Coalitional matchings

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Abstract

In a coalitional two-sided matching problem agents on each side of the market may form coalitions such as student groups and research teams who – when matched – form universities. We assume that each researcher has preferences over the research teams he would like to work in and over the student groups he would like to teach to. Correspondingly, each student has preferences over the groups of students he wants to study with and over the teams of researchers he would like to learn from. In this setup, we examine how the existence of core stable partitions on the distinct market sides, the restriction of agents' preferences over groups to strict orderings, and the extent to which individual preferences respect common rankings shape the existence of core stable coalitional matchings.

Keywords: coalitions, common rankings, core, stability, totally balanced games, two-sided matchings
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1 Introduction

Both hedonic coalition formation and matching models have been used to study a wide range of real-life situations. While the coalition formation literature focuses on the formation of groups on *one side of a market* – government formation, athletes forming teams, students forming student groups, researchers forming research teams, medical professionals forming practices – the matching literature investigates the "matching" of entities on *both sides of a market*, e.g., students choosing colleges, researchers choosing universities, patients choosing medical centers, medical interns choosing hospitals, etc. Many of these situations are, however, intrinsically interrelated: student groups and research teams when matched form universities; athletes form teams who match with a team of managers to form a sport club; and medical practitioners together with their patients comprise hospitals. For this reason, in this paper we integrate coalition formation and matching problems into a novel framework, which we call a *coalitional matching*. This allows us to analyze *stability* in two-sided matching problems where agents on each side of the market simultaneously form coalitions and "match" to coalitions on the other side.

Our work is immediately related to the literature on two-sided matching problems, where agents on one side of the market are matched to institutions on the other side, first defined by Gale and Shapley (1962). Their seminal contribution spurred a vast body of literature (for a thorough review, see Roth and Sotomayor (1990)). Among the key contributions, we note Shapley and Shubik (1972) and Crawford and Knoer (1981), who extend Gale and Shapley's framework to a transferable utility setting; Kelso and Crawford (1982), who provide sufficient conditions for the existence of core stable allocations; and Blair (1988), who proves that the set of stable matching is a lattice under a suitable ordering. More recently, Hatfield and Milgrom (2005) incorporate contracts in the analysis.

In this line of research it has also been recognized that agents' preferences over matchings may depend not only on the institutions they are matched with, but also on the other agents that are matched to the same institution, i.e., their colleagues. For example, the early study by Roth (1984), and the recent one by Klaus and Klijn (2005), investigate many-to-one matching problems in the presence of couples on the agent side of the market. In these models, however, the coalitions, i.e., the couples, are exogenously given. Dutta and Massó (1997) take the analysis one step further and study a many-to-one matching model in which agents' preferences are lexicographic and are defined over all institutions and all subsets of colleagues. These authors, however, restrict their analysis to situations in which institutions' preferences over agents satisfy a substitutability property, an assumption which might not be applicable to many real-life situations in which there are complementarities between agents as argued most recently by Pycia (2007). Pycia (2007) and Revilla (2007) move away from the lexicographic preferences assumption in the many-to-one matching problem with peer effects. In this respect, their contributions can be regarded as hedonic coalition formation problems with heterogeneous sets of actors: a set of institutions and a set of agents; and a restriction on the coalition structures such that a coalition may contain at most one institution. In a related piece of work, Echenique and Yenmez (2007) propose an algorithm to find a core stable matching, when it exists, in the general many-to-one matching problem with peer effects. To conclude this brief overview of the literature, we would like to mention that Dutta and Massó (1997), Pycia (2007), and Revilla (2007) all contain, under different names, a condition that imposes a degree of commonality of players' preferences over groups. As it will turn out, the spirit of commonality of players' preferences will be important for the analysis in this paper, too.

In this paper we depart from the existing literature, most notably, by allowing at the same time *coalition formation on both sides* of the market *and matching* between two coalitional entities. Throughout the paper we illustrate our concepts by considering a two-sided matching problem where *students* may form *student groups* and *researchers* may collaborate within *research teams* who when matched form *universities*. We assume that each researcher has preferences over research teams he would like to work in (and thus, a research team formation game is well defined) and over student groups he would like to teach to. Correspondingly, each student has preferences over groups of students she wants to study with (and thus, a student group formation game is well defined, too) and over groups of researchers she would like to learn from. In this setup, we study the existence of *core stable coalitional matchings*.

In our model, we consider *lexicographic* preference profiles as they allow us to clearly demarcate the coalition formation and matching aspects of the problem. Within this broad category, a *first possibility* is to assume that the agents' preferences over groups on *one* market side *dictate* their overall preferences over universities. In this case, if the market side is the same for all agents, then the existence of core stable coalitional matchings is determined by the existence of core stable partition of the agents on that side of the market. If, on the other hand, it coincides with an agent's own market side, then the existence of core stable coalitional matchings is determined by the existence of core stable partitions of students and researchers into student groups and research teams, respectively. If students judge universities according to their corresponding teaching teams and researchers judge universities according to their corresponding student groups, then a *common ranking property* (cf. Farrell and Scotchmer (1988)) assures the existence of core stable coalitional matchings. *Another possibility* to induce agents' preferences over universities is to assume that *priority* is given to groups on one of the market sides and then, in case of indifference, groups on the other market side also play a role. Depending on whether agents give priority to groups from one and the same side, their own side or the opposite market side, we show that the existence of core stable coalitional matchings requires appropriate selections from the following four properties. The *first* one is the existence of *core stable coalition structures* for the coalition formation games on separate market sides.[1] The *second* property is the *total balancedness* of the corresponding coalition formation games (cf. Bloch and Diamantoudi (2007)) requiring each restriction of these games to have a non-empty core. Although this condition is quite restrictive, many of the sufficient conditions for non-emptiness of the core of hedonic games guarantee that the game is in fact totally balanced (e.g., the common ranking property of Farrell and Scotchmer (1988) and the top coalition property of Banerjee et al. (2001)). The *third* and *fourth* properties that play a role in our analysis make the existence of core stable coalitional matchings dependent on whether individual preferences over groups are *strict* or not, and on whether these individual preferences *respect a common ranking over research teams* and *a common ranking over student groups*. The trade-off between these four properties determines the structure of the results presented in the main body of our work.

The rest of the paper is organized as follows. The next section introduces the basic concepts used in our analysis. Sections 3, 4, and 5 are devoted to the existence of core stable coalitional matchings when agents' preferences over universities are crucially shaped by their preferences over groups, respectively, on one and the *same* market side, on their *own* market side, and on the *opposite* market side. Each of these sections contains existence results with respect to the outlined induced preferences and provides examples that shed light on the importance of the identified (necessary and) sufficient conditions. We conclude in Section 6 with some final remarks. An appendix contains the proofs of all formal statements.

[1] We refer the reader to Banerjee et al. (2001) and Bogonolnaia and Jackson (2002) for different sufficient conditions with respect to this topic and would like to note that for our analysis it is not necessary to be more explicit on these conditions.

2 Notation and definitions

Our setup consists of the following basic ingredients.

Agents and overall preferences

There are two disjoint and finite sets of agents, the set R of researchers, and the set S of students. A research team T is a non-empty subset of R and a student group G is a non-empty subset of S. We denote by 2^R the set of all research teams, and by R_r the set of all teams containing researcher $r \in R$. Correspondingly, 2^S stands for the set of all student groups, while S_s is the set of all student groups containing student $s \in S$.

Each researcher and each student seek a research/teacher team and a student group. Thus, each student $s \in S$ has a complete and transitive preference $\succeq_s$ defined over $2^R \times S_s$, and each researcher $r \in R$ has a complete and transitive preference $\succeq_r$ defined over $R_r \times 2^S$. The corresponding strict preference and indifference relations are denoted, for $i \in R \cup S$, by $\succ_i$ and $\sim_i$, respectively.

Primitive preferences

We assume that each agent's overall preference over universities, i.e., over elements of $2^R \times 2^S$ that contain him, are induced by two corresponding primitive binary relations (assumed to be complete and transitive). More precisely, for each $s \in S$, these relation are $\succeq_s^G$ (defined over student groups containing s) and $\succeq_s^T$ (defined over all research teams). Correspondingly, for each $r \in R$, the relations are $\succeq_r^T$ (defined over all research teams containing r) and $\succeq_r^G$ (defined over all student groups). The different ways in which this primitive information is used to guide agents' overall preferences shape the domains we consider in the next sections.

Common rankings

For some of the results in the next sections to hold we need to assume the existence of a *common ranking*, i.e., a complete and transitive binary relation $\trianglerighteq^T$ over all research teams and of a *common ranking* $\trianglerighteq^G$ over all student groups, where the corresponding strict preference and indifference are denoted by $\triangleright^T$ ($\triangleright^G$) and $=^T$ ($=^G$), respectively. We say then that $\left(\succeq_i^T\right)_{i\in Z}$, $Z \subseteq R \cup S$, satisfies the *common ranking property* with respect to $\trianglerighteq^T$ or simply *respects* $\trianglerighteq^T$ (cf. Farell and Scotchmer (1988)) if, for all $i \in Z$, $T' \succeq_i^T T''$ if and only if $T' \trianglerighteq^T T''$ for all $T', T'' \in 2^R$. Correspondingly, $\left(\succeq_i^G\right)_{i\in Z}$, $Z \subseteq R \cup S$, *respects* $\trianglerighteq^G$ if, for all $i \in Z$, $G' \succeq_i^G G''$ if and only if $G' \trianglerighteq^G G''$ for all $G', G'' \in 2^S$.

Hedonic games

In a hedonic coalition formation game each player's preferences over coalitions depend only on the composition of members of his coalition (cf. Drèze and Greenberg (1980), Banerjee et al. (2001), Bogomolnaia and Jackson (2002)). The model of such a game consists of a complete and transitive preference, for each player, over the coalitions that player may belong to. The outcome of the game is a partition of the set of players into coalitions and it is supposed that each player compares two partitions based only on the comparison of the coalitions he is a member of in the two partitions. Notice then that $\left(R, (\succeq_r^T)_{r\in R}\right)$ and $\left(S, (\succeq_s^G)_{s\in S}\right)$ are well defined hedonic games. The *core* of such games consists of such partitions for which no group of players are able to form a coalition with each player being strictly better off with this new coalition compared to his corresponding coalition in the partition. In what follows, we denote by $Core\left(R, (\succeq_r^T)_{r\in R}\right)$ and $Core\left(S, (\succeq_s^G)_{s\in S}\right)$ the sets of core stable partitions of the games $\left(R, (\succeq_r^T)_{r\in R}\right)$ and $\left(S, (\succeq_s^G)_{s\in S}\right)$, respectively. Moreover, since the proofs of our existence results are obtained in a recursive manner, we need for some of them to assume that the corresponding hedonic games are totally balanced. Applied to the research team formation game, the corresponding definition reads as follows. For any $V \subseteq R$ and $r \in V$, let $\succeq_{r|V}^T$ denote the restriction of $\succeq_r^T$ on V. Then, $\left(R, (\succeq_r^T)_{r\in R}\right)$ is *totally balanced* (cf. Bloch and Diamantoudi (2007)) if any of its restrictions $\left(V, (\succeq_{r|V}^T)_{r\in V}\right)$ has a non-empty core. Note finally that if $(\succeq_r^T)_{r\in R}$ respects a common ranking $\trianglerighteq^T$, then $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$ (cf. Farell and Scotchmer (1988)) and $\left(R, (\succeq_r^T)_{r\in R}\right)$ is totally balanced.

Coalitional matchings

A *coalitional matching* is a function μ from $R \cup S$ into subsets of $R \cup S$, such that for all $r \in R$ and $s \in S$:

(1) $\mu(s) \in 2^R \times S_s$;

(2) $\mu(r) \in R_r \times 2^S$;

(3) If $\mu(i) = (T, G)$ for some $i \in R \cup S$, then $\mu(k) = (T, G)$ for all $k \in T \cup G$.

In what follows, we write $\mu(i) = (\mu(i)_1, \mu(i)_2)$ to denote the match of agent $i \in R \cup S$ under μ.

Notice that each coalitional matching μ induces a partition Π^μ of $R \cup S$ into coalitions (universities), i.e., $\Pi^\mu = \{\mu(i)_1 \cup \mu(i)_2 \mid i \in R \cup S\}$. For each $i \in R \cup S$, we denote by $\Pi^\mu(i)$ the coalition containing agent i in matching μ. Moreover, $\Pi_R^\mu = \{V \cap R \mid V \in \Pi^\mu\}$ is a partition of R into research teams, while $\Pi_S^\mu = \{Q \cap S \mid Q \in \Pi^\mu\}$ is a partition of S into student groups (both partitions being induced by μ).

We say that a pair (A, μ'), where $A \subseteq R \cup S$ and μ' is a coalitional matching, is *blocking* μ if

(1) For all $s \in A \cap S$ and $r \in A \cap R$, $\mu(s) \in 2^{A\cap R} \times (A \cap S)_s$ and $\mu(r) \in (A \cap R)_r \times 2^{A\cap S}$;

(2) For all $i \in A$, $\mu'(i) \succ_i \mu(i)$.

A coalitional matching μ is *core stable* if it cannot be blocked.

3 Same-sided priorities

We start our analysis by assuming that researchers' and students' preferences over groups on *one* and the *same* market side shape in a crucial way their overall preferences over universities. The first preference domain ($\mathcal{D}_1$) displays a situation where both researchers and students pay attention *only* to the research teams they can work in or learn from, respectively. In the second preference domain ($\mathcal{D}_2$) priority is given again to research teams but, in case an agent is indifferent between two research teams, the overall preference over universities *follows* the corresponding primitive preference over student groups. Thus, we have the following formal definitions.

$\mathcal{D}_1$:
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff $T' \succeq_r^T T''$;
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff $T' \succeq_s^T T''$.

$\mathcal{D}_2$:
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff (a) $T' \succ_r^T T''$ or (b) $T' \sim_r^T T''$ and $G' \succeq_r^G G''$;
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff (a) $T' \succ_s^T T''$ or (b) $T' \sim_s^T T''$ and $G' \succeq_s^G G''$.

Given the focus of agents' induced preferences in these two domains, it is easy to see that a necessary condition for the existence of a core stable coalitional matching μ is that $\Pi_R^\mu \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$; the reason is that all researchers (and students) look (first) at the corresponding research teams when comparing two universities. As it turns out, the existence of a core stable partition for the research team formation game is also a sufficient condition when the domain is $\mathcal{D}_1$.

Theorem 1 *Let $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_1$. Then a core stable coalitional matching exists if and only if $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$.*

However, as exemplified next, the existence of a core stable partition into research teams does not suffice for the existence of a core stable coalitional matching when agents' preferences are in $\mathcal{D}_2$. More precisely, the example shows that there may not be a core stable coalitional matching even if $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$, $Core\left(S, (\succeq_s^G)_{s\in S}\right) \neq \emptyset$ and agents' primitive preferences are strict, i.e., the induced rankings over universities are strict as well[2].

Example 1 *Consider a set of researchers* $R = \{r_1, r_2, r_3\}$ *and a set of students* $S = \{s_1, s_2, s_3\}$*. Let the preference profiles be as specified below:*

$$
\begin{aligned}
&\{r_1\} \succ^T_{r_1} \ldots; \{r_2\} \succ^T_{r_2} \ldots; \{r_3\} \succ^T_{r_3} \ldots;\\
&\{s_1, s_3\} \succ^G_{r_1} \emptyset \succ^G_{r_1} \ldots; \{s_1, s_2\} \succ^G_{r_2} \emptyset \succ^G_{r_2} \ldots; \{s_2, s_3\} \succ^G_{r_3} \emptyset \succ^G_{r_3} \ldots;\\
&\{s_1, s_2\} \succ^G_{s_1} \{s_1, s_3\} \succ^G_{s_1} \{s_1\} \succ^G_{s_1} \ldots;\\
&\{s_2, s_3\} \succ^G_{s_2} \{s_1, s_2\} \succ^G_{s_2} \{s_2\} \succ^G_{s_2} \ldots;\\
&\{s_2, s_3\} \succ^G_{s_3} \{s_1, s_3\} \succ^G_{s_3} \{s_3\} \succ^G_{s_3} \ldots;\\
&\{r_1\} \succ^T_{s_1} \{r_2\} \succ^T_{s_1} \emptyset \succ^T_{s_1} \ldots;\\
&\{r_2\} \succ^T_{s_2} \{r_3\} \succ^T_{s_2} \emptyset \succ^T_{s_2} \ldots;\\
&\{r_3\} \succ^T_{s_3} \{r_1\} \succ^T_{s_3} \emptyset \succ^T_{s_3} \ldots
\end{aligned}
$$

There is no core stable coalitional matching when $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_2$*. First notice that* $Core\big(R, (\succeq^T_r)_{r\in R}\big) = \{\{r_1\}, \{r_2\}, \{r_3\}\}$*. Further, any coalitional matching such that* $\mu(i)_2 = \emptyset$ *for all* $i \in R$ *is not core stable because it will be blocked by* (A, μ') *where* $A = \{r_1, s_1, s_3\}$ *and* $\mu'(i) = (\{r_1\}, \{s_1, s_3\})$ *for all* $i \in A$ *because* $\{s_1 s_3\} \succ^G_{r_1} \emptyset$*,* $\{r_1\} \succ^T_{s_1} \emptyset$*, and* $\{r_1\} \succ^T_{s_3} \emptyset$*. Next, consider a coalitional matching* μ *with* $\mu(r_1) = \mu(s_1) = \mu(s_3) = (\{r_1\}, \{s_1, s_3\})$*, and* $\mu(r_2)_2 = \mu(r_3)_2 = \mu(s_2)_1 = \emptyset$*. This matching is blocked by* (A, μ') *with* $A = \{r_3, s_2, s_3\}$ *and* μ' *such that* $\mu'(i) = (\{r_3\}, \{s_2, s_3\})$ *for all* $i \in A$ *because* $\{s_2, s_3\} \succ^G_{r_3} \emptyset$*,* $\{r_3\} \succ^T_{s_2} \emptyset$*; and* $\{r_3\} \succ^T_{s_3} \{r_1\}$*. Similarly, a coalitional matching* μ *with* $\mu(r_3) = \mu(s_2) = \mu(s_3) = (\{r_3\}, \{s_2, s_3\})$*, and* $\mu(r_1)_2 = \mu(r_2)_2 = \mu(s_1)_1 = \emptyset$ *is blocked by* (A, μ') *with* $A = \{r_2, s_1, s_2\}$ *and* μ' *such that* $\mu'(i) = (\{r_2\}, \{s_1, s_2\})$ *for all* $i \in A$*; and a coalitional matching* μ *with* $\mu(r_2) = \mu(s_1) = \mu(s_2) = (\{r_2\}, \{s_1, s_2\})$ *and* $\mu(r_1)_2 = \mu(r_3)_2 = \mu(s_3)_1 = \emptyset$ *is blocked by* (A, μ') *with* $A = \{r_1, s_1, s_3\}$ *and* μ' *such that* $\mu'(i) = (\{r_1\}, \{s_1, s_3\})$

[2] In all examples, the coalitions not listed are either not individually rational (less preferred than the corresponding singleton) or the empty set is preferred to any of them.

for all $i \in A$. In the same fashion, one can show that no other coalitional matching is core stable.

Notice that in the above example, in addition to $Core\left(S, (\succeq_s^G)_{s\in S}\right) \neq \emptyset$, the hedonic game $\left(S, (\succeq_s^G)_{s\in S}\right)$ is *totally balanced*. In order to prove our existence results for $\mathcal{D}_2$, we need to further assume a *common ranking property* to hold. More precisely, we have the following result.

Theorem 2 *Let $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_2$. Let $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$ and the game $\left(S, (\succeq_s^G)_{s\in S}\right)$ be totally balanced. Let $\left(\succeq_r^T\right)_{r\in R}$ be strict and $\left(\succeq_s^T\right)_{s\in S}$ respect $\trianglerighteq^T$ with $\trianglerighteq^T$ being strict. Then a core stable coalitional matching exists.*

Let us now allow for $\left(\succeq_s^T\right)_{s\in S}$ containing indifferences. The next example shows a coalitional matching situation with the following four features: (1) $(\succeq_i^T)_{i\in R\cup S}$ respects $\trianglerighteq^T$ (and thus, the hedonic game $\left(R, (\succeq_r^T)_{r\in R}\right)$ is totally balanced); (2) the primitive preferences $(\succeq_r^T)_{r\in R}$ are strict; (3) $(\succeq_r^G)_{r\in R}$ respects $\trianglerighteq^G$; (4) the hedonic game $\left(S, (\succeq_s^G)_{s\in S}\right)$ is totally balanced. Nevertheless, no core stable coalitional matching exists.

Example 2 *Consider a set of researchers $R = \{r_1, r_2\}$ and a set of students $S = \{s_1, s_2, s_3\}$. Let $(\succeq_i^T)_{i\in R\cup S}$ respect $\{r_1\} =^T \{r_2\} \triangleright^T \emptyset \triangleright^T \{r_1, r_2\}$ and $(\succeq_r^G)_{r\in R}$ respect*

$$\{s_1, s_2\} \triangleright^G \{s_1, s_3\} \triangleright^G \{s_2, s_3\} \triangleright^G \emptyset \triangleright^G \{s_1\} \triangleright^G \{s_2\} \triangleright^G \{s_3\} \triangleright^G \{s_1, s_2, s_3\}.$$

Last, let $(\succeq_r^G)_{r\in R}$ be as given below:

$$\begin{aligned}
&\{s_1, s_2, s_3\} \succ_{s_1}^G \{s_1, s_2\} \succ_{s_1}^G \{s_1, s_3\} \succ_{s_1}^G \{s_1\};\\
&\{s_1, s_2, s_3\} \succ_{s_2}^G \{s_2, s_3\} \succ_{s_2}^G \{s_1, s_2\} \succ_{s_2}^G \{s_2\};\\
&\{s_1, s_2, s_3\} \succ_{s_3}^G \{s_1, s_3\} \succ_{s_3}^G \{s_2, s_3\} \succ_{s_3}^G \{s_3\}.
\end{aligned}$$

There is no core stable coalitional matching when $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_2$. First, note that $\{r_1, r_2\}$ cannot be an element of a core stable coalitional matching because $\{r_1\} \succ_{r_1}^T \{r_1, r_2\}$. Next, consider the coalitional matching μ such *that* $\mu(r_1)_2 = \{s_1, s_2\}$*,* $\mu(s_1)_1 = \mu(s_2)_1 = \{r_1\}$ *and* $\mu(r_2)_2 = \mu(s_3)_1 = \emptyset$*. This matching is blocked by* (A, μ')*, with* $A = \{r_2, s_2, s_3\}$ *and* μ' *is such that* $\mu'(i) = (\{r_2\}, \{s_2, s_3\})$ *for all* $i \in A$*. Similarly, one can show that no other coalitional matching is core stable.*

In order to see that it is crucial that *both* $(\succeq_s^T)_{s\in S}$ and $(\succeq_r^T)_{r\in R}$ respect $\trianglerighteq^T$ when $\left(\succeq_r^T\right)_{r\in R}$ contains indifferences, let us consider a situation where (1) $(\succeq_r^T)_{r\in R}$ contains indifferences, does not respect $\trianglerighteq^T$ and it is such that $(R, \left(\succeq_r^T\right)_{r\in R})$ is totally balanced; (2) $\left(\succeq_r^G\right)_{r\in R}$ respects $\trianglerighteq^G$; (3) $\left(S, (\succeq_s^G)_{s\in S}\right)$ is totally balanced; (4) $(\succeq_s^T)_{s\in S}$ respects $\trianglerighteq^T$.

Example 3 *Let* $R = \{r_1, r_2, r_3\}$*,* $S = \{s_1\}$ *and* $(\succeq_r^T)_{r\in R}$ *be as follows:*

$$\begin{array}{l}\{r_1, r_2\} \succ_{r_1}^T \{r_1, r_3\} \succ_{r_1}^T \{r_1\} \succ_{r_1}^T \ldots;\\ \{r_2, r_3\} \succ_{r_2}^T \{r_1, r_2\} \succ_{r_2}^T \{r_2\} \succ_{r_2}^T \ldots;\\ \{r_1, r_3\} \succ_{r_3}^T \{r_2, r_3\} \sim_{r_3}^T \{r_3\} \succ_{r_3}^T \ldots\end{array}$$

Let $(\succeq_r^G)_{r\in R}$ *respect* $\{s_1\} \triangleright^G \emptyset$ *and* $\succeq_{s_1}^T$ *respect* $\{r_2, r_3\} \triangleright^T \{r_1, r_2\} \triangleright^T \{r_1, r_3\} \triangleright^T \emptyset \trianglerighteq^T \ldots$.

In this situation there is no core stable coalitional matching when $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_2$*. Consider the coalitional matching* $\mu(r_1) = \mu(r_2) = \mu(s_1) = (\{r_1, r_2\}, s_1)$*,* $\mu(r_3) = (\{r_3\}, \emptyset)$*. This matching is blocked by* (A, μ') *with* $A = \{r_2, r_3, s_1\}$ *and* $\mu(i) = (\{r_1, r_2\}, s_1)$ *for all* $i \in A$ *because* $\{r_2, r_3\} \succ_{r_2}^T \{r_1, r_2\}$*,* $\{r_2, r_3\} \sim_{r_3}^T \{r_3\}$*,* $\{s_1\} \succ_{r_3}^G \emptyset$*, and* $\{r_2, r_3\} \succ_{s_1}^T \{r_1, r_2\}$*. Next consider the coalitional matching* $\mu(r_1) = (\{r_1\}, \emptyset)$*,* $\mu(r_2) = \mu(r_3) = \mu(s_1) = (\{r_2, r_3\}, s_1)$*. This matching is blocked by* (A, μ') *with* $A = \{r_1, r_3\}$ *because* $\{r_1, r_3\} \succ_{r_1}^T \{r_1\}$ *and* $\{r_1, r_3\} \succ_{r_3}^T \{r_2, r_3\}$*. Similarly, one can show that no coalitional matching is core stable.*

As we show next, if, in addition to the properties of a coalitional matching problem outlined in the above example, one requires $(\succeq_s^G)_{s\in S}$ also to respect the common ranking over student groups, then a core stable coalitional matching exists also in the presence of indifferences. Notice that, since both $\left(\succeq_r^T\right)_{r\in R}$ and $\left(\succeq_s^G\right)_{s\in S}$ are required to respect the corresponding common rankings, the games $\left(R,(\succeq_r^T)_{r\in R}\right)$ and $\left(S,(\succeq_s^G)_{s\in S}\right)$ are totally balanced.

Theorem 3 *Let $(\succeq_i)_{i\in R\cup S}\in\mathcal{D}_2$, $\left(\succeq_i^T\right)_{i\in R\cup S}$ respect $\trianglerighteq^T$ and $\left(\succeq_i^G\right)_{i\in R\cup S}$ respect $\trianglerighteq^G$. Then a core stable coalitional matching exists.*

Our final example in this section shows the importance of the fact that the hedonic games on *both* market sides have to be totally balanced for the result in Theorem 3 to hold. Precisely, the example below illustrates that when (1) $\left(\succeq_s^T\right)_{s\in S}$ respects $\trianglerighteq^T$, (2) $\left(R,(\succeq_r^T)_{r\in R}\right)$ has a nonempty core but is not totally balanced, and (3) $\left(\succeq_i^G\right)_{i\in R\cup S}$ respects $\trianglerighteq^G$, there may not be a core stable coalitional matching.

Example 4 *Consider a set of researchers $R=\{r_1,r_2,r_3,r_4,r_5\}$ and a set of students $S=\{s_1\}$. Let $\left(R,(\succeq_r^T)_{r\in R}\right)$ be given as follows:*

$$\begin{aligned}
&\{r_1,r_2\}\sim_{r_1}^T\{r_1,r_3\}\succ_{r_1}^T\{r_1,r_4\}\succ_{r_1}^T\{r_1,r_5\}\succ_{r_1}^T\{r_1\};\\
&\{r_1,r_2\}\sim_{r_2}^T\{r_2,r_3\}\succ_{r_2}^T\{r_2\};\\
&\{r_2,r_3\}\succ_{r_3}^T\{r_1,r_3\}\succ_{r_3}^T\{r_3\};\\
&\{r_4,r_5\}\succ_{r_4}^T\{r_1,r_4\}\succ_{r_4}^T\{r_4\};\\
&\{r_1,r_5\}\succ_{r_5}^T\{r_4,r_5\}\succ_{r_5}^T\{r_5\}.
\end{aligned}$$

Let $\left(\succeq_i^G\right)_{i\in R\cup S}$ respect $\{s_1\}\rhd^G\emptyset$ and $\left(\succeq_s^T\right)_{s\in S}$ respect $\{r_2,r_3\}\rhd^T\{r_3\}\rhd^T\emptyset\trianglerighteq^T\ldots$.

There is no core stable coalitional matching when $(\succeq_i)_{i\in R\cup S}\in\mathcal{D}_2$. First notice that the only core stable partition for $\left(R,(\succeq_r^T)_{r\in R}\right)$ is $\Pi^R=\{\{r_1,r_2\},\{r_3\},\{r_4,r_5\}\}$, so no coalitional matching that induces a partition of the researcher set different from Π^R can be core stable. Next, consider the coalitional matching μ defined by $\mu(r_1)=\mu(r_2)=(\{r_1,r_2\},\emptyset)$, $\mu(r_3)=\mu(s_1)=(\{r_3\},\{s_1\})$, and $\mu(r_4)=\mu(r_5)=(\{r_4,r_5\},\emptyset)$. This matching is blocked by the pair (A,μ') with $A=\{r_2,r_3,s_1\}$ and μ' such that $\mu'(i)=(\{r_2,r_3\},\{s_1\})$ for all $i\in A$, because $\{r_1,r_2\}\sim_{r_2}^T\{r_2,r_3\}$ and

$\{s_1\} \succ_{r_2}^{G} \emptyset$, $\{r_2, r_3\} \succ_{r_3}^{T} \{r_3\}$, *and* $\{r_2, r_3\} \succ_{s_1}^{T} \{r_3\}$. *Similarly, one can show that no other coalitional matching is core stable.*

4 Own-sided priorities

Assume next that agents' preferences over universities are mainly shaped by the corresponding primitive preferences over groups on agents' *own* market side. For the domain $\mathcal{D}_3$ these primitive preferences *dictate* the overall preferences, while for $\mathcal{D}_4$ the primitive preferences over groups on the opposite market side *also* play a role.

$\mathcal{D}_3$:
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff $G' \succeq_s^G G''$;
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff $T' \succeq_r^T T''$.

$\mathcal{D}_4$:
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff
(a) $G' \succ_s^G G''$ or (b) $G' \sim_s^G G''$ and $T' \succeq_s^T T''$;
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff
(a) $T' \succ_r^T T''$ or (b) $T' \sim_r^T T''$ and $G' \succeq_r^G G''$.

Again, it is easy to see from the definitions of these two preference domains that a necessary condition for the existence of a core stable coalitional matching μ is that $\Pi_R^{\mu} \in Core\left(R, (\succeq_r^T)_{r \in R}\right)$ *and* $\Pi_S^{\mu} \in Core\left(S, (\succeq_s^G)_{s \in S}\right)$. The non-emptiness of the cores of these two coalition formation games turns out to be a sufficient condition when the domain is $\mathcal{D}_3$.

Theorem 4 *Let* $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_3$. *Then a core stable coalitional matching exists if and only if* $Core\left(R, (\succeq_r^T)_{r \in R}\right) \neq \emptyset$ *and* $Core\left(S, (\succeq_s^G)_{s \in S}\right) \neq \emptyset$.

Surprisingly, the non-emptiness of the two cores is necessary *and* sufficient for the existence of a core stable coalitional matching also when the domain is $\mathcal{D}_4$, provided that agents' preferences in the corresponding hedonic games are strict. The main reason for this result is that when research teams and student groups from two corresponding core stable partitions are matched, then the agent set in a blocking pair contains, along with an agent, also his coalition from the corresponding core stable partition. This fact, together with the properties of the preference domain, allows us to replace coalitions by players and then identify a stable matching in the corresponding standard two-sided matching problem. In turn, this stable matching induces in a natural way a core stable coalitional matching.

Theorem 5 *Let* $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_4$ *and* $\left(\succeq_r^T\right)_{r\in R}$ *and* $\left(\succeq_s^G\right)_{s\in S}$ *be strict. Then a core stable coalitional matching exists if and only if* $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$ *and* $Core\left(S, (\succeq_s^G)_{s\in S}\right) \neq \emptyset$*.*

However, in the presence of indifferences, one has similar problems to those identified in the previous section. Recall that Example 3 shows a situation with the following features: (1) there are indifferences in the preference profile $(\succeq_r^T)_{r\in R}$; (2) both games $(R, \left(\succeq_r^T\right)_{r\in R})$ and $(S, \left(\succeq_s^G\right)_{s\in S})$ are totally balanced; (3) $(\succeq_r^G)_{r\in R}$ respects $\trianglerighteq^G$ and $(\succeq_s^T)_{s\in S}$ respects $\trianglerighteq^T$. One can easily check that with $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_4$, no core stable coalitional matching exists. Again, the corresponding common rankings have to be respected by all agents in order for such a matching to exist.

Theorem 6 *Let* $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_4$*,* $\left(\succeq_i^T\right)_{i\in R\cup S}$ *respect* $\trianglerighteq^T$ *and* $\left(\succeq_i^G\right)_{i\in R\cup S}$ *respect* $\trianglerighteq^G$*. Then a core stable coalitional matching exists.*

5 Opposite-sided priorities

Finally, we consider a situation where the primitive preferences over groups from the *opposite* market side play the leading role in agents' overall preferences: for $\mathcal{D}_5$, this leading role is a dictatorial one, while for $\mathcal{D}_6$ agents' primitive preferences over groups from their own market side are *also* taken into account.

$\mathcal{D}_5$:
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff $T' \succeq^T_s T''$;
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff $G' \succeq^G_r G''$.

$\mathcal{D}_6$:
For all $s \in S$ and $(T', G'), (T'', G'') \in 2^R \times S_s$, $(T', G') \succeq_s (T'', G'')$ iff
(a) $T' \succ^T_s T''$ or (b) $T' \sim^T_s T''$ and $G' \succeq^G_s G''$;
For all $r \in R$ and $(T', G'), (T'', G'') \in R_r \times 2^S$, $(T', G') \succeq_r (T'', G'')$ iff
(a) $G' \succ^G_r G''$ or (b) $G' \sim^G_r G''$ and $T' \succeq^T_r T''$.

Notice that the properties of the corresponding hedonic games do not play any role when agents' preferences are in $\mathcal{D}_5$.

Theorem 7 *Let $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_5$, $\left(\succeq^T_s\right)_{s \in S}$ respect $\trianglerighteq^T$ and $\left(\succeq^G_r\right)_{r \in R}$ respect $\trianglerighteq^G$. Then a core stable coalitional matching exists.*

As exemplified below it is crucial the existing common rankings to be respected by the agents from both market sides. The example shows that if $(\succeq^G_r)_{r \in R}$ respects $\trianglerighteq^G$ but $(\succeq^T_s)_{s \in S}$ does not respect a common ranking over research teams, there might not be a core stable coalitional matching when $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_5$.

Example 5 *Let $R = \{r_1, r_2, r_3\}$ and $S = \{s_1, s_2, s_3\}$. Let $(\succeq^G_r)_{r \in R}$ respect $\{s_1, s_2\} \triangleright^G \{s_1, s_3\} \triangleright^G \{s_2, s_3\} \triangleright^G \emptyset \trianglerighteq^G \ldots$, and $(\succeq^T_s)_{s \in S}$ be as follows:*

$$\{r_1, r_2\} \succ^T_{s_1} \{r_1\} \succ^T_{s_1} \emptyset;\ \{r_3\} \succ^T_{s_2} \{r_1, r_2\} \succ^T_{s_2} \emptyset;\ \{r_1\} \succ^T_{s_3} \{r_3\} \succ^T_{s_3} \emptyset.$$

There is no core stable coalitional matching when $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_5$. First note that in any coalitional matching μ with $\mu(s_1)_1 = \mu(s_2)_1 = \mu(s_3)_1 = \mu(r_1)_2 = \mu(r_2)_2 = \mu(r_3)_2 = \emptyset$ the pair $(A, \mu'))$ with $A = \{r_3, s_2, s_3\}$ and $\mu'(s_2)_1 = \mu'(s_3)_1 = \{r_3\}$ blocks μ. Next consider the coalitional matching

$\mu(s_1)_1 = \mu(s_2)_1 = \{r_1, r_2\}$, $\mu(s_3)_1 = \emptyset$, $\mu(r_1)_2 = \mu(r_2)_2 = \{s_1, s_2\}$, $\mu(r_3)_2 = \emptyset$*: it is blocked by the pair* (A, μ') *with* $A = \{r_3, s_2, s_3\}$ *and* $\mu'(s_2)_1 = \mu'(s_3) = \{r_3\}$, $\mu'(r_3)_2 = \{s_2, s_3\}$. *Further consider the coalitional matching* $\mu(s_1)_1 = \emptyset$, $\mu(s_2)_1 = \mu(s_3)_1 = \{r_3\}$, $\mu(r_1)_2 = \mu(r_2)_2 = \emptyset$, $\mu(r_3)_2 = \{s_2, s_3\}$*: it is blocked by the pair* (A, μ') *with* $A = \{r_1, s_1, s_3\}$ *and* $\mu'(s_1)_1 = \mu'(s_3)_1 = \{r_1\}$, $\mu'(r_1)_2 = \{s_1, s_3\}$. *Last consider the coalitional matching* $\mu(s_1)_1 = \mu(s_3)_1 = \{r_1\}$, $\mu(s_2)_1 = \emptyset$, $\mu(r_1)_2 = \{s_1 s_3\}$, $\mu(r_2)_2 = \mu(r_3)_2 = \emptyset$*: it is blocked by the pair* (A, μ') *with* $A = \{r_1, r_2, s_1, s_2\}$ *and* $\mu'(s_1)_1 = \mu'(s_2)_1 = \{r_1, r_2\}$, *and* $\mu'(r_1)_2 = \mu'(r_2)_2 = \{s_1, s_2\}$. *All other matchings* μ *can shown to be blocked because at least one agent prefers to be matched to the empty set than to the coalition with which he is matched under* μ.

Although the properties of the corresponding coalition formation games do not play any role when the domain is $\mathcal{D}_5$, these properties are crucial when agents' preferences over universities are in $\mathcal{D}_6$ and especially in the presence of indifferences. In our last example, the hedonic games $\left(R, (\succeq_r^T)_{r\in R}\right)$ *and* $\left(S, (\succeq_s^G)_{s\in S}\right)$ are totally balanced, $(\succeq_r^G)_{r\in R}$ respects $\trianglerighteq^G$, $(\succeq_s^T)_{s\in S}$ respects $\trianglerighteq^T$, and there are indifferences in either $\trianglerighteq^T$ or $\trianglerighteq^G$.

Example 6 *Let the set of researchers be* $R = \{r_1, r_2\}$ *and the set of students* $S = \{s_1, s_2, s_3\}$. *Let the primitive preferences* $(\succeq_r^T)_{r\in R}$ *and* $(\succeq_s^G)_{s\in S}$ *be as follows:*

$$\begin{array}{l}
\{r_1\} \succ_{r_1}^T \{r_1, r_2\};\ \{r_2\} \succ_{r_2}^T \{r_1, r_2\} \qquad ; \\
\{s_1, s_2, s_3\} \succ_{s_1}^G \{s_1, s_2\} \succ_{s_1}^G \{s_1, s_3\} \succ_{s_1}^G \{s_1\}; \\
\{s_1, s_2, s_3\} \succ_{s_2}^G \{s_2, s_3\} \succ_{s_2}^G \{s_1, s_2\} \succ_{s_2}^G \{s_2\}; \\
\{s_1, s_2, s_3\} \succ_{s_3}^G \{s_1, s_3\} \succ_{s_3}^G \{s_2, s_3\} \succ_{s_3}^G \{s_3\}.
\end{array}$$

Let $(\succeq_s^T)_{s\in S}$ *respect* $\{r_1\} =^T \{r_2\} \triangleright^T \emptyset$, *and* $(\succeq_r^G)_{r\in R}$ *respect* $\{s_1, s_2\} \triangleright^G \{s_1, s_3\} \triangleright^G \{s_2, s_3\} \triangleright^G \emptyset \trianglerighteq^G \ldots$.
There is no stable coalitional matching when $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_6$. *Consider the coalitional matching* $\mu(r_1) = \mu(s_1) = \mu(s_2) = (\{r_1\}, \{s_1, s_2\})$, $\mu(r_2) = (\{r_2\}, \emptyset)$, *and* $\mu(s_3) = (\emptyset, \{s_3\})$. *This coalitional matching is blocked by the*

pair (A, μ') *with* $A = \{r_2, s_2, s_3\}$ *and* μ' *such that* $\mu'(i) = (\{r_2\}, \{s_2, s_3\})$ *because* $\{s_2, s_3\} \succ_{r_2}^G \emptyset$, $r_1 \sim_{s_2}^T r_2$ *and* $\{s_2, s_3\} \succ_{s_2}^G \{s_1, s_2\}$, *and* $\{r_2\} \succ_{r_3}^T \emptyset$. *Similarly, one can show that no other coalitional matching is core stable.*

In view of the above example, we obtain the following result.

Theorem 8 *Let* $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_6$ *and the games* $\left(R, (\succeq_r^T)_{r \in R}\right)$ *and* $\left(S, (\succeq_s^G)_{s \in S}\right)$ *be totally balanced. Let* $\left(\succeq_s^T\right)_{s \in S}$ *respect* $\trianglerighteq^T$ *and* $\left(\succeq_r^G\right)_{r \in R}$ *respect* $\trianglerighteq^G$ *with both* $\trianglerighteq^T$ *and* $\trianglerighteq^G$ *being strict. Then a core stable coalitional matching exists.*

As we show in our final result, when we allow for indifferences in the common rankings, then a core stable coalitional matching exists if all agents' preferences respect the corresponding common rankings.

Theorem 9 *Let* $(\succeq_i)_{i \in R \cup S} \in \mathcal{D}_6$, $\left(\succeq_i^T\right)_{i \in R \cup S}$ *respect* $\trianglerighteq^T$ *and* $\left(\succeq_i^G\right)_{i \in R \cup S}$ *respect* $\trianglerighteq^G$. *Then a core stable coalitional matching exists.*

6 Conclusion

The framework of coalitional matching enables us to study situations in which groups of agents are being formed on both sides of a market. It is recognized that an agent's preferences on either side of the market depend on his peers on the same side and on the identity of the agents with whom he is matched on the other side. In this context, we derive existence results for a number of possible lexicographic preferences profiles. These results allow us to see more clearly the connections between, first, the ways in which agents' overall preferences are induced and, second, the outlined sufficient conditions. Given the existence of core stable partitions on one of the market sides and the existence of a totally balanced game on the other, we highlight the trade-off between agents' preferences being strict and satisfying a corresponding common ranking property.

The latter property is admittedly restrictive, however, quite realistic. For instance, we observe in many industries the emergence of official rankings of institutions or participants, e.g., standardized tests such as SAT, GRE, and GMAT are used to rank students for admissions to universities. Worldwide rankings of academic institutions are produced in order to facilitate comparison between departments and happen to also facilitate academic job seekers (cf. Baltagi (2003) and Neary et al. (2003)). Moreover, in many countries national university ranking tables are developed which are then used by governments to allocate research funds and prospective students in higher education (see, e.g., Dill and Soo (2005)).

To illustrate our concepts, throughout this paper, we have used the example of students and researchers forming universities. The proposed framework, however, has a wider applicability and can be also used to study, for instance, hospital formations by medical staff and patient groups, sport club formation by athletes and coaching teams, and editorial boards and authors make up journals.

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Appendix

Proof of Theorem 1. We show first that if $Core\left(R,(\succeq_r^T)_{r\in R}\right)=\emptyset$, then each coalitional matching can be blocked. Fix a coalitional matching μ and let Π_R^μ be the partition of R into research teams induced by μ. Since $\Pi_R^\mu \notin Core\left(R,(\succeq_r^T)_{r\in R}\right)$, there exists $A\subseteq R$ such that $A\succ_r^T \Pi_R^\mu(r)$ for all $r\in A$. Define the coalitional matching μ' by $\mu'(r)=(A,\emptyset)$ for all $r\in A$. Since $(\succeq_r)_{r\in R}\in\mathcal{D}_1$, the pair (A,μ') is a blocking for μ.

Suppose next that $Core\left(R,(\succeq_r^T)_{r\in R}\right)\neq\emptyset$ and let $\Pi=\{T_1,\ldots,T_K\}\in Core\left(R,(\succeq_r^T)_{r\in R}\right)$. We construct in what follows a core stable coalitional matching. For each $s\in S$, let $T_s\in\Pi\cup\{\emptyset\}$ be such that $T_s\succeq_s^T T'$ for all $T'\in\Pi$. Further, for each $T'\in\Pi\cup\{\emptyset\}$, define $S^{T'}:=\{s\in S: T_s=T'\}$. Consider the coalitional matching μ defined as follows:

(1) For all $k=1,\ldots,K$, $\mu(s)=\left(T_k,S^{T_k}\right)$ for all $s\in S^{T_k}$,

(2) $\mu(s)=\left(\emptyset,S^{\emptyset}\right)$ for all $s\in S^{\emptyset}$,

(3) For all $k=1,\ldots,K$, $\mu(r)=\left(T_k,S^{T_k}\right)$ for all $r\in T_k$.

We show that there is no blocking for μ. Let, on the contrary, (A,μ') be such a blocking. If $A\cap R\neq\emptyset$, then, for $r\in A\cap R$, $\mu'(r)_1\succ_{r'}^T\mu(r')_1=\Pi(r')$ for all $r'\in\mu'(r)_1$ would imply that $\mu'(r)_1$ blocks Π in contradiction to $\Pi\in Core\left(R,\left(\succeq_r^T\right)_{r\in R}\right)$. If $A\subseteq S$ and $\mu'(s)_1\neq\emptyset$ for some $s\in A$ then, by $(\succeq_r)_{r\in R}\in\mathcal{D}_1$ and in order that all researchers in $\mu'(s)_1$ are strictly better off under μ', we would have $\emptyset\neq\mu'(s)_1\succ_r^T\mu(r)_1=\Pi(r)$ for all $r\in\mu'(s)_1$. Thus, $\mu'(s)_1$ would block Π in contradiction to $\Pi\in Core\left(R,\left(\succeq_r^T\right)_{r\in R}\right)$. Hence, $\mu'(s)_1=\emptyset$ should hold for all $s\in A\subseteq S$. Notice however that, by construction of μ, $\mu(s)_1\succ_s^T\emptyset=\mu'(s)_1$ for all $s\in\cup_{k=1}^K S^{T_k}$ and $\mu(s)_1=\emptyset$ for all $s\in S^{\emptyset}=S\setminus\left(\cup_{k=1}^K S^{T_k}\right)$. Thus, it is impossible that (A,μ') blocks μ. ■

Proof of Theorem 2. We construct a core stable coalitional matching. Let $\Pi^R=\{T_1,\ldots,T_K\}\in Core\left(R,(\succeq_r^T)_{r\in R}\right)$ be such that $T_k\rhd^T T_{k+1}$ for all $k=1,\ldots,K-1$, and let $q=\min\left\{k\in\{1,\ldots,K\}\mid T_k\rhd^T\emptyset\right\}$. Let Π^S be a collection of $M>0$ student groups which are pairwise disjoint and whose union is S and suppose that, w.l.o.g., $q\leq M$. The collection Π^S

is constructed as follows. Define $S_0 := S$, $G_0 := \emptyset$ and let, for all $l \in \{1,\ldots,q\}$, $G_l \subseteq S_l := S_{l-1} \setminus G_{l-1}$ be such that $G_l \succeq_r^G G'$ for some $r \in T_l$ and all $G' \in 2^{S_l}$; then, we let $\Pi^S := \{G_1,\ldots,G_q,G_{q+1},\ldots,G_M\}$, where $\{G_{q+1},\ldots,G_M\} \in Core\left(Q,(\succeq_{s|Q}^G)_{s\in Q}\right)$ with $Q := S \setminus (\cup_{l=1}^q G_l)$. We show now that the coalitional matching μ defined by

(1) $\mu(i) = (T_k, G_k)$ for all $i \in T_k \cup G_k$ and all $k \leq q$,

(2) $\mu(s) = (\emptyset, G_k)$ for all $s \in G_k$ and all $k \in \{q+1,\ldots,M\}$,

(3) $\mu(r) = (T_k, \emptyset)$ for all $r \in T_k$ and all $k \in \{q+1,\ldots,K\}$

is core stable.

Suppose on the contrary that there is a blocking (A,μ') for μ. If $A \subseteq S$, then there cannot be a student $s \in A$ such that $\mu(s) = (T_k, G_k)$ with $k \leq q$; the reason is that, in view of the construction of μ, $T_k \rhd^T \emptyset = \mu'(s)_1$ would imply $(T_k, G_k) \succ_s^T (\emptyset, \mu'(s)_2)$ which is a contradiction to (A,μ') being a blocking for μ. If $A \subseteq S$ and each student $s \in A$ is such that $\mu(s) = (\emptyset, G_{k_s})$ for some $k_s \in \{q+1,\ldots,M\}$, then $\mu'(s) = (\emptyset, \mu'(s)_2) \succ_s^T (\emptyset, G_{k_s})$ for all $s \in A$ is only possible if $\mu'(s)_2 \succ_s^G G_{k_s}$ for all $s \in A$. Notice also that $\mu(s) = (\emptyset, G_{k_s})$ for some $k_s \in \{q+1,\ldots,M\}$ and all $s \in A$ implies $s \in S \setminus (\cup_{k=1}^q G_k)$. Hence, we have then that $\mu'(s)_2$ blocks the partition $\{G_{q+1},\ldots,G_M\} \in Core\left(Q,(\succeq_{s|Q}^G)_{s\in Q}\right)$, a contradiction.

Suppose now that $A \subseteq R$ and $A \notin \Pi^R$. Take $r' \in A$ and note that it must be the case of $\mu(r')_1 \succ_r^T \Pi^R(r)$ for all $r \in \mu(r')_1$ because $(\succeq_r)_{r\in R} \in \mathcal{D}_2$ and the primitive preferences $(\succeq_r^T)_{r\in R}$ are strict. Thus, $\mu(r')_1$ blocks Π^R in contradiction to $\Pi^R \in Core\left(R,(\succeq_r^T)_{r\in R}\right)$.

Last, suppose that $A = T_k \cup G'$ with $T_k \in \Pi^R$ and $G' \subseteq S$. Notice first that, in order (A,μ') to be a blocking pair for μ, one should have $\mu'(r)_1 = T_k$ for all $r \in T_k$; otherwise, $\mu'(r)_1 \subset T_k$ for some $r \in T_k$ would imply, by $(\succeq_r)_{r\in R} \in \mathcal{D}_2$ and the primitive preferences $(\succeq_r^T)_{r\in R}$ being strict, that $\mu'(r)_1 \succ_{r'}^T T_k$ for all $r' \in \mu'(r)_1$. In the latter case $\mu'(r)_1$ would block Π^R in contradiction to $\Pi^R \in Core\left(R,(\succeq_r^T)_{r\in R}\right)$.

Notice further that $\mu'(r)_1 = T_k$ for all $r \in T_k$ requires, in order all researchers in T_k to be strictly better off under μ', that $\mu'(r)_2 \succ_r^G \mu(r)_2$ for all $r \in T_k$.

If $k \in \{1, \ldots, q\}$, then by definition of μ there exists at least one researcher $r_k \in T_k$ such that $\mu(r_k)_2 = G_k \succeq_r^G G'$ for all $G' \in 2^{\cup_{m=k}^M G_m}$; hence, r_k would be strictly better off under μ' only if $\mu'(r_k)_2$ contains at least one student $s_{r_k} \in \left(\cup_{m=1}^{k-1} G_m\right)$. However, it follows from $\mu'(r_k)_1 = T_k$ and $s_{r_k} \in \mu'(r_k)_2$ that $\mu'(s_{r_k})_1 = T_k$. Since $\mu(s_{r_k})_1 = T_{k'}$ for some $k' < k$, $(\succeq_s)_{s \in S} \in \mathcal{D}_2$ and $\left(\succ_s^T\right)_{s \in S}$ respects $\trianglerighteq^T$ with $\trianglerighteq^T$ being strict, we have that $\mu(s_{r_k})_1 = T_{k'} \succ_{s_{r_k}}^T T_k = \mu'(s_{r_k})_1$. Thus, (A, μ') cannot block μ.

If $k \in \{q+1, \ldots, K\}$, then $\mu(r) = (T_k, \emptyset)$ for all $r \in T_k$ and $\mu(s) = (\emptyset, G_k)$ with $k \in \{q+1, \ldots, M\}$ for all $s \in G'$. Notice first that, by the definition of q, $(\succeq_s)_{s \in S} \in \mathcal{D}_2$ and the fact that $\left(\succeq_s^T\right)_{s \in S}$ respects $\trianglerighteq^T$, we should have $G' \subseteq \left(\cup_{m=q+1}^M G_m\right)$. Then, in order all researchers in A (i.e., in T_k) to be strictly better off under μ', one should have $\mu'(r)_2 \succ_r^G \emptyset$ for all $r \in T_k$ since, as already shown, $\mu'(r)_1 = T_k$ for all $r \in T_k = A \cap R$. Suppose that this is indeed the case. Notice first that, for all $r \in T_k$, we have $\emptyset \neq \mu'(r)_2 \subseteq G'$ and $\mu'(r)_1 = T_k = \mu'(s_r)_1$ for all $s_r \in \mu'(r)_2$. Thus, there are two possibilities for $s_r \in \mu'(r)_2$ to strictly better off under μ'. The first one is that $\mu'(s_r)_1 = T_k \succ_s^T \emptyset = \mu_1(s_r)$ which leads to a contradiction since, by $\left(\succeq_s^T\right)_{s \in S}$ respecting $\trianglerighteq^T$ and the definition of q, there is no $s \in S$ such that $T_k \succ_s^T \emptyset$. Hence, the only remaining second possibility for $s_r \in \mu'(r)_2$ to be strictly better off under μ' (and hence, the only possibility for all $s_r \in \mu'(r)_2$ to be strictly better off under μ') is that one has $\mu'(s_r)_1 = T_k \sim_s^T \emptyset = \mu(s_r)_1$ and $\mu'(s_r)_2 \succ_{s_r}^G G_k = \mu(s_r)_2$ which is not possible because of $\left(\succeq_s^T\right)_{s \in S}$ respecting $\trianglerighteq^T$ and $\trianglerighteq^T$ being strict. ■

Proof of Theorem 3. We construct a core stable coalitional matching. Let $\Pi^R = \{T_1, \ldots, T_K\}$ be a partition of R with $T_k \trianglerighteq^T T'$ for all $T' \in 2^{R_k}$ for all $k \in \{1, \ldots, K\}$, where $R_0 := R$, $T_0 := \emptyset$, $T_k \subseteq R_k := R_{k-1} \setminus T_{k-1}$. Similarly, let $\Pi^S = \{G_1, \ldots, G_M\}$ be a partition of S with $G_m \trianglerighteq^G G'$ for all $G' \in 2^{S_m}$ for all $m \in \{1, \ldots, M\}$, where $S_0 := S$, $G_0 := \emptyset$, $G_m \subseteq S_m := S_{m-1} \setminus G_{m-1}$.

Note that $\Pi^R \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$ and $\Pi^S \in Core\left(S, (\succeq_s^G)_{s\in S}\right)$. Further, let $q^R = \min\{k \in \{1,\dots,K\} \mid T_k \rhd^T \emptyset\}$, $q^S = \min\{m \in \{1,\dots,M\} \mid G_m \rhd^G \emptyset\}$, and suppose, w.l.o.g., that $q^R \leq q^S$. We show now that the coalitional matching μ defined by

(1) $\mu(i) = (T_k, G_k)$ for all $i \in T_k \cup G_k$ and all $k \leq q^R$,

(2) $\mu(s) = (\emptyset, G_k)$ for all $s \in G_k$ and all $k \in \{q^R+1,\dots,M\}$,

(3) $\mu(r) = (T_k, \emptyset)$ for all $r \in T_k$ and all $k \in \{q^R+1,\dots,K\}$

is core stable.

Suppose on the contrary that there is a blocking pair (A, μ') for μ. First, suppose that $A \subseteq R$. Fix $r \in A$ and note that, since $(\succeq_r)_{r\in R} \in \mathcal{D}_2$, we should have, for each $r' \in \mu'(r)_1 \subseteq A$, either $\mu'(r)_1 = \mu'(r')_1 \succ_{r'}^T \mu(r')_1 = \Pi^R(r')$ or $\mu'(r)_1 = \mu'(r')_1 \sim_{r'}^T \mu(r')_1 = \Pi^R(r')$ and $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ_{r'}^G \mu(r')_2$. If $\mu'(r)_1 \succ_{r'}^T \Pi^R(r')$ holds for all $r' \in \mu'(r)_1$, then $\mu'(r)_1$ blocks Π^R in contradiction to $\Pi^R \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$. Therefore, there must be a researcher $r' \in \mu'(r)_1$ with $\mu'(r)_1 = \mu'(r')_1 \sim_{r'}^T \mu(r')_1 = \Pi^R(r')$ and $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ_{r'}^G \mu(r')_2$. Note that it must be that $\mu(r')_2 \neq \emptyset$ which, by construction of μ, is possible only if $\mu(r')_2 = G_{k_{r'}}$ for some $k_{r'} \leq q^R$. Since $\left(\succeq_r^G\right)_{r\in R}$ respects $\trianglerighteq^G$, we have $\emptyset \rhd^G G_{k_{r'}}$ which is a contradiction to the definition of q^R and $k_{r'} \leq q^R \leq q^S$.

Next, suppose that $A \subseteq S$. Fix $s \in A$ and note that, since $(\succeq_s)_{s\in S} \in \mathcal{D}_2$, we should have, for each $s' \in \mu'(s)_2 \subseteq A$, either $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ_{s'}^T \mu(s')_1$ or $\mu'(s)_1 = \mu'(s')_1 = \emptyset \sim_{s'}^T \mu(s')_1$ and $\mu'(s)_2 = \mu'(s')_2 \succ_{s'}^G \mu(s')_2 = \Pi^S(s')$. Note first that, by construction of μ, $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ_{s'}^T \mu(s')_1 \neq \emptyset$ can hold for some $s' \in \mu'(s)_2$ only if $\mu(s')_1 = T_{k_{s'}}$ for some $k_{s'} \leq q^R$. Since $\left(\succeq_s^T\right)_{s\in S}$ respects $\trianglerighteq^T$, we have $\emptyset \rhd^T T_{k_{s'}}$ which is a contradiction to the definition of q^R and $k_{s'} \leq q^R$. Therefore, for all $s' \in \mu'(s)_2 \subseteq A$ it must be that $\mu'(s)_1 = \mu'(s')_1 = \emptyset \sim_{s'}^T \mu(s')_1$ and $\mu'(s)_2 = \mu'(s')_2 \succ_{s'}^G \mu(s')_2 = \Pi^S(s')$. Thus, $\mu'(s)_2$ blocks Π^S in contradiction to $\Pi^S \in Core\left(S, (\succeq_s^G)_{s\in S}\right)$.

Last, suppose that $A \cap R \neq \emptyset$ and $A \cap S \neq \emptyset$. If $A \cap T_1 \neq \emptyset$ then $\mu(r)_2 = G_1$ for $r \in A \cap T_1$. Since $\left(\succeq_r^T\right)_{r\in R}$ respects $\trianglerighteq^T$ and $\left(\succeq_r^G\right)_{r\in R}$ respects $\trianglerighteq^G$, and by construction $T_1 \trianglerighteq^T T'$ for all $T' \in 2^R$ and $G_1 \trianglerighteq^G G'$ for all $G' \in 2^S$, it is not possible, by $(\succeq_r)_{r\in R} \in \mathcal{D}_2$, that $r \in A \cap T_1$ is strictly better off under μ'. By an analogous argument, no student in $A \cap G_1$ can be made strictly better off under μ'. Hence, we should have $A \subseteq (R \setminus T_1) \cup (S \setminus G_1)$.

Similarly, for $k \in \{2, 3, \dots, q^R\}$, we can show that $A \cap T_k = \emptyset$ and $A \cap G_k = \emptyset$. Therefore, $A \subseteq \left(R \setminus \left(\cup_{k=1}^{q^R} T_k\right)\right) \cup \left(S \setminus \left(\cup_{k=1}^{q^R} G_k\right)\right)$. Moreover, for all $r, s \in A$ holds then $\mu(r)_2 = \emptyset$ and $\mu(s)_1 = \emptyset$. Since by definition $\emptyset \trianglerighteq^T T'$ for all $T' \in 2^{R \setminus T_q}$ with $T_q = \cup_{k=1}^{q^R} T_k$, and $\left(\succeq_s^T\right)_{s\in S}$ respects $\trianglerighteq^T$, it follows that $\mu'(s)_1 \sim_s^T \mu(s)_1$ holds for all $s \in A$. Fix $s \in A$ and notice that, in order μ' to be a blocking for μ, it must be, for all $s' \in \mu'(s)_2 \subseteq A$, that $\mu'(s)_2 = \mu'(s')_2 \succ_{s'}^G \mu(s')_2 = G_{k_{s'}}$ for some $k_{s'} \in \{q^R + 1, \dots, M\}$. The latter means that $\mu'(s)_2$ blocks $\left\{G_{q^R+1}, \dots, G_M\right\}$ which establishes a contradiction since, by $\left(\succeq_s^G\right)_{s\in S}$ respecting $\trianglerighteq^G$ and the construction of Π^S, $\left\{G_{q^R+1}, \dots, G_M\right\} \in Core\left(S_q, (\succeq_{s|S_q}^G)_{s\in S_q}\right)$ with $S_q = S \setminus \cup_{k=1}^{q^R} G_k$. Hence, A does not contain any students. By an analogous argument, and since both researchers' and students' primitive preference over student groups respect $\trianglerighteq^G$, A does not contain any researcher either. We conclude then that no blocking for μ exists. ■

Proof of Theorem 4. Let $\Pi^R \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$ and $\Pi^S \in Core\left(S, (\succeq_s^G)_{s\in S}\right)$. Since $(\succeq_i)_{i\in R\cup S} \in \mathcal{D}_3$, it is trivial to see that the coalitional matching μ defined by $\mu(r) = \left(\Pi^R(r), \emptyset\right)$ for all $r \in R$, and $\mu(s) = \left(\emptyset, \Pi^S(s)\right)$ for all $s \in S$ is core stable. Suppose next that, w.l.o.g., $Core\left(R, (\succeq_r^T)_{r\in R}\right) = \emptyset$. As already shown in the first part of the proof of Theorem 1, a blocking for μ does exist in this case. ■

Proof of Theorem 5. We show first that if, w.l.o.g., $Core\left(R, (\succeq_r^T)_{r\in R}\right) = \emptyset$, then each coalitional matching can be blocked. Fix a coalitional matching μ and let Π_R^μ is the partition of R into research teams induced by μ. Since $\Pi_R^\mu \notin Core\left(R, (\succeq_r^T)_{r\in R}\right)$, there exists $A \subseteq R$ such that $A \succ_r^T \Pi_R^\mu(r)$ for all $r \in A$. Define the matching μ' by $\mu'(r) = (A, \emptyset)$ for all $r \in A$. Since $(\succeq_r)_{r\in R} \in \mathcal{D}_4$, the pair (A, μ') is a blocking for μ.

Suppose next that $Core\left(R, (\succeq_r^T)_{r\in R}\right) \neq \emptyset$ and $Core\left(S, (\succeq_s^G)_{s\in S}\right) \neq \emptyset$ and we show that a core stable coalitional matching exists. In particular, we will show that the existence of such a matching follows from the existence of a stable matching in a standard two-sided matching problem as shown by Sotomayor (1996).

Let $\Pi^R = \{T_1, \ldots, T_K\} \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$ and $\Pi^S = \{G_1, \ldots, G_M\} \in Core\left(S, (\succeq_s^G)_{s\in S}\right)$. Let μ be a coalitional matching of the following type: for all $r \in R$, $\mu(r)_1 = \Pi^R(r)$ and $\mu(r)_2 = \Pi^S(s) \cup \emptyset$ for some $s \in S$; for all $s \in S$, $\mu(s)_2 = \Pi^S(s)$ and $\mu(s)_1 = \Pi^R(r) \cup \emptyset$ for some $r \in R$. Suppose that (A, μ') is a blocking for μ.

We show first that if $A \cap R \neq \emptyset$, then $\Pi^R(r) \subseteq A$ and $\mu'(r)_1 = \Pi^R(r)$ for all $r \in A$. Fix $r \in A$ and note that, since $(\succeq_r)_{r\in R} \in \mathcal{D}_4$, we should have, for each $r' \in \mu'(r)_1 \subseteq A$, either $\mu'(r)_1 = \mu'(r')_1 \succ_{r'}^T \mu(r')_1 = \Pi^R(r')$ or $\mu'(r)_1 = \mu'(r')_1 \sim_{r'}^T \mu(r')_1 = \Pi^R(r')$ and $\mu'(r)_2 = \mu'(r')_2 \succ_{r'}^G \mu(r')_2$. If $\mu'(r')_1 \succ_{r'}^T \Pi^R(r')$ holds for all $r' \in \mu'(r)_1$, then $\mu'(r)_1$ would be blocking Π^R in contradiction to $\Pi^R \in Core\left(R, (\succeq_r^T)_{r\in R}\right)$. Therefore, there must be a researcher $r' \in \mu'(r)_1$ with $\mu'(r)_1 = \mu'(r')_1 \sim_{r'}^T \mu(r')_1 = \Pi^R(r')$. Since $\left(\succeq_r^T\right)_{r\in R}$ is a profile of strict preferences, we have then $\mu'(r)_1 = \mu'(r')_1 = \mu(r')_1 = \Pi^R(r')$. Thus, we have $r \in \mu(r')_1$ and hence, $\mu'(r)_1 = \Pi^R(r') = \Pi^R(r) \subseteq A$. Similarly, one can conclude that for all $s \in A$, $\Pi^S(s) \subseteq A$ and $\mu'(s)_2 = \Pi^S(s)$.

For each $T_k \in \Pi^R$, fix $r_{T_k} \in T_k$ and for each $G_m \in \Pi^S$, fix $s_{G_m} \in G_m$. Let $R^{\Pi^R} = \{r_{T_1}, \ldots, r_{T_K}\}$ and $S^{\Pi^S} = \{s_{G_1}, \ldots, s_{G_M}\}$. For each $r_{T_k} \in R^{\Pi^R}$, let $\geq_{r_{T_k}}$ be a complete and transitive preference relation on $S^{\Pi^S} \cup \{r_{T_k}\}$ defined as follows: for all $m_1, m_2 \in \{1, \ldots, M\}$, $s_{G_{m_1}} \geq_{r_{T_k}} s_{G_{m_2}}$ if and only if $G_{m_1} \succeq_{r_{T_k}}^G G_{m_2}$ and, for all $m \in \{1, \ldots, M\}$, $r_{T_k} \geq_{r_{T_k}} s_{G_m}$ if and only if $\emptyset \succeq_{r_{T_k}}^G G_m$. For each $s_{G_m} \in S^{\Pi^S}$, let $\geq_{s_{G_m}}$ be a complete and transitive preference relation on $R^{\Pi^R} \cup \{s_{G_m}\}$ defined in an analogous way. Notice then that the sets R^{Π^R} and S^{Π^S} together with the corresponding preferences form a well defined standard two-sided matching problem. As shown by Gale and

Shapley (1962) and Sotomayor (1996), a stable matching in this problem always exists.

Take now a stable matching ν in the two-sided matching problem described above. Notice that ν induces a coalitional matching μ^{ν} as follows: for all $r \in R$, $\mu^{\nu}(r)_1 = \Pi^R(r)$ and $\mu^{\nu}(r)_2 = G_m \in \Pi^S \cup \{\emptyset\}$ with $\nu\left(r_{\Pi^R(r)}\right) = s_{G_m}$; for all $s \in S$, $\mu^{\nu}(s)_2 = \Pi^S(s)$ and $\mu^{\nu}(s)_1 = T_k \in \Pi^R \cup \{\emptyset\}$ with $\nu\left(s_{\Pi^S(s)}\right) = r_{T_k}$. We show that μ^{ν} is core stable.

Suppose on the contrary that there is a blocking (B, μ'') for μ^{ν}. From the analysis above we know that for all $r \in B$, $\mu''(r)_1 = \mu^{\nu}(r)_1 = \Pi^R(r)$. Therefore, for μ'' to be blocking μ^{ν}, it must be that $\mu''(r)_2 \succ_r^G \mu^{\nu}(r)_2 \in \Pi^S \cup \{\emptyset\}$ holds for all $r \in B$ and in particular for $r_{\Pi^R(r)} \in \Pi^R(r)$ with $\Pi^R(r) \subseteq B$. Thus, we should have that $\mu''\left(r_{\Pi^R(r)}\right)_2 \succ^G_{r_{\Pi^R(r)}} \mu^{\nu}\left(r_{\Pi^R(r)}\right)_2 = G_m \in \Pi^S \cup \{\emptyset\}$ with $\nu\left(r_{\Pi^R(r)}\right) = s_{G_m}$. Similarly, we have $\mu''(s)_2 = \mu^{\nu}(s)_2 = \Pi^S(s)$ and thus, $\mu''(s)_1 \succ_s^T \mu^{\nu}(s)_1 \in \Pi^R \cup \{\emptyset\}$ should hold for all $s \in B$ and, in particular, for $s_{\Pi^S(s)} \in \Pi^S(s)$ with $\Pi^S(s) \subseteq B$. Hence, we have $\mu''\left(s_{\Pi^S(s)}\right)_1 \succ^T_{s_{\Pi^S(s)}} \mu^{\nu}\left(s_{\Pi^S(s)}\right)_1 = T_k \in \Pi^R \cup \{\emptyset\}$ with $\nu\left(s_{\Pi^S(s)}\right) = r_{T_k}$.

First consider the case when $B \subseteq R$. The analysis above implies that for some $r_{\Pi^R(r)} \in \Pi^R(r)$ with $\Pi^R(r) \subseteq B$, it holds that $\mu''\left(r_{\Pi^R(r)}\right)_2 = \emptyset$. Notice that $\emptyset \succ^G_{r_{\Pi^R(r)}} \mu^{\nu}\left(r_{\Pi^R(r)}\right)_2$ is possible only if $\mu^{\nu}\left(r_{\Pi^R(r)}\right)_2 = G_m \in \Pi^S$ with $\nu\left(r_{\Pi^R(r)}\right) = s_{G_m}$. By construction, we have then $r_{\Pi^R(r)} >_{r_{\Pi^R(r)}} s_{G_m}$ (i.e., $r_{\Pi^R(r)}$ prefers to stay alone than to be matched (as he is under ν) to s_{G_m}) in contradiction to the fact that ν is stable for the above defined standard two-sided matching problem.

Next consider the case when $B \subseteq S$. The analysis above implies that for some $s_{\Pi^S(s)} \in \Pi^S(s)$ with $\Pi^S(s) \subseteq B$, it holds that $\mu''\left(s_{\Pi^S(s)}\right)_1 = \emptyset$. Notice that $\emptyset \succ^T_{s_{\Pi^S(s)}} \mu^{\nu}\left(s_{\Pi^S(s)}\right)_1$ is possible only if $\mu^{\nu}\left(s_{\Pi^S(s)}\right)_1 = T_k \in \Pi^R$ with $\nu\left(s_{\Pi^S(s)}\right) = r_{T_k}$. By construction, we have then $s_{\Pi^S(s)} >_{s_{\Pi^S(s)}} r_{T_k}$ (i.e., $s_{\Pi^S(s)}$ prefers to stay alone than to be matched (as he is under ν) to r_{T_k}) in contradiction to the fact that ν is stable for the above defined standard two-sided matching problem.

Last consider the case when $B \cap R \neq \emptyset$ and $B \cap S \neq \emptyset$. The analysis above again implies that there are $r_{\Pi^R(r)} \in \Pi^R(r)$ with $\Pi^R(r) \subseteq B$ and $s_{\Pi^S(s)} \in \Pi^S(s)$ with $\Pi^S(s) \subseteq B$ such that $\mu''\left(r_{\Pi^R(r)}\right)_1 = \mu^\nu\left(r_{\Pi^R(r)}\right)_1$ and $\mu''\left(s_{\Pi^S(s)}\right)_2 = \mu^\nu\left(s_{\Pi^S(s)}\right)_2$. Moreover, for (B, μ'') to be blocking μ^ν it must also hold that $\mu''\left(r_{\Pi^R(r)}\right)_2 = \mu^\nu\left(s_{\Pi^S(s)}\right)_2 \succ \mu^\nu\left(r_{\Pi^R(r)}\right)_2$ and $\mu''\left(s_{\Pi^S(s)}\right)_1 = \mu^\nu\left(r_{\Pi^R(r)}\right)_1 \succ \mu^\nu\left(s_{\Pi^S(s)}\right)_1$. Since by construction $\mu^\nu\left(r_{\Pi^R(r)}\right)_2 = G_m \in \Pi^S \cup \{\emptyset\}$ with $\nu(r_{\Pi^R(r)}) = s_{G_m}$ and $\mu^\nu\left(s_{\Pi^S(s)}\right)_1 = T_k \in \Pi^R \cup \{\emptyset\}$ with $\nu(s_{\Pi^S(s)}) = r_{T_k}$, this implies that $s_{\Pi^S(s)} >_{r_{\Pi^R(r)}} s_{G_m}$ and $r_{\Pi^R(r)} >_{s_{\Pi^S(s)}} r_{T_k}$ in contradiction to the fact that ν is stable for the above defined standard two-sided matching problem. ■

Proof of Theorem 6. Let μ be the coalitional matching constructed in the proof of Theorem 3. If there is a blocking pair (A, μ') for μ, then notice that reaching a contradiction goes in the same way as in the corresponding parts of the proof of Theorem 3, except for $A \subseteq S$. For this case, fix $s \in A$ and note that, since $(\succeq_s)_{s \in S} \in \mathcal{D}_4$, we should have, for each $s' \in \mu'(s)_2 \subseteq A$, either $\mu'(s)_2 = \mu'(s')_2 \succ_{s'}^G \mu(s')_2 = \Pi^S(s')$ or $\mu'(s)_2 = \mu'(s')_2 \sim_{s'}^G \mu(s')_2 = \Pi^S(s')$ and $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ_{s'}^T \mu(s')_1$. If $\mu'(s)_2 \succ_{s'}^G \Pi^S(s')$ holds for all $s' \in \mu'(s)_2$, then $\mu'(s)_2$ blocks Π^S in contradiction to $\Pi^S \in Core\left(S, (\succeq_s^G)_{s \in S}\right)$. Therefore, there must be a student $s' \in \mu'(s)_2$ with $\mu'(s)_2 = \mu'(s')_2 \sim_{s'}^G \mu(s')_2 = \Pi^S(s')$ and $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ_{s'}^T \mu(s')_1$. Note that it must be that $\mu(s')_1 \neq \emptyset$ which, by construction of μ, is possible only if $\mu(s')_1 = T_{k_{s'}}$ for some $k_{s'} \leq q^R$. Since $\left(\succeq_s^T\right)_{s \in S}$ respects $\trianglerighteq^T$, we have $\emptyset \triangleright^T T_{k_{s'}}$ which is a contradiction to the definition of q^R and $k_{s'} \leq q^R$. ■

Proof of Theorem 7. Let $\{T_1, \ldots, T_K\}$ be a collection of research teams which are pairwise disjoint and $\cup_{k=1}^K T_k = R$ with $T_k \trianglerighteq^T T'$ for all $k = 1, \ldots, K$ and all $T' \subseteq R \setminus \left(\cup_{k'=1}^{k-1} T_{k'}\right)$, and let $\{G_1, \ldots, G_L\}$ be a collection of student groups which are pairwise disjoint and $\cup_{l=1}^L G_l = S$ with $G_l \trianglerighteq^G G'$ for all $l = 1, \ldots, L$ and all $G' \subseteq S \setminus \left(\cup_{l'=1}^{l-1} G_{l'}\right)$. Further, let $q^R = \min\left\{k \in \{1, \ldots, K\} \mid T_k \triangleright^T \emptyset\right\}$ and $q^S = \min\left\{l \in \{1, \ldots, L\} \mid G_l \triangleright^G \emptyset\right\}$, and suppose, w.l.o.g., that $q^R \leq q^S$. We show that the coalitional matching μ defined by

(1) $\mu(i) = (T_q, G_q)$ for all $i \in T_q \cup G_q$ and all $q \leq q^R$,
(2) $\mu(s) = \left(\emptyset, S \setminus \cup_{l=1}^{q^R} G_l\right)$ for all $s \in S \setminus \cup_{l=1}^{q^R} G_l$,
(3) $\mu(r) = \left(R \setminus \cup_{k=1}^{q^R} T_k, \emptyset\right)$ for all $r \in R \setminus \cup_{k=1}^{q^R} T_k$

is core stable.

Suppose on the contrary that there is a blocking pair (A, μ') for μ. First, suppose that $A \subseteq R$. Since $(\succeq_r)_{r\in R} \in \mathcal{D}_5$, we should have $\mu'(r)_2 = \emptyset \succ_r^G \mu(r)_2$ for all $r \in A$. Note that it must be that $\mu(r)_2 \neq \emptyset$ which, by construction of μ, is possible only if $\mu(r)_2 = G_{q_r}$ for some $q_r \leq q^R$. Since $\left(\succeq_r^G\right)_{r\in R}$ respects $\trianglerighteq^G$, we have $\emptyset \triangleright^G G_{q_r}$ which is a contradiction to the definition of q^R and $q_r \leq q^R \leq q^S$.

Next, suppose that $A \subseteq S$. Since $(\succeq_s)_{s\in S} \in \mathcal{D}_5$, we should have $\mu'(s)_1 = \emptyset \succ_s^T \mu(s)_1$ for all $s \in A$. Note that it must be that $\mu(s)_1 \neq \emptyset$ which, by construction of μ, is possible only if $\mu(s)_1 = T_{q_s}$ for some $q_s \leq q^R$. Since $\left(\succeq_s^T\right)_{r\in R}$ respects $\trianglerighteq^T$, we have $\emptyset \triangleright^T T_{q_s}$ which is a contradiction to the definition of q^R and $q_s \leq q^R$.

Last, suppose that $A \cap R \neq \emptyset$ and $A \cap S \neq \emptyset$. If $A \cap T_1 \neq \emptyset$ then $\mu(r)_2 = G_1$ for $r \in A \cap T_1$. Since $\left(\succeq_s^T\right)_{s\in S}$ respects $\trianglerighteq^T$ and $\left(\succeq_r^G\right)_{r\in R}$ respects $\trianglerighteq^G$, and by construction $T_1 \trianglerighteq^T T'$ for all $T' \in 2^R$ and $G_1 \trianglerighteq^G G'$ for all $G' \in 2^S$, it is not possible, by $(\succeq_r)_{r\in R} \in \mathcal{D}_5$, that $r \in A \cap T_1$ be strictly better off under μ'. By an analogous argument, no student in $A \cap G_1$ can be made strictly better off under μ'. Hence, we should have $A \subseteq (R \setminus T_1) \cup (S \setminus G_1)$.

Similarly, for $q \in \{2, \ldots, q^R\}$, we can show that $A \cap T_q = \emptyset$ and $A \cap G_q = \emptyset$. Therefore, $A \subseteq \left(R \setminus \left(\cup_{q=1}^{q^R} T_q\right)\right) \cup \left(S \setminus \left(\cup_{q=1}^{q^R} G_q\right)\right)$. Moreover, for all $r, s \in A$ holds then $\mu(r)_2 = \emptyset$ and $\mu(s)_1 = \emptyset$. Since by definition $\emptyset \trianglerighteq^T T'$ for all $T' \in 2^{R\setminus T''}$ with $T'' = \cup_{q=1}^{q^R} T_q$, and $\left(\succeq_s^T\right)_{s\in S}$ respects $\trianglerighteq^T$, it follows that $\mu'(s)_1 \sim_s^T \mu(s)_1$ holds for all $s \in A$. Hence, A does not contain any students. By an analogous argument, A does not contain any researcher either. We conclude then that no blocking for μ exists. ■

Proof of Theorem 8. We construct a core stable coalitional matching. Let $\{T_1, \ldots, T_K\}$ be a collection of research teams which are pairwise disjoint and $\cup_{k=1}^{K} T_k = R$ with $T_k \rhd^T T'$ for all $k = 1, \ldots, K$ and all $T' \subseteq R \setminus \left(\cup_{k'=1}^{k-1} T_{k'}\right)$, and $\{G_1, \ldots, G_L\}$ be a collection of student groups which are pairwise disjoint and $\cup_{l=1}^{L} G_l = S$ with $G_l \rhd^G G'$ for all $l = 1, \ldots, L$ and all $G' \subseteq S \setminus \left(\cup_{l'=1}^{l-1} G_{l'}\right)$. Further, let $q^R = \min\left\{k \in \{1, \ldots, K\} \mid T_k \rhd^T \emptyset\right\}$ and $q^S = \min\left\{l \in \{1, \ldots, L\} \mid G_l \rhd^G \emptyset\right\}$, and suppose, w.l.o.g., that $q^R \leq q^S$. Let $Q := S \setminus \cup_{l=1}^{q^R} G_l$, $V := R \setminus \cup_{k=1}^{q^R} T_k$, $\Pi^Q \in Core\left(Q, (\succeq^T_{s|Q})_{s\in Q}\right)$ and $\Pi^V \in Core\left(V, (\succeq^G_{r|V})_{r\in V}\right)$.

We show that the coalitional matching μ defined by

(1) $\mu(i) = (T_q, G_q)$ for all $i \in T_q \cup G_q$ and all $q \leq q^R$,

(2) $\mu(s) = \left(\emptyset, \Pi^Q(s)\right)$ for all $s \in Q$,

(3) $\mu(r) = \left(\Pi^V(r), \emptyset\right)$ for all $r \in V$

is core stable.

Suppose on the contrary that there is a blocking pair (A, μ') for μ. First, suppose that $A \subseteq R$. Fix $r \in A$ and note that, since $(\succeq_r)_{r\in R} \in \mathcal{D}_6$ and the common ranking over research teams is linear, we should have, for each $r' \in \mu'(r)_1 \subseteq A$, either $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ^G_{r'} \mu(r')_2$ or $\mu'(r)_2 = \mu'(r')_2 = \emptyset = \mu(r')_2$ and $\mu'(r)_1 = \mu'(r')_1 \succ^T_{r'} \mu(r')_1$. If $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ^G_{r'} \mu(r')_2 \neq \emptyset$ holds for some $r' \in \mu'(r)_1$ then, since $\left(\succeq^G_r\right)_{r\in R}$ respects $\rhd^G$, we have $\emptyset \rhd^G \mu(r')_2 \neq \emptyset$. However, $\mu(r')_2 \neq \emptyset$ implies that $r' \in T_{q_{r'}}$ for some $q_{r'} \leq q^R$ which is not possible, since, by the construction of μ, the definition of q^S and $q^R \leq q^S$, we have that $\mu(r')_2 \rhd^G \emptyset$. Therefore, there must be the case that for all $r' \in \mu'(r)_1$, $\mu'(r)_2 = \mu'(r')_2 = \emptyset = \mu(r')_2$ and $\mu'(r)_1 = \mu'(r')_1 \succ^T_{r'} \mu(r')_1$. Note that $\mu'(r)_2 = \mu'(r')_2 = \emptyset = \mu(r')_2$ for all $r' \in \mu'(r)_1$ implies, by the construction of μ, that $\mu(r')_1 = \Pi^V(r')$ for all $r' \in \mu'(r)_1$. Thus, $\mu'(r)_1$ blocks Π^V in contradiction to $\Pi^V \in Core\left(V, \left(\succeq^G_{r|V}\right)_{r\in V}\right)$.

Next, suppose that $A \subseteq S$. Fix $s \in A$ and note that, since $(\succeq_s)_{s\in S} \in \mathcal{D}_6$ and the common ranking over student groups is linear, we should have, for each $s' \in \mu'(s)_2 \subseteq A$, either $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ^T_{s'} \mu(s')_1$ or $\mu'(s)_1 =$

$\mu'(s')_1 = \emptyset = \mu(s')_1$ and $\mu'(s)_2 = \mu'(s')_2 \succ^G_{s'} \mu(s')_2$. Note first that, by construction of μ, $\mu'(s)_1 = \mu'(s')_1 = \emptyset \succ^T_{s'} \mu(s')_1 \neq \emptyset$ can hold for some $s' \in \mu'(s)_2$ only if $\mu(s')_1 = T_{q_{s'}}$ for some $q_{s'} \leq q^R$. Since $\left(\succeq^T_s\right)_{r\in R}$ respects $\rhd^T$, we have $\emptyset \rhd^T T_{q_{s'}}$ which is a contradiction to the definition of q^R and $k_{s'} \leq q^R$. Therefore, for all $s' \in \mu'(s)_2 \subseteq A$ it must be that $\mu'(s)_1 = \mu'(s')_1 = \emptyset = \mu(s')_1$ and $\mu'(s)_2 = \mu'(s')_2 \succ^G_{s'} \mu(s')_2$. Note that $\mu'(s)_1 = \mu'(s')_1 = \emptyset = \mu(s')_1$ for all $s' \in \mu'(s)_2$ implies, by the construction of μ, that $\mu(s')_2 = \Pi^Q(s')$ for all $s' \in \mu'(s)_2$. Thus, $\mu'(s)_2$ blocks Π^S in contradiction to $\Pi^S \in Core\left(S, (\succeq^G_s)_{s\in S}\right)$.

Last, suppose that $A \cap R \neq \emptyset$ and $A \cap S \neq \emptyset$. If $A \cap T_1 \neq \emptyset$ then $\mu(r)_2 = G_1$ for $r \in A \cap T_1$. Since $\left(\succeq^G_r\right)_{r\in R}$ respects the linear common ranking $\rhd^G$, $(\succeq_r)_{r\in R} \in \mathcal{D}_6$, and by construction $G_1 \rhd^G G'$ for all $G' \in 2^S$, $r \in A \cap T_1$ can be made better off under μ' only if $\mu'(r)_2 = G_1 = \mu(r)_2$ and $\mu'(r)_1 \succ^T_r \mu(r)_1 = T_1$, i.e., $G_1 \subset A$ should hold too. Notice that, since $\left(\succeq^T_s\right)_{s\in S}$ respects the linear common ranking $\rhd^T$, $(\succeq_s)_{s\in S} \in \mathcal{D}_6$ and by construction $T_1 \rhd^G T'$ for all $T' \in 2^R$, we should have $\mu'(s)_1 = T_1 = \mu(s)_1$ and $\mu'(s)_2 \succ^G_s \mu(s)_2 = G_1$ in order that all $s \in G_1 = \mu'(r)_2 = \mu(r)_2$ to be strictly better off under μ'. Thus, we have $\mu'(r)_2 = G_1$ for $r \in A \cap T_1$ and $\mu'(s)_1 = T_1$ for all $s \in G_1$. The latter fact implies however that $\mu'(s) = (T_1, G_1) = \mu(s)$ for all $s \in G_1 \subset A$ in contradiction to (A, μ) being a blocking for μ. We conclude that $A \cap T_1 = \emptyset$. By an analogous argument, $A \cap G_1 = \emptyset$ holds too. Hence, we should have $A \subseteq (R \setminus T_1) \cup (S \setminus G_1)$.

Similarly, for $k \in \{2, \ldots, q^R\}$, we can show that $A \cap T_k = \emptyset$ and $A \cap G_k = \emptyset$. Therefore, $A \subseteq \left(R \setminus \left(\cup_{k=1}^{q^R} T_k\right)\right) \cup \left(S \setminus \left(\cup_{k=1}^{q^R} G_k\right)\right)$. Moreover, for all $r, s \in A$ holds then $\mu(r)_2 = \emptyset$ and $\mu(s)_1 = \emptyset$. Since by definition $\emptyset \rhd^T T'$ for all $T' \in 2^{R\setminus T_v}$ with $T_v = \cup_{k=1}^{q^R} T_k$, and $\left(\succeq^T_s\right)_{s\in S}$ respects the linear common ranking $\rhd^T$, it follows that $\mu'(s)_1 = \mu(s)_1 = \emptyset$ holds for all $s \in A$. Fix $s \in A$ and notice that, in order μ' to be a blocking for μ, it must be, for each $s' \in \mu'(s)_2 \subseteq A$, that $\mu'(s)_2 = \mu'(s')_2 \succ^G_{s'} \mu(s')_2 = \Pi^Q(s')$. The latter means that $\mu'(s)_2$ blocks Π^Q in contradiction to $\Pi^Q \in Core\left(Q, (\succeq^T_{s|Q})_{s\in Q}\right)$. Hence, A does not contain any students. By an analogous argument, A does not contain any researcher either. We conclude then that no blocking for μ exists. ■

Proof of Theorem 9. Let μ be the coalitional matching constructed in the proof of Theorem 3. If there is a blocking pair (A, μ') for μ, then notice that reaching a contradiction goes in the same way as in the corresponding parts of the proof of Theorem 3, except for $A \subseteq R$. For this case, fix $r \in A$ and note that, since $(\succeq_r)_{r\in R} \in \mathcal{D}_6$, we should have, for each $r' \in \mu'(r)_1 \subseteq A$, either $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ^G_{r'} \mu(r')_2$ or $\mu'(r)_2 = \mu'(r')_2 = \emptyset \sim^G_{r'} \mu(r')_2$ and $\mu'(r)_1 = \mu'(r')_1 \succ^T_{r'} \mu(r')_1$. If $\mu'(r)_2 = \mu'(r')_2 = \emptyset \succ^G_{r'} \mu(r')_2 \neq \emptyset$ holds for some $r' \in \mu'(r)_1$ then, since $\left(\succeq^G_r\right)_{r\in R}$ respects $\trianglerighteq^G$, we have $\emptyset \rhd^G \mu(r')_2 \neq \emptyset$. However, $\mu(r')_2 \neq \emptyset$ implies that $r' \in T_{q_{r'}}$ for some $q_{r'} \leq q^R$ which is not possible, since, by the construction of μ, the definition of q^S and $q^R \leq q^S$, we have that $\mu(r')_2 \rhd^G \emptyset$. Therefore, there must be the case that for all $r' \in \mu'(r)_1$, $\mu'(r)_2 = \mu'(r')_2 = \emptyset \sim^G_{r'} \mu(r')_2$ and $\mu'(r)_1 = \mu'(r')_1 \succ^T_{r'} \mu(r')_1$. Note that $\mu'(r)_2 = \mu'(r')_2 = \emptyset \sim^G_{r'} \mu(r')_2$ for all $r' \in \mu'(r)_1$ implies, since $\left(\succeq^G_r\right)_{r\in R}$ respects $\trianglerighteq^G$, that $\emptyset =^G \mu(r')_2$ holds for all $r' \in \mu'(r)_1$. Thus, by the construction of μ, we have $\mu(r')_1 = \Pi^R(r')$ for all $r' \in \mu'(r)$; hence, $\mu'(r)_1$ blocks Π^R in contradiction to $\Pi^R \in Core\left(R, (\succeq^T_r)_{r\in R}\right)$. ■