisfies, for each t, (8) with equality, then agent i would want to sell the bubble at every date.

4.3 On the necessity of bubbles

A large class of AD allocations must be implemented with bubbles. We provide general results that are illustrated by Examples 1 and 2. In this results, the equilibrium of the sequential economy must satisfy (10) with equality. Additionally, x^i must be optimal in the sequential choice set $B_P(q, \omega^i, z_0^i)$. For this it suffices that $B_P(q, \omega^i, z_0^i)$ is contained in the AD budget set.

We look at portfolio constraints that mimic transversality conditions. When A4 and A5 hold at z^i and $x^i = x^i(z^i)$, the necessary transversality condition demands $\lim \mu_t^i q_t z_t^i = \nu^i(x^i - \omega^i)$, for the common ℓ^1 component μ^i of all supergradients of U^i at x^i and the pure charge component ν^i of one of these supergradients. We consider constraints that require, for every feasible portfolio plan z, $\lim \mu_t^i q_t z_t \geq \nu^i(x^i(z) - \omega^i)$ for one of those pure charges. Such constraints prevent a non-financed asymptotic excess of consumption over endowments (that is, precludes asymptotic bankruptcy). A possible choice is to pick the pure charge of the supergradient that satisfies the first-order condition of the AD equilibrium (see condition (9)). This is equivalent (dividing both sides by ρ^i) to require, at q in the set $Q(p)$ of asset price sequences for which p is a non-arbitrage deflator,

$$\lim p_t q_t z_t \geq \nu(x^i(z) - \omega^i) \tag{11}$$

for any z. The constraint $p_t q_t z_t \geq -\sum_{s>t} p_s \omega_s^i + \nu(x^i(z) - \omega^i)$ implies (11). Next we show, for a suitable choice of q, that (11) implies the inclusion of $B_P(q, \omega^i, z_0^i)$ in the AD budget set[27] and that, when $\sum_i z_0^i > 0$ and $R \ggg 0$, a bubble is necessary to implement an equilibrium $(\pi, (x^i)_i)$ with a pure charge component in π.

[27] Also, this inclusion implies (11) when the direction $q_1 e_1 - \sum_{t>1} R_t e_t$ is left-admissible (when one can buy some arbitrarily small amount of the asset and hold this additional position forever).

Theorem 1: Necessity of Bubbles

Let $(\pi, (x^i)_i)$ *be an AD equilibrium, with* $\pi = p + \nu$ *with* $p \in \ell^1$, ν *a positive pure charge and,* $\forall\, i$*, at* $x^i \ggg 0$*, A4 holds. If the portfolio constraint is (11), then*

(i) If $R \gg 0$, $(\pi, (x^i)_i)$ *is implemented with asset prices* q *such that* $\lim_s p_s q_s = \nu(R)$*. In this case,* $B_P(q, \omega^i, z_0^i) = B_{AD}(\pi, W^i)\ \forall i$*;*

(ii) If $\sum z_0^i > 0$ *and* $(\pi, (x^i))_i$ *is implemented with* $q \in Q(p)$*, then* $\lim_s p_s q_s \geq \nu(R)$*. Thus, there is a bubble when* $R \ggg 0$*;*

(iii) If $R = 0$, $(\pi, (x^i)_i)$ *is implemented if* $z_0^i = 0\ \forall i$*. In this case, we could use any* $q \in Q(p)$ *such that* $\lim_s p_s q_s > 0$*. Further,* $B_P(q, \omega^i, z_0^i) = B_{AD}(\pi, W^i)\ \ \forall i$*.*

Proof: (i) Define q by $p_t q_t = \sum_{s>t} p_s R_s + \nu(R) > 0$ (that is, $\lim_s p_s q_s = \nu(R)$). Since $x^i = x^i(z^i) \in B_{AD}(\pi, W^i)$, Lemma 2 implies $\lim p_s q_s z_s^i = \nu(x^i - \omega^i)$, so, $x^i \in B_P(q, \omega^i, z_0^i)$. If $x \in B_P(q, \omega^i, z_0^i)$, satisfies condition (11) and, therefore, (10).
(ii) If $(\pi, (x^i))_i$ is implemented with prices q', by Lemma 2, $\nu(x^i - \omega^i) - \lim_t p_t q_t z_t^i = z_0^i(\nu(R) - \lim p_t q_t)$. Since (x^i, z^i) satisfy (11), we get $z_0^i(\nu(R) - \lim p_t q_t) \leq 0$. As z_0^i is positive for some i, $\lim p_t q_t \geq \nu(R)$.
(iii) For $k > 0$, define $p_t q_t = k$ (so, $\lim_s p_s q_s = k$). As $z_0^i = 0\ \forall i$, the conditions (10) and (11) become the same. Thus, $B_P(q, \omega^i, z_0^i) = B_{AD}(\pi, W^i)$. *Q.E.D.*

Remark 5: Example 2 illustrates Theorem 1. The results in this theorem still hold when $x^i \ggg 0$ fails to hold but μ^i is still colinear with $p = \pi - \nu$, as it is the case in Example 3 (see Subsection 4.4). Example 1 could have been presented also with constraint (11) but we picked instead the related constraint (4), requiring $\lim_t p_t q_t z_t \geq \beta \limsup_t (x(z)_t - \omega_t^i)$, which illustrates our next theorem.

Let us address the occurrence of bubbles for non-specified portfolio constraints, satisfying A5, when net trades $(x^i - \omega^i)$ converge and consumers have a well defined way of valuing distant consumption, more precisely:

Assumption A6: At $x \in \ell_+^\infty$, $\lim_n \delta^+ U^i(x; \mathbb{1}_{E_n}) = \lim_n \delta^- U^i(x; \mathbb{1}_{E_n})$, and we call this common value μ_∞^i the marginal utility of consumption at infinity.

This assumption is satisfied by preferences given by (7) when infimum is never attained ($\mu_\infty^i = \beta(u^i)'(\underline{x})$) or the infimum is not a cluster point ($\mu_\infty^i = 0$).

Theorem 2: More on the Necessity of Bubbles

Let $(\pi, (x^i)_i)$ *be an AD equilibrium such that* $\pi = p + \nu$ *where* $p \in \ell^1$, ν *is a positive pure charge and,* $\forall i$, $x^i \ggg 0$ *satisfies A4, A5 and A6. Suppose* $(\pi, (x^i)_i)$ *is implemented using an asset with prices* q *and* $\sum_i z_0^i > 0$. *If* $(x^i - \omega^i)$ *converges* $\forall i$, *then* $q \in Q(p)$ *and* $\lim p_t q_t = \nu(R)$ *(there is a bubble whenever* $R \ggg 0$*)*[28].

Proof: A4 and A6 imply that for each $T \in \partial U^i(x^i)$ there is a generalized limit LIM^T such that $T = \mu^i + \mu^i_\infty \mathrm{LIM}^T$ (see Lemma 6 and Lemma 8 in Appendix A). As $(x^i - \omega^i)$ converges, $\mathrm{LIM}^T(x^i - \omega^i) = \mu^i_\infty \lim(x^i - \omega^i)$, and we can make two remarks. First, Proposition 6 give us $\lim_s \mu^i_s q_s z^i_s = \mu^i_\infty \lim(x^i - \omega^i)$. Since $x^i \ggg 0$, we get, by conditon (9), $\exists \rho^i$ such that $\rho^i(p + \nu) \in U^i(x^i)$. Then, $\rho^i \nu(x^i - \omega^i) = \mu^i_\infty \lim(x^i - \omega^i) = \lim_s \rho^i p_s q_s z^i_s$. Summing over i, $\lim_s p_s q_s = \nu(R)$. Secondly, $\delta U^i(x^i; y(t))$ exists $\forall t$, and, so it must be equal to zero. As $z^i_t v(t) = y(t) - y(t+1)$, $z^i_t T(v(t)) = T(y(t)) - T(y(t+1)) = 0\ \forall T \in \partial U^i(x^i)$, $\forall t$. Given t, in equilibrium, $z^i_t > 0$ for some i. Thus, $\delta U^i(x^i; v(t)) = 0$ for this i, which implies $\mu^i_t q_t = \mu^i_{t+1}(q_{t+1} + R_{t+1})$. From (9), $p_t q_t = p_{t+1}(q_{t+1} + R_{t+1})\ \forall t$, so p is a deflator. *Q.E.D.*

4.4 Coexistence with an Impatient Agent Does Not Kill the Bubble

If some agents are impatient and other agents are not, we can still obtain efficient bubbles in the prices of assets in positive net supply.

Huang and Werner [21] already gave an example where the impatient agent, with time separable linear preferences, consumed zero after the initial date and the AD price had a pure charge induced by the preferences of the other agent. Sequential implementation was obtained for portfolio constraints that require asset positions to be constant after some date. In this example, the impatient agent chose zero consumption after the initial date and this was compatible with the pure charge in prices. One may wonder what are, in general, the features of the impatient agent's problem that allow for bubbles under other constraints.

If U^i is Mackey continuous, then, at $x \ggg 0$, $\partial U^i(x) \subset \ell^1$ (see Lemma 1). In this case, x could not be the optimal choice for an AD price $\pi = p + \nu$ with a positive pure charge ν. In fact, the necessary and sufficient (together with the

[28] For portfolio constraints that allow an agent with a long position at t to reduce it by a small enough amount, the result in Theorem 2 holds assuming A5, A6 and the convergence of net trade for just one agent.

budget constraint) optimality condition (see Zeidler [37], p.391, Theorem 47.C) is that, for some $\rho^i > 0$, we have

$$\rho^i \pi \in \partial U^i(x) + \partial \chi_{\ell^\infty_+}(x) \tag{12}$$

where $\chi_{\ell^\infty_+}$ is the extended real valued functional that takes value 0 on ℓ^∞_+ and $-\infty$ elsewhere. Now, by Lemma 12 in Appendix D.5, the superdifferential $\partial\chi_{\ell^\infty_+}(x)$ is the set $\{T \in ba : T(y) \geq 0 \ \forall y \in \ell^\infty_+ \text{ and } T(x) = 0\}$. Hence, if $x \ggg 0$, then $\partial\chi_{\ell^\infty_+}(x) = \{0\}$ and, therefore, as $\partial U^i(x) \subset \ell^1$, π could not have a pure charge.

That is, when the AD price has a pure charge, impatient agents must choose bundles with subsequence converging to 0. This holds trivially in the example by Huang and Werner [21], but it is compatible with a larger class of preferences, namely when U is strictly concave and Inada conditions hold, so that the agent is not consuming only at the first date, in spite of being impatient. An agent with Mackey continuous utility can satisfy (12) at a norm boundary point $x \gg 0$ of ℓ^∞_+ for an AD price with a pure charge ν. In fact, $\partial\chi_{\ell^\infty_+}(x)$ only contains pure charges[29] and the pure charge η of the element of $\partial\chi_{\ell^\infty_+}(x)$ cancels out the price pure charge ($\rho^i\nu = \eta$)[30]. Let us illustrate this.

Example 3: Coexistence with an Impatient Agent

Utilities are $U^1(x) = \sum_{t=1}^\infty \zeta_1(t)\sqrt{x_t}$ and $U^2(x) = \sum_{t=1}^\infty \zeta_2(t)\sqrt{x_t} + 6\inf_t \sqrt{x_t}$. The discount factors are given by $\zeta_1(t) = (\frac{1}{4})^{t-1}\frac{t}{1+t}$ and $\zeta_2(t) = (\frac{1}{2})^{t-1}\frac{t}{1+t}$, which are both in ℓ^1. Agents have the same endowments, given by $W^i_t = \frac{1}{2}(1+7(\frac{1}{4})^{t-1})(\frac{1+t}{t})^2$. We show that an AD equilibrium is given by $x^1_t = 7\left(\frac{1}{4}\right)^{t-1}(\frac{1+t}{t})^2$, $x^2_t = (\frac{1+t}{t})^2$ and prices π such that, for $x \in \ell^\infty_+$, $\pi(x) = \frac{1}{2}\sum_{t=1}^\infty (\frac{1}{2})^{t-1}(\frac{t}{1+t})^2 x_t + 3\mathrm{B}(x)$.

Let us start by checking that the budget constraint holds for agent 1 (which implies also that the budget constraint of agent 2 holds). Now $\frac{1}{2}\sum_{t=1}^\infty (1/2)^{t-1}(\frac{t}{1+t})^2(x^1_t - W^1_t) + 3\mathrm{B}(x^1 - W^1) = \frac{1}{4}\sum_{t=1}^\infty (\frac{1}{2})^{t-1}(7(\frac{1}{4})^{t-1} - 1) - 3/2 = \frac{1}{4}(8-2) - 3/2 = 0$.

The first order optimality conditions of agents 1 and 2 are given by (12). The latter holds (by making $\rho^2 = 1$ and noticing that $x^2 \ggg 0$) since π is according to

[29] Take $T \in \partial\chi_{\ell^\infty_+}(x)$, with countably additive and pure charge components $\gamma \geq 0$ and $\eta \geq 0$, respectively. $Tx = 0$ implies $\gamma x = \eta(-x) \leq -\alpha \liminf x$ for some $\alpha > 0$ (by Lemma 6 in Appendix A.2). Hence, $\gamma x = 0$ and since $\gamma e_t \geq 0, \forall t$, we get $\gamma = 0$.

[30] By Mackey continuity, the countably additive component of any supergradient of U at x is a supergradient (see Lemma 14 in the Appendix D.5) and can replace the supergradient satisfying condition (12) (by moving the associated pure charge to the supergradient of $\chi_{\ell^\infty_+}$).

Proposition 4. To check the former, recall that we can make the price pure charge cancel out with the pure charge of the element of $\partial\chi_{\ell^\infty_+}(x^1)$. Hence, it suffices to find $\rho^1 \geq 0$ verifying $\rho^1\pi(e_t) = \frac{\zeta_1(t)}{2\sqrt{x^1_t}}$ for each t. That is, we must have $\rho^1 = 1/\sqrt{7}$.

Although $W^1 \ggg 0$, consumption of agent 1 tends to zero. This happens since the wary agent 2 is placing a high value on consumption at arbitrarily large dates, inducing the impatient agent 1 to sell endowments at distant dates.

The AD equilibrium can be implemented imposing the transversality condition of the wary agent on the portfolio of both[31] or (for an appropriate choice of $(z^i_0)_i$) by imposing on each agent a constraint that mimics the respective transversality condition. In the former we use (11) (that is, denoting by p the ℓ^1 component of π, $\lim_t p_t q_t z_t \geq 3\mathrm{B}(x(z) - \omega^i))$ on both agents and, as seen in Subsection 4.3 (see Remark 5), $B_P(q, \omega^i, z^i_0) \subset B_{AD}(\pi, W^i)$. In the latter, agent 1 faces the usual constraint $\lim_t p_t q_t z_t = 0$. Let, for $t > 1$, $R_t = (\frac{1+t}{t})^2$ and define q by $p_1 q_1 = pR + 3$ and $p_{t-1}q_{t-1} = p_t(q_t + R_t)$ for $t > 1$. Thus, $\lim p_t q_t = \nu(R) = 3\mathrm{B}(R) = 3$. We choose $z^1_0 = 1/2$. Since $x^1 \in B_{AD}(\pi, W^1)$ and $\nu(x^1 - \omega^1) = \nu(x^1 - W^1) + z^1_0 \nu(R) = -3/2 + 3z^1_0 = 0$, Lemma 2 implies $\lim p_t q_t z^1_t = 0$. Moreover, as p is colinear with $\mu^1 \in \partial U^1(x^1)$ (see Lemma 14 in Appendix D.5), individual optimality holds: $U^1(x(z)) - U^1(x^1) \leq \mu^1(x(z) - \omega^1) + \mu^1(\omega^1 - x^1) = \lim \mu^1_t q_t z^1_t - \lim \mu^1_t q_t z_t \leq 0$.

5 Two Date Economies and Bubbles

The AD equilibria can be implemented as equilibria of a two date economy with a complete set of assets traded at the initial date and paying returns on the countable infinite set of states at the second date. When the AD price is not in ℓ^1 and the complete set of assets consists of Arrow securities, the price of an asset whose returns are uniformly bounded away from zero will exceed the series of returns weighted by state prices (given by the Arrow security prices). Even for other asset structures, the series of returns, deflated by marginal rates of substitution, should be interpreted as the fundamental value (see Proposition 10) and the asset price has a bubble, which is related to the pure charge component in the AD price evaluated at the returns stream.

[31] Given a constraint choice, agent 2 demands all the asset supply, as $t \to \infty$. So, if one is to impose the same constraint on both agents, it is reasonable to choose one that mimics agent 2 transversality condition.

Our previous examples could be adapted to this two date setting, but we chose to look at a more provocative case, when a durable good (say gold) that has no pure charge in the respective marginal utility turns out to have a bubble due to its role in hedging against fluctuations in the endowment of the other commodity. That is, consumers are not specifically wary about the consumption of gold (as the partial utility function on gold is Mackey continuous) but gold has a bubble.

The asset structure is described by the returns linear operator $R : \ell^\infty \to \ell^\infty$, mapping a portfolio $z \in \ell^\infty$ of countably many assets into a bounded sequence of payoffs over the countably many states. Asset j is identified by its returns $R(e_j)$. Let $q \in ba_+$ be the vector of asset prices. We allow for initial holdings $z_0^i \geq 0$.

At date 0, besides the choice a portfolio $z \in \ell^\infty$, it is possible to trade and consume a perishable good (the numeraire) and gold. The numeraire endowment is $\omega_0^i > 0$ and gold endowment is $e_0^i > 0$. Consumptions of the numeraire and gold are denoted by c_0 and g_0, respectively. Date 0 budget constraint of agent i is

$$c_0 + \rho_0 g_0 + q(z) \leq \omega_0^i + \rho_0 e_0^i + q(z_0^i), \tag{13}$$

where $\rho_0 > 0$ is the gold relative price. Each unit of gold consumed at $t = 0$ is still available at date 1. This durable good can be seen as an commodity-money that serves to transfer wealth and hedge against variations in the numeraire.

At the next date, there are state-dependent endowments of the numeraire $\omega_1^i \in \ell_+^\infty$ and gold $e_1^i \in \ell_+^\infty$. Let $\rho_1 \in \ell_+^\infty$ be the state-dependent relative price of gold. State-dependent consumptions of the numeraire and gold, c_1 and g_1, lie in ℓ_+^∞.

Date 1 budget constraints: at each $s \in \mathbb{N}$

$$c_{1s} + \rho_{1s} g_{1s} \leq \omega_{1s}^i + \rho_{1s}(e_{1s}^i + g_0) + R(z)_s, \tag{14}$$

Denoting $\rho = (\rho_0, \rho_1)$ and analogously for the other date-dependent variables, let $B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i) \equiv \{(c, g) : \text{(13) and (14) hold for some } z \in \ell^\infty\}$.

Preferences are represented by a concave and monotonous utility function U such that for each $T \in \partial U(c^*, g^*)$, with $(c^*, g_0^*) \ggg 0$, the partial supergradient on gold has no pure charge (as the next example illustrates). That is, there exist $\mu_0^N, \mu_0^G \in (0, +\infty)$, $\mu^N, \mu^G \in \ell_+^1$ and a positive pure charge ν^N so that $T(c, g) = \mu_0^N c_0 + \mu_0^G g_0 + (\mu^N + \nu^N)(c_1) + \mu^G(g_1)$ for all (c, g). This leads us to:

Proposition 10: Asset Pricing

Suppose $x^* = (c_0^*, c_1^*, g_0^*, g_1^*)$, with $(c_0^*, c_1^*, g_0^*) \ggg 0$, maximizes U on $B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i)$. For each $\mu^N + \nu^N + \mu^G \in \partial U(x^*)$,

(i) if there exists $\delta U(x^*, v_j)$ for $v_j = (-q(e_j), R(e_j), 0, 0)$, then

$$\mu_0^N q(e_j) = \sum_{s=1}^{\infty} \mu_s^N R(e_j)_s + \nu^N(R(e_j));$$

(ii) if there exists $\delta U(x^*, v)$ for $v = (-\rho_0, \rho_1, 1, 0)$, then

$$\mu_0^N \rho_0 = \mu_0^G + \sum_{s=1}^{\infty} \mu_s^N \rho_s + \nu^N(\rho).$$

(The proof is immediate as, for each $T \in \partial U(x^*)$ and any of the directions w considered, $T \cdot w = \delta U(x^*; w) = 0$; the directional derivatives exist in the Example 4 below but even when these do not exist we can establish inequality conditions)

Remark 6: The weights μ_s^N are the state prices, whereas $\nu^N(R(e_j))/\mu_0^N$ and $\nu^N(\rho)/\mu_0^N$ (when positive) are the price bubbles in the financial assets and gold, respectively. The bubble on gold is the excess of its price over both the marginal rate of substitution between gold and the numeraire at date 0 and the series of its deflated prices at date 1[32].

Definition: An *equilibrium for the two date economy* is a vector $(\rho, q, (c^i, g^i, z^i)_i)$ such that: (i) $\forall i$, (c^i, g^i) maximizes U^i on $B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i)$ and (c^i, g^i, z^i) satisfies (13) and (14); (ii) $\sum_{i=1}^{I}(c_0^i - \omega_0^i) = 0$ and $\sum_{i=1}^{I}(c_1^i - \omega_1^i - R(z_0^i)) = 0$; (iii) $\sum_{i=1}^{I}(g_0^i - e_0^i) = 0$ and $\sum_{i=1}^{I}(g_1^i - e_1^i - g_0^i) = 0$; (iv) $\sum_{i=1}^{I}(z^i - z_0^i) = 0$.

We will relate the two date economy and an associate AD economy whose consumption bundles are of the form (c_0, c_1, g_0, g_1). AD endowments are related to the two date endowments by $W_0^i = \omega_0^i$, $W_1^i = \omega_1^i + R(z_0^i)$, $E_0^i = e_0^i$ and $E_1^i = e_1^i + e_0^i \mathbb{1}$. AD prices (taking c_0 as numeraire) are specified by prices for (c_1, g_0, g_1) given by $(\pi, \gamma_0, \gamma_1)$, where $\pi \in ba$, $\gamma_0 \in (0, +\infty)$ and $\gamma_1 \in ba$. The AD budget set $B_{\mathrm{AD}}(\pi, \gamma, W^i, E^i)$ is the set of bounded vectors $(c_0, c_1, g_0, g_1) \geq 0$ satisfying

$$c_0 - W_0^i + \gamma_0(g_0 - E_0^i) + \pi(c_1 - W_1^i) + \gamma_1(g_1 - E_1^i) \leq 0. \tag{15}$$

[32] Asset pricing could have been characterized using a non-arbitrage deflator, not necessarily given by the marginal rates of substitution. Without gold, in the absence of arbitrage opportunities, there exists π in the dual of ℓ^∞ such that $q \cdot z = \pi \circ R(z)$. Non-arbitrage in the presence of a durable good could be characterized also (see Araujo, Fajardo and Páscoa [2] for the finite state case).

Proposition 11: AD and Two Date Budgets

(i) If $(c, g) \in B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i)$ with (13) and (14) holding with equality at some z and there exists π is such that $q(y) = \pi \circ R(y)\ \forall y \in \ell^\infty$, then $(c, g) \in B_{\mathrm{AD}}(\pi, \gamma, W^i, E^i)$, where $\gamma_0 = \rho_0 - \pi(\rho_1)$ and $\gamma_1(\widetilde{g}_1) = \pi((\rho_{1s}\widetilde{g}_{1s})_s)\ \forall \widetilde{g}_1 \in \ell^\infty$.

(ii) Assume that R is onto and z_0^i is such that $W_1^i - R(z_0^i) \geq 0$.
If $(c, g) \in B_{\mathrm{AD}}(\pi, \gamma, W^i, E^i)$ and $\exists\, \rho_1 \in \ell^\infty : \gamma_1(\widetilde{g}_1) = \pi((\rho_{1s}\widetilde{g}_{1s})_s)\ \forall \widetilde{g}_1 \in \ell^\infty$, then we have $(c, g) \in B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i)$ where $\rho_0 = \gamma_0 + \pi(\rho_1)$ and $q(y) = \pi \circ R(y)\ \forall y \in \ell^\infty$.

Remark 7: Proposition 11 (proven in Appendix E) can be used to relate AD and two date equilibria. Under injectivity (besides surjectivity) of R, the former induces the latter. In fact, by item (ii), at the AD allocation, consumption plans lie in $B_{\mathrm{G}}(\rho, q, \omega^i, e^i, z_0^i)$; optimality within this set follows by item (i) and, finally, asset market clearing holds by injectivity of R since $\sum_i (c_{1s}^i - W_{1s}^i + \rho_{1s}(g_{1s}^i - E_{1s}^i + e_0^i - g_0^i)) = 0\ \forall s$ implies $R(\sum_i (z^i - z_0^i)) = 0$. The converse is also true (by using items (i) and (ii) in the reverse order).

Example 4: A Bubble in Gold

Consider two consumers $i = 1, 2$ whose preferences are described by

$$U^i(c, g) = \sqrt{c_0} + g_0 + (1-\epsilon)\sum_{s=1}^{\infty} \mu_s(\sqrt{c_{1s}} + g_{1s}) + \epsilon \inf_{s \in \mathbb{N}} \sqrt{c_{1s}},$$

where $\epsilon \in (0, 1)$ and $\mu_s = (\frac{1}{2})^{s-1}$ for any $s \geq 1$. Numeraire endowments are $W_0^i = 1, W_1^1 = ((\frac{s+1}{s})^2 + \psi_s))_{s\in\mathbb{N}}, W_1^2 = ((\frac{s+1}{s})^2 - \psi_s))_{s\in\mathbb{N}}$, with $\psi_s = -1/4$ if s is even and $\psi_s = 1/2$ if s is odd. Gold endowments are $E_0^i = 1/2$, $E_1^i = \mathbb{1}$. We claim that AD prices are defined by

$$\widetilde{\pi}(c, g) = c_0 + 2g_0 + (1-\epsilon)\sum_{s=1}^{\infty} \mu_s \left(\frac{s}{s+1}c_s + 2g_s\right) + \epsilon(B(c_1) + B(\rho_1 g_1)),$$

where $\rho_1 = (2\frac{(s+1)}{s})_{s\in\mathbb{N}}$ and B is a Banach limit. AD consumption plans are $x^i = (c^i, g^i)$ with $c^i = (1, ((\frac{s+1}{s})^2)_{s\in\mathbb{N}})$ and $g^1 = (\frac{1}{2}, 2\mathbb{1}_O)$, $g^2 = (\frac{1}{2}, 2\mathbb{1}_{O^c})$, where O is the set of odd natural numbers.

In fact, since $\frac{1}{2}\widetilde{\pi} \in \partial U^i(x^i) + \partial \chi_{\ell_+^\infty}(x^i)$ and $\widetilde{\pi}(x^i - (W^i, E^i)) = 0$ holds for an appropriate choice of ϵ (see Appendix E), x^i is optimal by condition (12). Notice that the absence of a pure charge in the partial supergradient of U^i with respect to gold is compensated by a pure charge in the respective supergradient of $\chi_{\ell^\infty_+}$ (as g^1_1 is on the norm boundary of the orthant). Market-clearing is immediate.

At last, using Remark 7, we get an equilibrium for the two date economy with a bubble in gold. Selling gold helps to hedge the numeraire shocks.

When markets are incomplete and there is just one commodity, defining the consumption set of each agent i as $(R(\ell^\infty) + W^i) \cap \ell^\infty_+$ and restricting utility functions to these sets, AD equilibrium prices will be bounded linear functionals on the set of net trades $R(\ell^\infty)$[33]. Existence of AD equilibrium with these consumption sets follows from Theorem 1 in Bewley (1972), provided that $R(\ell^\infty)$ is Mackey closed. Redoing Propositions 10 and 11 for the one good case, when $\dim R(\ell^\infty)$ is not finite, pure charges in AD prices induce asset bubbles[34].

6 Concluding Remarks and Further Extensions

We show that when infinite lived consumers are wary, positive net supply assets can have bubbles, even under complete markets and finite present value of wealth. Transversality conditions no longer prevent a creditor at infinity. In our examples, the wary attitude is formulated as a maxmin problem. Agents maximize the minimum series of discounted utilities, over a certain class of discount factors (more specifically, a Choquet integral with respect to a convex capacity not continuous at the full set). This is *an aversion to ambiguity in discount factors.* A large class of efficient allocations (illustrated in the examples and characterized in the two theorems) are sequentially implementable with a bubble in the price of the infinite lived asset that complete the markets. Thus, this bubble is efficient. Moreover, the bubble is essential, as the allocations can not be implemented without it.

Similar examples, in a two date context, with a countable infinite set of states, exhibit an analogous *aversion to ambiguity in beliefs.* In both contexts, Arrow-Debreu prices fail to be countably additive and are implemented with asset prices above the series of deflated returns, even when present values of wealth are finite and assets are in positive net supply. We addressed the precautionary bubble in a durable good (as gold) that plays the role of a commodity-money, even when

[33]The restriction of an element in $(\ell^\infty)^*$ to $R(\ell^\infty)$ is a bounded linear functional on $R(\ell^\infty)$.

[34]However, as not all Arrow securities are available, some states will have more than one state price (given by different non-arbitrage deflators).

agents are not wary with respect to its consumption. Actually, if agents were concerned about the infimum of the consumption of the durable good itself (as in the case of housing), it could much easier to give an example of a bubble.

A study of the bursting of bubbles was beyond the scope of this paper. However, it seems to us to be a promising issue, as the realization of some events may change the precautionary attitude. For instance, in a stochastic sequential economy with preferences analogous to the ones in the guiding examples, if the precaution coefficient β is path-dependent or in a subtree consumption is higher at infinity than at some date, the bubble could burst.

In spite of difficulties, evidenced by Theorems 1 and 2, to deal with assets whose returns are not strictly positive, we intend to explore this framework to study fiat money and monetary equilibria as well as other macroeconomic issues such as Ricardian equivalence and taxation.

APPENDIX

A Notation and Basic Concepts

A.1 The Space ℓ^∞

The space ℓ^∞ is the Banach space of real bounded sequences equipped with the norm defined by $\|x\| = \sup_t |x_t|$. The space ℓ^1 is the Banach space set of absolutely convergent real sequences equipped with the norm defined by $\|x\|_1 = \sum_{t=1}^{\infty} |x_t|$.

Given $x \in \ell^\infty$, we denote by x_s its s-th term. We say that x is *nonnegative* (and write $x \geq 0$) when $x_s \geq 0 \; \forall s \in \mathbb{N}$. We write $x \gg 0$, when $x_s > 0 \; \forall s$, and $x \ggg 0$, when $\exists \, h > 0$ such that $x_s \geq h$ for each s. Given x and x' in ℓ^∞, we write $x > x'$ if $(x - x') \geq 0$ and $x \neq x'$. The *positive orthant* of a Banach space X with respect to a pre-order is the subset X_+ of elements that dominate the origin (in particular, $\ell^\infty_+ = \{x \in \ell^\infty : x \geq 0\}$). The set $\mathrm{int}_{\|.\|}\ell^\infty_+$ denotes the interior of ℓ^∞_+ with respect to the norm topology. Now, $x \in \mathrm{int}_{\|.\|}\ell^\infty_+$ if and only if $x \ggg 0$. Moreover, $\langle y, x \rangle$ denotes $\sum_{s=1}^{\infty} y_s x_s$, when this series is defined. We denote by e_t the t-th *canonical direction*, that is, the sequence such that $(e_t)_t = 1$ and $(e_t)_s = 0$ otherwise.

When ℓ^∞ is endowed with a particular topology Γ, its *(topological) dual* with respect to Γ is the set of linear Γ-continuous functionals on ℓ^∞. If Γ is the norm topology, the dual is denoted by $(\ell^\infty)^*$. A coarser topology is the *Mackey topology*, defined as the strongest topology on ℓ^∞ for which [35] the dual is ℓ^1. A net (x^α) converges to x in this topology if and only if, for any weakly compact subset A of ℓ^1, $\langle x^\alpha, y\rangle \to \langle x, y\rangle$ uniformly on $y \in A$. We give now the proof of Lemma 3 (that will be used often and is crucial to understand why continuity of preferences with respect to the Mackey topology characterizes impatience):

Lemma 3: *$\forall x \in \ell^\infty$, given a decreasing sequence $(A_n)_n$ of subsets of $\mathbb{N}$ with $\cap_n A_n = \emptyset$, $x_{A_n} \to 0$ in the Mackey topology.*

Proof: We have to show that, for any weakly compact subset K of ℓ^1, $\langle y, x_{A_n}\rangle$ tends to zero, uniformly on $y \in K$. Denote by y^+ and y^- the positive and the negative parts of y, so that $y = y^+ - y^-$ and $|y| = y^+ + y^-$. Let $\overline{x}$ and $\underline{x}$ be the supremum and the infimum, respectively, of the sequence x. Then $\underline{x}\,\langle y^+, \mathbb{1}_{A_n}\rangle - \overline{x}\,\langle y^-, \mathbb{1}_{A_n}\rangle \leq \langle y, x_{A_n}\rangle \leq \overline{x}\,\langle y^+, \mathbb{1}_{A_n}\rangle - \underline{x}\,\langle y^-, \mathbb{1}_{A_n}\rangle$.
We know that $\langle y, \mathbb{1}_{A_n}\rangle$ converges to zero uniformly in $y \in K$ (see Bewley [5], Remark 24, p.534). Now, $\langle |y|, \mathbb{1}_{A_n}\rangle = \langle y, \mathbb{1}_{A_n}\rangle + 2\langle y^-, \mathbb{1}_{A_n}\rangle$, where $\langle y^-, \mathbb{1}_{A_n}\rangle = \sum_{t\in A_n} y_t^- \leq \sum_{t\in A_n} |y|_t$. As K is weakly compact, the set $\{|y|\}_{y\in K}$ will be weakly sequentially compact (see Corollary 10 in Dunford and Schwartz [13], p.293). Hence, $\sum_{t\geq n} |y|_t$ converges uniformly to zero on $y \in K$ (by Theorem 9 in Dunford and Schwartz [13], p.292).
Now, for each n let $m(n) = \min A_n$. Then, $A_n \subset \{m(n), m(n)+1, ...\}$ and $m(n) \to \infty$ (otherwise $\cap_n A_n$ would not be empty). Then $\sum_{t\geq m(n)} |y|_t$ converges uniformly to zero on $y \in K$ and, as $\sum_{t\in A_n} |y|_t \leq \sum_{t\geq m(n)} |y|_t$, the result follows since $y^+ = \frac{1}{2}(y + |y|)$ and $y^- = \frac{1}{2}(|y| - y)$. *Q.E.D.*

Proof of Proposition 1:
The condition is sufficient, since $y_{E_n^c} \to y$ in the Mackey topology and $x \not\succsim y$. Now, in order to show the necessity, suppose that the lower contour set of x is not Mackey closed. There is a net (y^α) converging to some y, in the Mackey topology, such that $y \succ x$ and $x \succsim y^\alpha\ \forall\alpha$. Thus, $\langle y^\alpha - y, z\rangle \to 0$ uniformly in z belonging to any weakly compact subset of ℓ^1. Given n, let $K_n = \{e_1, ..., e_n\}$, then, for $1 \leq j \leq n$, the real net y_j^α converges to y_j. Moreover, $x \succsim y^\alpha_{E_n^c}\ \forall\alpha$. On the other

[35] Observe that each $x \in \ell^1$ can be identified with a linear functional by the rule $y \mapsto \langle x, y\rangle$.

hand, $y^{\alpha}_{E_n^c} \to y_{E_n^c}$ in the norm topology, so $x \succsim y_{E_n^c}$ (since the lower contour set of x is norm closed). *Q.E.D.*

LEMMA 4: *Under A1, preferences are not usc at x if and only if $\exists\, y \prec x$ and $\exists\, m \in I\!R_+$ such that, $\forall n$, $x \precsim (y_1, ..., y_n, m, m, ...)$.*

PROOF: It is easy to adapt the proof of Proposition 1. Sufficiency is immediate and necessity follows by noticing that when the net (y^{α}) converges in the Mackey topology to y, with $x \precsim y^{\alpha}$ for each α, this net is bounded (as it is weak* convergent), say by $m \in I\!R_+$. So $x \precsim y^{\alpha}_{E_n^c} + m1\!\!1_{E_n}$ $\forall \alpha$. *Q.E.D.*

A.2 The Norm Dual of ℓ^{∞}, the Space of Charges

The norm dual $(\ell^{\infty})^*$ is larger than ℓ^1. Let us recall its characterization.

Given a set Ω and a field $\mathcal{F}$ of its subsets, a set function $\mu : \mathcal{F} \to I\!R$ is said to be a *charge* (or *bounded finitely additive*) when (i) $\mu(\emptyset) = 0$, (ii) there is $m \in I\!R_+$ such that $|\mu(A)| \leq m$ $\forall A \in \mathcal{F}$ and (iii) if $A, B \in \mathcal{F}$ are such that $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$. Besides, we say that μ is *countably additive* whenever $A_1, A_2, ..., A_n, ... \in \mathcal{F}$, with $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$ and $A_j \cap A_k = \emptyset$, for $k \neq j$, implies $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

We say that a charge is nonnegative (and write $\mu \geq 0$) when μ is such that $\mu(A) \geq 0$ for all $A \in \mathcal{F}$. If $\mu \geq 0$ and there is $A \in \mathcal{F}$ such that $\mu(A) > 0$, we say that μ is positive (and write $\mu > 0$). If a charge $\mu \geq 0$ is countably additive, it is called a *measure* on $(\Omega, \mathcal{F})$. A measure μ with $\mu(\Omega) = 1$ is said to be a *probability measure*. Denote by $ba(\Omega, \mathcal{F})$ and $ca(\Omega, \mathcal{F})$, the sets of charges and of countably additive set functions on $(\Omega, \mathcal{F})$, respectively (so, $ca(\Omega, \mathcal{F}) \subset ba(\Omega, \mathcal{F})$). When $\Omega = \mathbb{N}$ and $\mathcal{F} = 2^{\mathbb{N}}$, we let $\mu_n = \mu(\{n\})$ and $ba = ba(\mathbb{N}, 2^{\mathbb{N}})$ (and $ca = ca(\mathbb{N}, 2^{\mathbb{N}})$).

The space ba can be put in a one-to-one isometric correspondence with $(\ell^{\infty})^*$ by associating with $\mu \in ba$ the functional $x \mapsto \int_{\mathbb{N}} x d\mu \quad \forall x \in \ell^{\infty}$. Similarly, ca can be put in a one-to-one isometric correspondence with ℓ^1 by associating $\mu \in ca$ with $y \in \ell^1$ by the rule $\langle y, x \rangle = \int_{\mathbb{N}} x d\mu \quad \forall x \in \ell^{\infty}$. On the first correspondence, the operator $\int \cdot\, d\mu$ denotes the Dunford integral, whereas on the second one, it denotes the Lebesgue integral [36]. These two isomorphisms allow us to use the notation $\mu(x)$, $x \in \ell^{\infty}$, for a given charge μ.

[36] See Dunford and Schwartz [13] on these two results, p.258 and p.176, respectively.

A charge $\nu \geq 0$ is a *pure charge* when $[\lambda \in ca_+,\ \nu \geq \lambda \Rightarrow \lambda \equiv 0]$. More generally, a charge ν is a pure charge if there are pure charges $\nu^+, \nu^- \geq 0$ such that $\nu = \nu^+ - \nu^-$. Denote by *pch* the set of pure charges on $(\mathbb{N}, 2^{\mathbb{N}})$. We have the following decomposition result due to Yosida-Hewitt (see Bhaskara Rao and Bhaskara Rao [7], Theorem 10.2.1, p.241):

PROPOSITION 12: *Any $\pi \in ba$ can be written in the form $\pi = \mu + \nu$ where $\mu \in ca$ and $\nu \in pch$. Furthermore, this decomposition is unique.*

REMARK 8: If $\pi \geq 0$ and $\pi = \mu + \nu$ with $(\mu, \nu) \in ca \times pch$, then $\mu \geq 0$ and $\nu \geq 0$.

Finally, we have the following useful properties.

LEMMA 5: *Let B be a finite subset of $\mathbb{N}$. If $\nu \in pch_+$, then $\nu(B) = 0$.*

PROOF: By Theorem 10.3.2 in Bhaskara Rao and Bhaskara Rao [7], given $\lambda \in ca$ and $\varepsilon > 0$, there is $A_{\lambda,\varepsilon} \subset \mathbb{N}$ such that $\lambda(A^C_{\lambda,\varepsilon}) < \varepsilon$ and $\nu(A_{\lambda,\varepsilon}) = 0$. Let us take λ_0 defined by $\lambda_0(A_1) = \sum_{n \in B \cap A_1} \frac{1}{\#B}$ for $A_1 \subset \mathbb{N}$ and $\varepsilon_0 < \frac{1}{\#B}$. In this case, the set $A_{\lambda_0,\varepsilon_0}$ must contain B. Thus $0 = \nu(A_{\lambda_0,\varepsilon_0}) \geq \nu(B) \geq 0$, as claimed. *Q.E.D.*

LEMMA 6: *Let $\nu > 0$ be a pure charge such that $\nu(\mathbb{1}) = 1$. Then, $\nu(x) \in [\liminf x, \limsup x]$, for any $x \in \ell^\infty$. In other words, ν is a generalized limit.*

PROOF: Given $x \in \ell^\infty$ and $\varepsilon > 0$, $\exists\, n_0$ such that $\limsup x + \varepsilon \geq x_n \geq \liminf x - \varepsilon$ for all $n > n_0$. So, $(\limsup x + \varepsilon)\mathbb{1} - x_{E_n} \geq 0$ for n large enough. Now, $\nu((\limsup x + \varepsilon)\mathbb{1} - x_{E_n}) = \int [(\limsup x + \varepsilon)\mathbb{1} - x_{E_n}]\, d\nu \geq 0$ for n large enough (see Dunford and Schwartz [13], Lemma 14, p.108). Thus, $\limsup x + \varepsilon = (\limsup x + \varepsilon)\nu(\mathbb{1}) \geq \nu(x_{E_n}) = \nu(x)$, since ν is a pure charge, by Lemma 5. As $\varepsilon > 0$ is arbitrary, $\limsup x \geq \nu(x)$. Similarly, $\liminf x \leq \nu(x)$. *Q.E.D.*

LEMMA 7: *Given an infinite ordered subset N' of $\mathbb{N}$, there exists a generalized limit* LIM *such that, for each $x \in \ell^\infty$,* $\mathrm{LIM}(x) = \lim_{n \in N'} x_n$ *when this limit exists.*

PROOF: Define a continuous linear functional $\nu : \ell^\infty \to I\!R$ in the following way: consider the collection of points $y \in \ell^\infty$ such that the subsequence (y_{n_i}) with $n_i \in N'$ converges. The function mapping each of these points y into $\lim_{n_i \in N'} y_{n_i}$ is linear, nonnegative and continuous in the norm topology, so it can be extended to a linear, nonnegative and continuous functional ν on the whole space (see Schaefer [31], p.227, Corollary 2). Now ν is a pure charge if and only if for every $\lambda \in \ell^1$ and $\varepsilon > 0$ there is $B \subset \mathbb{N}$ such that $\nu(\mathbb{1}_B) = 0$ and $\lambda(\mathbb{1}_{B^c}) < \varepsilon$ (again by Theorem 10.3.2 in Bhaskara Rao and Bhaskara Rao [7]). This holds since we can find a finite set B such that $\lambda(\mathbb{1}) - \sum_{t \in B} \lambda_t < \epsilon$ and, moreover, $\nu(\mathbb{1}_B) = 0$ by definition of ν. *Q.E.D.*

A.3 On Supergradients

Let U be a concave extended real valued function on ℓ^∞. A supergradient of U at x is a functional $T \in (\ell^\infty)^*$ such that $U(x+h) - U(x) \leq Th$ for any $h \in \ell^\infty$. The set of all supergradients of U at x is called the superdifferential of U at x and is denoted by $\partial U(x)$.

Given $x \in D$ and $v \in \ell^\infty$, $\lim_{h \to 0} \frac{U(x+hv)-U(x)}{h}$, when it exists, is called the *directional derivative* of U at x along (the direction) v and it is denoted by $\delta U(x; v)$. The limit evaluated only for $h > 0$ (or only for $h < 0$) always exists for $x \in D$ and is called the *right-directional derivative*, with notation $\delta^+ U(x; v)$ (respectively, the *left-directional derivative*, with notation $\delta^- U(x; v)$).

Let us see an additional property for pure charge components of a supergradient. We saw in Lemma 6 that each positive pure charge is a distortion of a generalized limit. Now, we will say more about the distortion coefficient.

LEMMA 8: *Let $T = \mu + \nu \in \partial U(x)$ such that $(\mu, \nu) \in ca \times pch$. There are a generalized limit* LIM *and a positive constant $\alpha \in [\lim_n \delta^+ U(x; \mathbb{1}_{E_n}), \lim_n \delta^- U(x; \mathbb{1}_{E_n})]$ such that $\nu(x) = \alpha \,\mathrm{LIM}(x) \;\; \forall \; x \in \ell^\infty$.*

PROOF: We just need to show that α belongs to the mentioned interval. Given $n \in \mathbb{N}$, it is true that $\delta^+ U(x; \mathbb{1}_{E_n}) \leq T(\mathbb{1}_{E_n}) \leq \delta^- U(x; \mathbb{1}_{E_n})$. Moreover, $T(\mathbb{1}_{E_n}) = \sum_{t > n} \mu_t + \nu(\mathbb{1}_{E_n})$. Since, $\forall n$, $\nu(\mathbb{1}_{E_n}) = \nu(\mathbb{1}) = \alpha$ and $\lim_n \sum_{t>n} \mu_t = 0$, we get $\lim_n \delta^+ U(x; \mathbb{1}_{E_n}) \leq \alpha \leq \lim_n \delta^- U(x; \mathbb{1}_{E_n})$. *Q.E.D.*

Notice that the constant α in the statement of this lemma is actually the norm of the pure charge: $\alpha = \|\nu\|_{ba} = \sup\{\nu(x) : \|x\| \leq 1\} = \nu(\mathbb{1})$.

B On Subsection 3.2.2

B.1 On Ambiguity Aversion: Proofs

Let $M(\mathbb{N})$ be the set of all probability measures on $(\mathbb{N}, 2^{\mathbb{N}})$. Suppose that $u : I\!R_+ \to I\!R$ is continuous. We can state:

LEMMA 9: *If $\mu \in M(\mathbb{N})$, $\epsilon \in [0,1)$ and ν is the ϵ-contamination capacity associated, then*

$$\min_{\eta \in \mathrm{core}(\nu)} \int_{\mathbb{N}} u \circ x \, d\eta = \inf_{\substack{\eta \in M(\mathbb{N}) \\ \eta \geq \nu}} \int_{\mathbb{N}} u \circ x \, d\eta = (1-\epsilon) \int_{\mathbb{N}} u \circ x \, d\mu + \epsilon \inf_{s \in \mathbb{N}} u(x_s) \tag{16}$$

is true for all $x \in \ell^\infty_+$.

PROOF: Denote $I(\nu) = \min_{\eta \in \mathrm{core}(\nu)} \int u \circ x \, d\eta$ and $F(\nu) = \inf_{\substack{\eta \in M(\mathbb{N}) \\ \eta \geq \nu}} \int u \circ x \, d\eta$. It is clear that $F(\nu) \geq I(\nu)$. Let $\eta \in ba$ such that $\eta \geq \nu$ and $\eta(\mathbb{N}) = 1$. Thus $(\eta - (1-\epsilon)\mu) \in ba_+$. So, we get $\int u \circ x \, d\eta = (1-\epsilon) \int u \circ x \, d\mu + \int u \circ x \, d(\eta - (1-\epsilon)\mu) \geq (1-\epsilon) \int u \circ x \, d\mu + \epsilon \inf u \circ x$, hence $I(\nu) \geq (1-\epsilon) \int u \circ x \, d\mu + \epsilon \inf u \circ x$.
On the other hand, let (y_n) be a sequence in $x(\mathbb{N})$ such that $y_n \to \inf x$. Define $\varsigma_n = (1-\epsilon)\mu + \epsilon\, \theta_n$, where θ_n is the Dirac probability measure with mass at $s \in \mathbb{N}$ such that $x_s = y_n$. So $\varsigma_n \in M(\mathbb{N})$, $\varsigma_n \geq \nu$ and $I_n := \int u \circ x \, d\varsigma_n = (1-\epsilon) \int u \circ x \, d\mu + \epsilon\, u(y_n) \geq F(\nu)$. Since $I_n \to (1-\epsilon) \int u \circ x \, d\mu + \epsilon \inf u \circ x$, we are done because we get $(1-\epsilon) \int u \circ x \, d\mu + \epsilon \inf u \circ x \geq F(\nu) \geq I(\nu) \geq (1-\epsilon) \int u \circ x \, d\mu + \epsilon \inf u \circ x$. *Q.E.D.*

C On Subsection 3.3 and Examples 1 and 5

PROOF OF LEMMA 1:
We have $U(e_t + x) > U(x)$ for any t. For any $\alpha \in I\!R$, $e_t + x - \alpha \mathbb{1}_{E_n}$ converges in the Mackey topology to $e_t + x$. Let us pick a positive α such that $x - \alpha \mathbb{1}_{E_n} \in \ell^\infty_+$. Hence, for n sufficiently large, without loss of generality bigger than t, it is true that $U(e_t + x - \alpha \mathbb{1}_{E_n}) > U(x)$. However, for any $T \in \partial U(x)$ and n large enough, we know that $0 < U(e_t + x - \alpha \mathbb{1}_{E_n}) - U(x) \leq T(e_t - \alpha \mathbb{1}_{E_n})$. Now, $T = \mu + \nu$, where $\mu \in \ell^1$ and ν is a pure charge. If $T \notin \ell^1$, then, as U is increasing, μ and ν are positively valued on $\mathrm{int}_{\|.\|} \ell^\infty_+$. This implies that, for n large enough, $0 < \mu_t - \alpha(\sum_{s>n} \mu_s + \nu(\mathbb{1}_{E_n})) < \mu_t - \alpha \nu(\mathbb{1}_{E_n})$. As ν is a pure charge, we have $\nu(\mathbb{1}_{E_n}) = \nu(\mathbb{1}) > 0$. Thus, we can choose t sufficiently large such that $\mu_t - \alpha \nu(\mathbb{1}_{E_n}) < 0$, a contradiction. *Q.E.D.*

C.1 The Superdifferential of Preferences (7)

Let us characterize ∂U for U given by (7). By Lemma 6, for each $T \in \mathrm{ba}_+$ there exist a real $\alpha^T \geq 0$ and a generalized limit LIM^T such that, at every $x \in \ell^\infty$, $T(x) = \sum_{t+1}^{\infty} T(e_t)x_t + \alpha^T \mathrm{LIM}^T(x)$. Denote by $\widehat{\partial} U(x) \subset \mathrm{ba}$ the set of linear and norm continuous operators T such that $T(e_t) = u'(x_t)(\zeta_t + \gamma_t \beta)$ and $\alpha^T = \sigma\beta u'(\underline{x})$ where (i) $\gamma_t \geq 0 \; \forall\, t \geq 1$, (ii) $\gamma_t = 0$, if $x_t > \underline{x}$, (iii) $\sigma \geq 0$ is zero when $\underline{x}$ is not a cluster point of the sequence x and (iv) $\sum_{t=1}^{\infty} \gamma_t + \sigma = 1$.

If $\underline{x}$ is a cluster point of x, there exists an infinite ordered set $\mathbb{N}_1 \subset \mathbb{N}$ such that $(x_n)_{n \in \mathbb{N}_1}$ converges to $\underline{x}$. Define $\widetilde{\partial} U(x)$ as the set of charges T that fulfill (i) through (iv) with the additional condition that $\mathrm{LIM}^T(y) = \lim_{n \in \mathbb{N}_1} y_n$ when this limit exists. We state that:

LEMMA 10: *If $\underline{x}$ is a cluster point of the sequence $x \ggg 0$, then $\widetilde{\partial} U(x) \subset \partial U(x) \subset \widehat{\partial} U(x)$. Otherwise, $\partial U(x) = \widehat{\partial} U(x)$.*

PROOF: First, we will show that (in any case) $\partial U(x) \subset \widehat{\partial} U(x)$. Let us define $U^t : I\!R \to I\!R \cup \{-\infty\}$ by $U^t(z) = U(x_1, ..., x_{t-1}, z, x_{t-2}, ...)$. Given T an operator in $\partial U(x)$, we have, for each t, $T(e_t) \in \partial U^t(x_t) \equiv [\zeta_t u'(x_t), (\zeta_t + \beta)u'(x_t)]$ and, so, there is $\gamma_t \in [0, 1] \; : \; T(e_t) = (\zeta_t + \gamma_t \beta)u'(x_t)$. When $x_t > \underline{x}$, the directional derivative $\delta U(x; e_t)$ there exists and it is equal to $\zeta_t u'(x_t)$, which implies $\gamma_t = 0$. As observed, the pure charge component of T can be written as $\alpha^T \mathrm{LIM}^T$. Define $\sigma = \alpha^T/\beta u'(\underline{x}) \geq 0$ and $\underline{\mathbb{N}}$ be the ordered subset of $\mathbb{N}$ composed by all the indices $t \; : \; x_t = \underline{x}$. It is true that $T(1\!\!1) = \sum_{t \geq 1}(\zeta_t + \gamma_t)u'(x_t) + \sigma\beta u'(\underline{x}) = \sum_{t \geq 1} \zeta_t u'(x_t) + (\sum_{t \in \underline{\mathbb{N}}} \gamma_t + \sigma)\beta u'(\underline{x})$. Now $T(1\!\!1) = \delta U(x; 1\!\!1)$ where $\delta U(x; 1\!\!1) = \sum_{t \geq 1} \zeta_t u'(x_t) + \beta u'(\underline{x})$. So, $\sum_{t=1}^{\infty} \gamma_t + \sigma = \sum_{t \in \underline{\mathbb{N}}} \gamma_t + \sigma = 1$. It remains to show that $\sigma = 0$ when $\underline{x}$ is not a cluster point. Suppose $\sigma > 0$. There are $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ such that $x_t > \underline{x} + \varepsilon$ for all $t > t_0$. Given $n \in \mathbb{N}$, let $\widetilde{x}^n$ be the sequence $(x_1, ..., x_n, \underline{x} + \frac{\varepsilon}{2}, \underline{x} + \frac{\varepsilon}{2}, ...)$. Then $\widetilde{x}^n_t - x_t < -\varepsilon/2$ for all $t > t_0$. As $T \in \partial U(x)$, the inequality $U(\widetilde{x}^n) - U(x) \leq T(\widetilde{x}^n - x)$ holds for each n. If $n > t_0$, it is true that $U(\widetilde{x}^n) - U(x) = \sum_{t > n} \zeta_t (u(\widetilde{x}^n_t) - u(x_t))$ and the left hand side converges to zero when n goes to infinity. Then, $\liminf_n T(\widetilde{x}^n - x) \geq 0$. However, for all n large enough, it is true that $T(\widetilde{x}^n - x) \leq -\frac{\varepsilon}{2} \sum_{t > n} T(e_t) + \sigma\beta u'(\underline{x})\mathrm{LIM}^T(\widetilde{x}^n - x) < \sigma\beta u'(\overline{x})\mathrm{LIM}^T(\widetilde{x}^n - x)$. As $\mathrm{LIM}^T(\widetilde{x}^n - x) \leq \mathrm{LIM}^T(-\frac{\varepsilon}{2} 1\!\!1) < 0$, we get a contradiction.

Now, suppose that $\underline{x}$ is not a cluster point and take an operator $T \in \widehat{\partial} U(x)$. Since $\sigma = 0$ and, so, $\sum_{t\in N_1} \gamma_t = 1$, it is clear that, for every $y \in \ell^\infty_+$, we have $U(y) \le \sum_{t\ge1}(\zeta_t + \gamma_t\beta)u(y_t)$ and that $U(x) = \sum_{t\ge1}(\zeta_t + \gamma_t\beta)u(x_t)$, which implies $U(y) - U(x) \le \sum_{t\ge1}(\zeta_t + \gamma_t\beta)(u(y_t) - u(x_t))$. Since $u(y_t) - u(x_t) \le u'(x_t)(y_t - x_t)$ holds for each t, we get $U(y) - U(x) \le \sum_{t\ge1}(\zeta_t + \gamma_t\beta)u'(x_t)(y_t - x_t) = T(y - x)$. If $y \in (\ell^\infty_+)^c$, $U(y) = -\infty$, so $U(y) - U(x) \le T(y - x)$ is immediate. Thus, we conclude that, in this case, $\widehat{\partial} U(x) = \partial U(x)$.

At last, suppose that $\underline{x}$ is a cluster point of x. Given $T \in \widetilde{\partial} U(x)$, if $\sigma = 0$, the same argument done in the last paragraph implies $T \in \partial U(x)$. Let us address the case $\sigma > 0$. Notice that, for $y \in \ell^\infty_+$, we get once more $\sum_{t\ge1}(\zeta_t + \gamma_t\beta)(u(y_t) - u(x_t)) \le \sum_{t\ge1}(\zeta_t + \gamma_t\beta)_t u'(x_t)(y_t - x_t)$. Now, we will show that $\inf(u \circ y) - \inf(u \circ x) \le u'(\underline{x})\mathrm{LIM}^T(x - y)$, what implies the desired inequality $U(y) - U(x) \le T(y - x)$. In fact, $\inf(u \circ y) - u(x_n) \le u(y_m) - u(x_n)$ $\forall n \in \mathbb{N}_1$ and $\forall m \in \mathbb{N}$. So, $\inf(u \circ y) - u(x_n) \le u'(x_n)(y_m - x_n)$ holds too. Making $n \in \mathbb{N}_1$ go to infinite and using that u is of class C^1 at $(0, +\infty)$, we get $\inf(u \circ y) - \inf(u \circ x) \le u'(\underline{x})(y_m - \underline{x})$. Finally, making $m \to +\infty$, we obtain $\inf(u \circ y) - \inf(u \circ x) \le u'(\underline{x})(\liminf y - \underline{x}) \le u'(\underline{x})(\mathrm{LIM}^T(y) - \mathrm{LIM}^T(x))$. *Q.E.D.*

C.2 On Examples 1 and 5

We present first the missing details of the computation of AD equilibria in Example 1 of Section 2 and, next, we give another example.

Example 1:

The price functional (given by Proposition 4) is induced by marginal utilities, which are the same for the two agents. In fact, for each i, $(u^i)'(x^i_t) = \frac{t}{t+8}$ and $(u^i)'(\inf x^i) = 1$. Hence, equation (2) holds with $\kappa = 1/2$. Finally, we have that $\pi W^2 = \pi x^2$ (and, therefore, $\pi W^1 = \pi x^1$) for some $\beta > 0$. In fact, $\pi(x^2 - W^2) = \sum_{t\ge1}(\frac{1}{2})^{t-1}\frac{t}{t+8}[\frac{t+8}{t} - h - \frac{1}{4}(\frac{t+8}{t})^2] + \beta(1 - h - \frac{1}{4}) = 2 - \sum_{t\ge1}(\frac{1}{2})^{t-1}[h\frac{t}{t+8} + \frac{1}{4}(\frac{t+8}{t})] + \beta(\frac{3}{4} - h)$. Since $3/4 > h$ and $[h\frac{t}{t+8} + \frac{1}{4}(\frac{t+8}{t})] > 0$ $\forall t \ge 1$ and bigger than 2 for $t = 1$, we can pick $\beta = \frac{1}{3/4-h}\{\sum_{t\ge1}(\frac{1}{2})^{t-1}[h\frac{t}{t+8} + \frac{1}{4}(\frac{t+8}{t})] - 2\} > 0$ in order to make $\pi(x^2 - W^2) = 0$. Thus, (π, x^1, x^2) is an AD equilibrium.

Example 5: Charge-Expected Utility

Consider a representative consumer economy where preferences are described by the following Dunford integral $U(x) = \int u \circ x \, d\nu$ for any $x \in \ell^\infty_+$, where $u : I\!R_+ \to I\!R$ is a concave strictly increasing function, of class C^1 on $(0, +\infty)$, and $\nu \in ba$ is not countably additive and for which $\nu_t > 0$ for any t. Loosely speaking, U is an expected utility with respect to a "probability" which is just finitely additive [37]. There exist $\mu \in ca$ and $\eta \in pch$ such that $\nu = \mu + \eta$. Moreover $\mu > 0$ and $\eta > 0$. Then

$$U(x) = \sum_{t=1}^{\infty} u(x_t)\mu_t + \eta(u \circ x). \tag{17}$$

Now, given $x \ggg 0$, denote by $\eta(u'(x)y)$ the value of η at the sequence with general term $u'(x_t)y_t$. Notice that $u(y_t) - u(x_t) \le u'(x_t)(y_t - x_t)$ implies $\eta(u \circ y) - \eta(u \circ x) \le \eta(u'(x)y) - \eta(u'(x)x)$. Let T be the linear functional defined by $T(y) = \sum_{t=1}^{\infty} u'(x_t)\mu_t y_t + \eta(u'(x)y)$ for any $y \in \ell^\infty$.

As the hypothesis made imply that $u'(x_t) \ge h$ for some $h > 0$ and all t, the functional $y \mapsto \eta(u'(x)y)$ is a positive pure charge. In fact, the functional is clearly linear and continuous in the sup norm, so it belongs to the dual of ℓ^∞ and, furthermore, its countably additive part is zero (as e_t is mapped into $\eta(u'(x)e_t) = 0$ since η is a pure charge). Finally, $\eta(u'(x)\mathbb{1}) \ge \eta(h\mathbb{1}) > 0$ and the positivity follows. Then, $T \in \partial U(x)$ and, therefore, given endowments $W = x$, we have that x is maximal on the budget set $\{y \in \ell^\infty_+ : T(y - W) \le 0\}$. So, the AD equilibrium price $T \notin$ ca.

Remark 9: Comparing with Examples 1 and 2, we did not need in this example to assume certain features for the optimal consumption bundles in order to obtain a pure charge in the Arrow-Debreu price (recall that in the previous examples, we had to suppose that the infimum consumption was a cluster point). Notice also that the utility function in this example is not just Mackey lower semi-discontinuous but is also Mackey upper semi-discontinuous [38] (and, therefore, the existence theorem in Bewley [5] could not be used to guarantee that AD equilib-

[37] The idea that countable additivity is just a regularity hypothesis and not an integral part of the probability concept dates back to de Finetti (in several papers from the 1930's), as Dubins and Savage [11] recall (p.10), sharing this view

[38] To show the upper semi-discontinuity, we need to show that the upper contour set might not be Mackey closed. Given a positive scalar m and a bounded sequence $y \ggg 0$, let $x^n_t = y_t - \varepsilon$ for $t \le n$ and $x^n_t = y_t + m$ otherwise. Then, $x^n \to y - \varepsilon\mathbb{1}$ in the Mackey topology (see Lemma 3). Now, $\exists h_y > 0 : u(y_t + m) - u(y_t) \ge h_y, \forall t$ and, therefore, $\eta((u(y_t + m) - u(y_t))_t) \ge \eta(\mathbb{1})h_y > 0$. So $\varepsilon > 0$ can be chosen small enough so that $\eta((u(y_t + m))_t) > \eta((u(y_t))_t) + \mu(y) - \mu(y - \varepsilon\mathbb{1})$. Then, $U(x^n) > U(y)$ but $U(y - \varepsilon\mathbb{1}) < U(y)$. Changing the signs of ε and m we could show the lower semi-discontinuity.

ria exist). In Araujo [1] it was proven that the Mackey topology is the strongest topology for which continuity of preferences (under the other assumptions in Bewley [5]) always implies the existence of equilibrium. This not precludes some particular and important examples of existence without Mackey lower or upper semi-continuity.

D On Section 4

D.1 On Necessary Optimality Conditions: Proofs

Proof of Proposition 5:

Consider the general case, when A4 is not assumed, stated in Remark 3 (ii). For $h > 0$, $\frac{U(x^*+hv(t))-U(x^*)}{h} \leq 0$ so $\delta^+U(x^*; v(t)) = \lim_{h\downarrow 0} \frac{U(x^*+hv(t))-U(x^*)}{h} \leq 0$. We know that there exists $T \in \partial U(x^*)$ such that $T(v(t)) = \delta^+U(x^*; v(t))$ since $\delta^+U(x^*; v(t)) = \inf\{L(v(t)) : L \in \partial U(x^*)\}$, where the infimum can be replaced by the minimum, as U is norm continuous at x^* and therefore $\partial U(x^*)$ is weak* compact (see Zeidler [37], Theorem 47.A, p.387). *Q.E.D.*

Lemma 11: *Let $x^* \ggg 0$ be maximal for U subject to $B_P(q, \omega, z_0)$ and z^* such that $x^* = x(z^*)$. Then,*

(i)
$$\lim_n \nu^n(x^* - \omega) \geq \limsup \mu_n^n q_n z_n^*$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components of the n-th term of some weak converging sequence $(T^n) \subset \partial U(x^*)$;*

(ii) If the direction $y(n)$ is right-admissible, for some n, then the following transversality condition holds:

$$\lim_n \nu^n(x^* - \omega) \leq \liminf \mu_n^n q_n z_n^*$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components of the n-th term of some weak converging sequence $(T^n) \subset \partial U(x^*)$;*

(iii) If the directional derivative $\delta U(x^*; y(n))$ exists and $y(n)$ is right-admissible, for some n, then, for every $\widetilde{T} \in \partial U(x^*)$, the following transversality condition holds

$$\nu(x^* - \omega) = \lim \mu_n q_n z_n^*$$

where $(\mu, \nu) \in ca \times pch$ is such that $\widetilde{T} = \mu + \nu$.

Proof: (i) For each n,

$0 \le \lim_{h\uparrow 0} \frac{U(x^*+hy(n))-U(x^*)}{h} = T^n(\sum_{t>n}^{\infty}(q_t(z^*_{t-1}-z^*_t)+R_t z^*_{t-1})e_t) - q_n z^*_n T^n e_n$,

where T^n is some supergradient of U at point x^* (since the limit is $\delta^- U(x^*; v_t)$, which is equal to $\max\{L(v_t) : L \in \partial U(x^*)\}$, as U is norm continuous at x^*). Now, $T^n = \mu^n + \nu^n$, where $\mu^n \in \ell^1$ and ν^n is a pure charge. Hence,

$$\sum_{t>n}^{\infty}(q_t(z^*_{t-1}-z^*_t)+R_t z^*_{t-1})\mu^n_t+\nu^n(\sum_{t>n}^{\infty}(q_t(z^*_{t-1}-z^*_t)+R_t z^*_{t-1})e_t)-q_n z^*_n \mu^n_n \ge 0. \quad (18)$$

Let $n \to \infty$. The sequence (T^n) lies in the weak* compact set $\partial U(x^*)$ and therefore has a subsequence converging, in the weak* topology, to some T in this set. Without loss of generality, we take this subsequence to be the initial sequence. Let us show that the associated pure charges sequence (ν^n) lies in a bounded set and, therefore, in a weak* compact set. Now, as (T^n) lies in a weak* compact set, there is $N > 0$ such that $\|T^n\|_{ba} \equiv \max\{|T^n(g)| : \|g\| \le 1\} \le N$. For every g in the unit ball of ℓ^∞, g_{E_m} is also in the unit ball. So, $\left|\sum_{k>m}\mu^n_k (g_{E_m})_k + \nu^n(g_{E_m})\right| = \left|T^n(g_{E_m})\right| \le N$. Taking the limit as m goes to ∞, $|\nu(g)| \le N$ since $\nu(g) = \nu(g_{E_m})$ for every m. Then we can take a subsequence of (ν^n) converging in the weak* topology to some ν and along this subsequence the associated countably additive components converges also to some μ. Again, without loss of generality, we stick to the same sequence. Then, (μ^n) converges in the weak topology of ℓ^1 to μ and, therefore, the union set of terms of the sequence and its limit constitutes a weakly compact set.

Let $\widetilde{y}(n) = y(n) + q_n z^*_n e_n$. By Lemma 3, the sequence $(\widetilde{y}(n))$ tends, as n goes to ∞, to zero in the Mackey topology (and, therefore, uniformly on weakly compact sets in ℓ^1). Hence, the series in inequality (18) tends to zero, as n goes to ∞. Notice that, for every n, $\nu^n(y(n)) = \nu^n(y(1))$ and, as (ν^n) converges in the weak* topology, $\lim \nu^n(y(1))$ exists. This completes the proof of item (i). The proof of item (ii) is analogous. To see (iii), an equality holds now in (18), for any $T \in \partial U(x^*)$ decomposable into a countably additive part μ and a pure charge ν. The series also tends to zero as $n \to \infty$. By the argument at the end of the proof of item (i) we complete the proof. *Q.E.D.*

D.2 AD Budget Set and Implementation

Proof of Lemma 2:

As $x \leq x^i(z)$, it suffices to show that $x^i(z) \in B_{AD}(\pi, W^i)$ is equivalent to (10). In fact, $x^i(z)$ satisfies $\pi(x^i(z) - W^i) \leq 0$ if and only if $\sum_{t=1}^{\infty} p_t(q_t(z_{t-1} - z_t) + R_t z_{t-1}) + \nu(x^i(z) - \omega^i) - z_0^i \pi(R) \leq 0$ where the series is equal to $p_1 q_1 z_0^i - \lim p_t q_t z_t$ and $p_1 q_1 = p(R) + \lim p_t q_t$. *Q.E.D.*

D.3 On Standard Portfolio Constraints

Now, let us see that, given a deflator $\lambda \in \ell^1$, portfolio constraints of type (b) or (c) imply, in equilibrium, $\lim \lambda_t q_t z_t^i = 0$.

Each constraint implies that any feasible portfolio z satisfies $\liminf \lambda_t q_t z_t \geq 0$ as $\omega^i \in \ell^\infty_+ \ \forall i$. On the other hand, multiplying, at each t, the budget constraint by λ_t and summing over t gives us $\sum_t \lambda_t(x_t^i - \omega_t^i) = \lambda_1 q_1 z_0^i - \lim_t \lambda_t q_t z_t^i$. Adding over i, we get $(\sum_i z_0^i)(\sum_t \lambda_t R_t) = \lambda_1 q_1 (\sum_i z_0^i) - \sum_i \lim_t \lambda_t q_t z_t^i$. Either for $\sum_i z_0^i > 0$ (as there is no bubble) or $\sum_i z_0^i = 0$, $\sum_i \lim_t \lambda_t q_t z_t^i = 0$, so, $\lim_t \lambda_t q_t z_t^i = 0 \ \forall i$.

D.4 Impatient Agents Sell the Bubble: Proof

Proof of Proposition 9:
By A4 and Lemma 1, $\partial U^i(x) = \{\mu^i\}$. Thus,

$$\lim_{h \downarrow 0} \frac{1}{h}(U^i(x + h(q_t e_t - \sum_{\tau > 1} R_\tau e_\tau) - U^i(x)) = \mu^i(q_t e_t - \sum_{\tau > t} R_\tau e_\tau)$$

As $\mu_s^i q_s = \mu_{s+1}^i (q_{s+1} + R_{s+1})$ for each s, μ^i is a deflator. Since there is a bubble and the markets are complete, $\mu^i(q_t e_t - \sum_{\tau > t} R_\tau e_\tau) = \mu_t^i q_t - \sum_{\tau > t} \mu_\tau^i R_\tau > 0$ which implies $U^i(x + h(q_t e_t - \sum_{\tau > 1} R_\tau e_\tau) > U^i(x)$ for $h > 0$ sufficiently small. *Q.E.D.*

D.5 Coexistence with an Impatient Agent

Lemma 12: *For $x \in \ell^\infty_+$, the superdifferential $\partial \chi_{\ell^\infty_+}(x)$ is the set $\{T \in ba : T(y) \geq 0 \ \forall y \in \ell^\infty_+$ and $T(x) = 0\}$.*

Proof: Using Example 47.9 in Zeidler [37], p.385, we have, for $x \in \ell^\infty_+$, that $\partial \chi_{\ell^\infty_+}(x)$ is the set $\{T \in ba : T(y) \geq T(x) \ \forall y \in \ell^\infty_+\}$[39]. Now, applying T to 0 and to $2x$, we get, using the above inequality, $T(x) = 0$. *Q.E.D.*

Lemma 1 does not extend to boundary points. Actually, we have the following result (that does not depend on U being Mackey continuous):

[39] We have changed the sign of $\chi_{\ell^\infty_+}$ to make it concave and then we work with the superdifferential instead of using the subdifferential.

Lemma 13: *If* $U : \ell^\infty \to I\!R \cup \{-\infty\}$ *is an increasing function with effective domain* ℓ^∞_+ *and* x *is a norm boundary point of* ℓ^∞_+ *such that* $\partial U(x) \neq \emptyset$*, then* $\partial U(x)$ *is not contained in* ℓ^1*.*

Proof: Given x not uniformly bounded away from zero and $T \in \partial U(x)$, it is always possible to find a nonnegative pure charge ν such that $\nu(x) = 0$, which implies that $T + \nu \in \partial U(x)$. Take the subsequence (x_{n_i}) converging to zero and let N' the ordered set of natural numbers constituted by the respective indices. By Lemma 7, there is a generalized limit LIM such that $\mathrm{LIM}(x) = 0$. It remains to show that $T + \mathrm{LIM} \in \partial U(x)$. Now, $U(x') - U(x) \leq T(x' - x)$, together with $\nu(x) = 0$ and $\nu(x') \geq 0$ for $x' \in \ell^\infty_+$, implies that $U(x') - U(x) \leq (T + \mathrm{LIM})(x' - x)$, as desired. *Q.E.D.*

We can nevertheless say the following about superdifferentials, even on the boundary points of ℓ^∞_+.

Lemma 14: *Under A1 and A3, suppose* $U|D$ *is a Mackey continuous function. Given* $x \in \ell^\infty_+$ *for any* $\mu + \nu \in \partial U(x)$ *with* $(\mu, \nu) \in ca \times pch$*, we have* $\mu \in \partial U(x)$*. Moreover,* $\nu(x) = 0$*.*

Proof: Let us show that μ is also a supergradient. Take any $y \in \ell^\infty$, we know that y_{E_n} converges in the Mackey topology to 0. Now, given $m \in \mathbb{N}$, there exists n_0 such that, for $n > \max\{n_0, m\}$, $U(y) \leq U(y_{E_n^c}) + 1/m$. Hence, $U(y) - U(x) \leq \sum_{t=1}^{n} \mu_t(y_t - x_t) + 1/m$. Taking the limit as m goes to ∞, we see that $U(y) - U(x) \leq \sum_{t=1}^{\infty} \mu_t(y_t - x_t)$. As the preferences are monotonous, we know that ν is a nonnegative operator. Suppose $\nu(x) > 0$. Let $x^n = (x_1, ..., x_n, \frac{x_{n+1}}{2}, \frac{x_{n+2}}{2}, ...)$, then $U(x^n) - U(x) \leq (\mu + \nu)(x^n - x)$. Now, x^n converges to x in the Mackey topology, so the left hand side tends to zero. However, the right hand side is equal to $-\frac{1}{2}\sum_{t>n} \mu_t x_t - \nu(\frac{1}{2}x) \leq -\frac{1}{2}\nu(x) < 0$, a contradiction. *Q.E.D.*

E On Section 5

Proof of Proposition 11:

(i) $R(z) = (c_{1s} - \omega^i_{1s} + \rho_{1s}(g_{1s} - e^i_{1s} - g_0))_{s \in \mathbb{N}}$, so $q(z) = \pi((c_{1s} - \omega^i_{1s} + \rho_{1s}(g_{1s} - e^i_{1s} - g_0))_{s \in \mathbb{N}})$. By (13), $c_0 + \rho_0 g_0 + \pi((c_{1s} - \omega^i_{1s} + \rho_{1s}(g_{1s} - e^i_{1s} - g_0))_s) = \omega^i_0 + \rho_0 e^i_0 + \pi \circ R(z^i_0)$.

Thus, $(c_0-\omega_0^i)+\rho_0(g_0-e_0^i)+\pi((c_{1s}-\omega_{1s}^i-R(z_0^i))_s)+\pi(\rho_{1s}(g_{1s}-(e_{1s}^i+e_0^i))_s)+(e_0^i-g_0)\pi(\rho_1)=0$, that is, (c,g) satisfies (15).

(ii) Let us define $z\in\ell^\infty$ by $c_{1s}-\omega_{1s}^i+\rho_{1s}(g_{1s}-e_{1s}^i-g_0)=R(z)_s,\ \forall s\in\mathbb{N}$. It remains to show that date 0 budget constraint holds. In fact, $(c_0-\omega_0^i)+\rho_0(g_0-e_0^i)+q(z-z_0^i)=(c_0-\omega_0^i)+(\gamma_0+\pi(\rho_1))(g_0-e_0^i)+\pi((c_{1s}-\omega_{1s}^i+\rho_{1s}(g_{1s}-e_{1s}^i-g_0))_s-R(z_0^i))=(c_0-\omega_0^i)+\gamma_0(g_0-e_0^i)+\pi((c_{1s}-W_{1s}^i+\rho_{1s}(g_{1s}-E_{1s}^i))_s)$ and this last expression is less than or equal to zero, since $(c,g)\in B_{\mathrm{AD}}(\pi,\gamma,W^i,E^i)$. *Q.E.D.*

Let us prove now that, for an appropriate choice of ϵ, we have $\widetilde{\pi}(x^1-(W^1,E^1))=0$ in Example 4. Notice that $\widetilde{\pi}(x^1-(W^1,E^1))=(1-\epsilon)\sum_{s\geq 1}(\frac{1}{2})^{s-1}(-\frac{s}{s+1}\psi_s+2(\mathbb{1}_O-\mathbb{1}_{O^c}))+\epsilon(B(-\psi)+B(\rho(\mathbb{1}_O-\mathbb{1}_{O^c})))$. Now, $B(\rho(\mathbb{1}_O-\mathbb{1}_{O^c}))=0$ whereas $B(-\psi)<0$. It suffices to show that $\sum_{s\geq 1}(\frac{1}{2})^{s-1}(-\frac{s}{s+1}\psi_s+2(\mathbb{1}_O-\mathbb{1}_{O^c}))>0$. This holds as in this series, the sum an odd term and the next even term is $(\frac{1}{2})^{s-1}(-\frac{1}{2}\frac{s}{s+1}+\frac{1}{8}\frac{s+1}{s+2}+2-1)>0$ (since $\frac{s}{s+1}<1$).

References

[1] A. Araujo, "Lack of Pareto Optimal Allocations in Economies with Infinitely Many Commodities: The Need for Impatience", *Econometrica* **53** (1985), 455-461.

[2] A. Araujo, J. Fajardo and M. Páscoa, "Endogenous Collateral", *Journal of Mathematical Economics* **41** (2005), 439-462.

[3] K. Arrow, "Essays in the Theory of Risk-Bearing", North Holland, Amsterdam, 1970.

[4] R. Barrios, "Equilíbrio, Impaciência e Neutralidade à Incerteza em uma Economia Tipo Bewley", PhD Thesis, IMPA, Rio de Janeiro, 1996.

[5] T. Bewley, "Existence of Equilibrium in Economies with Infinitely Many Commodities", *Journal of Economic Theory* **4** (1972), 514-540.

[6] T. Bewley, The Optimum Quantity of Money, in "Models of Monetary Economics" (J. Kareken and N.Wallace, Eds.), Federal Reserve Bank, Minneapolis, 1980.

[7] K. Bhaskara Rao and M. Bhaskara Rao, "Theory of Charges", Academic Press, New York, 1983.

[8] D. Brown and L. Lewis, "Myopic Economic Agents", *Econometrica* **49** (1981), 359-368.

[9] A. Chateauneuf and Y. Rébillé, "Some Characterizations of Non-Additive Multi-Period Models", *Mathematical Social Sciences* **48** (2004), 235-250.

[10] J. Dow and S. Werlang, "Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio", *Econometrica* **60** (1992), 197-204.

[11] L. Dubins and L. Savage, "How to Gamble If You Must (Inequalities for Stochastic Processes)", McGraw-Hill, London, 1965.

[12] J. Dubra, M. Macheroni and E. Ok, "Expected Utility Theory without the Completeness Axiom", *Journal of Economic Theory* **115** (2004), 118-133.

[13] N. Dunford and J. Schwartz, "Linear Operators, Part I", Interscience, New York, 1958.

[14] L. Epstein and T. Wang, "Uncertainty, Risk-Neutral Measures and Security Price Booms and Crashes", *Journal of Economic Theory* **67** (1995), 40-82.

[15] I. Gilboa, "Expectation and Variation in Multi-period Decisions", *Econometrica* **57** (1989), 1153-1169.

[16] I. Gilboa and D. Schmeidler, "Maxmin Expected Utility with a Non-Unique Prior", *Journal of Mathematical Economics* **110** (1989), 605-639.

[17] C. Gilles and S. LeRoy, "Bubbles and Charges", *International Economic Review* **33** (1992), 323-339.

[18] L. Hansen and T. Sargent, "Robust Control and Model Uncertainty", *American Economic Review* **91** (2001), 60-66.

[19] L. Hansen and T. Sargent, "Robustness", Princeton University Press, 2008.

[20] J. Hodges and E. Lehmann, "The Use of Previous Experience in Reaching Statistical Decisions", *Annals of Mathematical Statistics* **23** (1992), 396-407.

[21] K. Huang and J. Werner, "Asset Price Bubbles in Arrow-Debreu and Sequential Equilibrium", *Economic Theory* **15** (2000), 253-278.

[22] D. Kreps, "A Representation Theorem for Preference for Flexibility", *Econometrica* **47** (1979), 565-577.

[23] N. Kocherlakota, "Injecting Rational Bubbles", *Journal of Economic Theory* **142** (2008), 218-232.

[24] M. Kurz and M. Majumdar, "Efficiency Prices in Infinite Dimensional Spaces: A Synthesis", *The Review of Economic Studies* **39** (1972), 147-158.

[25] M. Magill and M. Quinzii, "Incomplete Markets Over an Infinite Horizon: Long Lived Securities and Speculative Bubbles", *Journal of Mathematical Economics* **26** (1996), 133-170.

[26] M. Marinacci, "An Axiomatic Approach to Complete Patience and Time Invariance", *Journal of Economic Theory* **83** (1998), 105-144.

[27] E. Prescott and R. Lucas, "A Note on Price Systems in Infinite Dimensional Space", *International Economic Review* **13** (1972), 416-422.

[28] R. Radner, "Efficiency Prices for Infinite Horizon Production Programmes", *The Review of Economic Studies* **34**(1967), 51-66.

[29] M. Santos and M. Woodford, "Rational Asset Pricing Bubbles", *Econometrica* **65** (1997), 19-57.

[30] C. Sawyer, "When Prices Are in l_1", Southern Illinois University Carbondale, IL, mimeograph, 1997.

[31] H. Schaeffer, "Topological Vector Spaces", Macmillan, New York, 1966.

[32] D. Schmeidler, "Subjective Probability and Expected Utility Without Additivity", *Econometrica* **57** (1989), 571-587.

[33] J. Tirole, "Asset Bubbles and Overlapping Generations", *Econometrica* **53** (1989), 1499-1528.

[34] A. Tversky and D. Kahneman, "Loss Aversion in Riskless Choice: A Reference Dependent Model", *Quarterly Journal of Economics* **106** (1991), 1039-1061.

[35] C. Villegas, "On Qualitative Probability σ-Algebras", *The Annals of Mathematics and Statistics* **35** (1964), 787-796.

[36] J. Werner, "Arbitrage, Bubbles, and Valuation", *International Economic Review* **38** (1997), 453-464.

[37] E. Zeidler, "Nonlinear Functional Analysis and Its Applications", Vol. III, Springer, New York, 1984.