An Investor's Martingale Walk

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The purpose of this article is to show how martingale convergence theorem applied to simplest of martingale encodes very well and at once explains some of the main causes of current global fiancial predicament and combined with the simplest version of the fundamental theorem asset price theory shows a way out at least at a theoretical level. The article does not use any advanced mathematics and should be accessible to any one with a knowledge of rudiments of probability.

We shall use the word investor to mean any individual, institution or a body which has to take probability related risk in doing its work, business or transactions. Thus an investor, may be an entrepreneur, an financial institution such as bank or an insurance company, a farmer etc. It also includes a gambler, whose risk taking is of perverse kind, and not a risk taking dictated by need. Mathematics of risk taking is, however, impersonal and independent of who is taking risk or for what purpose. In this article we shall treat investor as a female individual, but the word should be given the above broad interpretation.

Section 1. An investor's Martingale walk: the case of a single investor.

We will consider a rather simple model of investor's walk which may be described as follows:
The walk is confined to the initial segment of the $\{1, 2, \cdots, N\}$ of non-negative integers. At time 0 the investor invests an amount a, $0 < a < N$, with the hope of receiving a fixed higher amount b, $a < b \leq N$, at time 1 but is aware that she may loose in the process, and is willing to receive a smaller amount c, $0 \leq c < a$. The quantities a, b and c are chosen by the investor at the start. If she receives the amount b at time 1 then her gain g is $b - a$, while if she receives the amount c at time 1 then her loss l is $a - c$. Note that $b - c = b - a + a - c = g + l$. The probabilities $x = p(b)$, $y = p(c)$ that the investor receives the amount b or c at time 1 are determined by the simultaneous equations:

$$x + y = 1, \quad (1)$$

$$bx + cy = a, \quad (2)$$

which gives

$$x = p(b) = \frac{a-c}{b-c} = \frac{l}{g+l}, \ y = \ p(c) = \frac{b-a}{b-c} = \frac{g}{g+l}.$$

The requirement $bp(b)+cp(c) = a$ is known as *'fair game'* or *'martingale'* condition in probability theory. It is a consequence of *'no arbitrage opportunities'* requirement in financial mathematics. ('This definition of fairness is some what arbitrary, although hallowed by tradition.' (Doob, [**4**] p 299))
We can rewrite this condition as $gp(b) - lp(c) = 0$. Since a is fixed, we may think of $p(b)$ also as the probability $P(g)$ of the investor making a gain of amount g while $p(c)$ may be interpreted as the probability $P(l)$ that the investor makes a loss of amount l, so that the fair game or martingale condition becomes

$$gP(g) - lP(l) = 0,$$

which is interpreted to mean that on an average the investor will break even.

From the equations

$$P(g) = \frac{l}{g+l}, P(l) = \frac{g}{g+l}$$

we note at once that $l \leq g$ if and only if $P(g) \leq P(l)$, so that under martingale condition a 'large' gain is possible only with 'small' probability, while a 'small' gain is possible with 'high' probability, in which case the loss is 'big' should the investor be unlucky to loose. Thus, these equations may be viewed as equations of the statutory warning in fast forward and small print: 'mutual fund investments are subject to market risk, read the offer document carefully before investing'.

The term *'fair game'* comes from classical betting considerations (see Doob [**4**], p 299], Feller [**5**]) where a gambler is supposed to be playing against a gambling house, and the role of the investor is replaced by that of the gambler. If a is the amount the gambler bets, and receives an amount $b > a$ if she wins and an amount $c < a$ if she looses, then the game is said to be *favorable* to the gambler if $bp(b) + cp(c) > a$, it is said to be favorable to the gambling house if $bp(b)+cp(c) < a$, and, fair if the equality holds. Here, as before, $p(b), p(c)$ denote the probabilities of the gambler receiving the amount b and c respectively. We note that the 'fair game' condition also seems to be fair between any two investors, rich or poor, because the probabilities $P(g)$ and $P(l)$ of gain and loss depend entirely on g and l and not on the initial investment a. However, these conclusions of fairness between the market forces and the investor or between two investors are deceptive as we will see.

If the investor chooses to attempt to make full gain, i.e., chooses $b = N$, and is lucky enough to receive the amount N at time 1, then she is contented and does not invest any more. Similarly if she chooses $c = 0$ and is unlucky enough to loose, then at time 1 she has no capital to invest, so that she does not invest anymore. In case she receives an amount d at time 1 which can be either, b or c, and if $0 < d < N$, then she invests the amount d hoping to obtain a fixed higher amount e, $d < e \leq N$ at time 2, but is willing to receive a fixed smaller amount f, $0 \leq f < a$, should she loose. The probabilities $p(e), p(f)$, whose sum is one, are again determined by the 'fair game' condition $fp(f)+ep(e) = d$. The process continues. Some care is required to describe the situation at time n. Let x_i, denote the amount the investor receives at time i, or already has this amount at time i, (which is the case if $x_{i-1} = 0$ or N). If at time n the investor has amount x_n and if $x_n = 0$ or N, then she does not invest anymore. Otherwise $0 < x_n < N$, in which case she invests this amount again choosing new quantities, say α and β, $0 \leq \alpha < x_n$, $x_n < \beta \leq N$, which she is willing to receive at time $n+1$. These quantities can depend on $x_0 = a, x_1, x_2, \cdots, x_n$, since the investor chooses α and β keeping in mind the history of the market and the world up to time n. The probabilities $p(\alpha)$ and $p(\beta)$ with which these values are realized satisfy (1) and (2), i.e,

$$p(\alpha) + p(\beta) = 1 \tag{3}$$

$$\alpha p(\alpha) + \beta p(\beta) = x_n \quad \text{(fair game condition)}. \tag{4}$$

Thus,

$$p(\alpha) = \frac{\beta - x_n}{\beta - \alpha}, \quad p(\beta) = \frac{x_n - \alpha}{\beta - \alpha}. \tag{5}$$

Let

$$p(x_1, x_2, \cdots, x_n)$$

denote the probability that the investor receives an amount x_1 at time 1, x_2 at time 2, and in general, that she receives an amount x_n at time n. Let

$$p(x_{n+1} \mid x_1, x_2, \cdots, x_n)$$

denote the conditional probability that the investor receives an amount x_{n+1} at time $n+1$ given that she has received an amount x_1 at time 1, x_2 at time 2, and in general that she has received an amount x_n at time n.

Recall that for any two events A and B, the probability of A and B happening together, i.e., $P(A \cap B)$ satisfies, $P(A \cap B) = P(A \mid B) \cdot P(B)$. Thus,

$$p(x_1, x_2, \cdots, x_n) = p(x_n \mid x_1, x_2, \cdots, x_{n-1}) p(x_1, x_2, \cdots, x_{n-1}).$$

On iteration we have

$$p(x_1, x_2, \cdots, x_n) = \Pi_{i=1}^{n} p(x_i \mid x_1, x_2, \cdots, x_{i-1}), \qquad (6)$$

where $p(x_1 \mid x_0)$ is interpreted to mean $p(x_1 \mid a) = p(x_1)$ since $p(a) = 1$. Now, from (5), we see that $p(x_i \mid x_1, x_2, \cdots, x_{i-1})$ is a ratio of two positive integers between 1 and N, so that

$$p(x_i \mid x_1, x_2, \cdots, x_{i-1}) \geq \frac{1}{N},$$

hence from (6),

$$p(x_1, x_2, \cdots, x_n) \geq (\frac{1}{N})^n.$$

Let A_n denote the set of paths $(x_1, x_2, \cdots, x_n)$ with $x_n = 0$ or $x_n = N$, and let $B_n =$ remaining set of paths, namely those path $(x_1, x_2, \cdots, x_n)$ of length n for which $0 < x_n < N$. Note that for a path $(x_1, x_2, \cdots, x_n)$ in B_n, $0 < x_i < N$, for all $i, 1 \leq i \leq n$.

There is a non-increasing path $a \geq x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq \cdots$ wherein there are strict inequalities until an x_i is 0, after which they are all equalities. Moreover, the first i for which $x_i = 0$ is at most equal to $a < N$. There is a non-decreasing path $a \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \cdots$ wherein there are strict inequalities until an x_i is N, after which they are all equalities. Moreover the first i for which $x_i = N$ is at most equal to $N - a < N$. So the probability of the set of paths $(x_1, x_2, \cdots, x_N)$ with $x_N = 0$ or N is $\geq 2(\frac{1}{N})^N$, i.e.,

$$p(A_N) \geq 2(\frac{1}{N})^N,$$

whence

$$p(B_N) < 1 - 2(\frac{1}{N})^N.$$

Consider now a path $(x_1, x_2, \cdots, x_N)$, of length N, with $0 < x_N < N$. Then the probability of the set of paths starting at x_N at time N and not hitting 0 or N during time points $(N+1, N+2, \cdots, 2N)$ is again $< 1 - 2(\frac{1}{N})^N$. This implies that

$$p(B_{2N} \mid B_N) < 1 - 2(\frac{1}{N})^N.$$

We see therefore that

$$p(B_{2N}) = p(B_{2N} \mid B_N) \cdot p(B_N) < (1 - 2(\frac{1}{N})^N)^2.$$

In general we have,

$$p(B_{kN}) < (1 - 2(\frac{1}{N})^N)^k, k = 1, 2, \cdots .$$

Hence

$$p(B_{kN}) \to 0 \ \text{ as } n \to \infty.$$

In addition, $p(B_n)$ is non-increasing in n, hence $p(B_n) \to 0$ as $n \to \infty$.

This implies that

$$p(\bigcap_{n \geq 1} B_n) = \lim_n p(B_n) = 0.$$

But

$$B \equiv \bigcap_{n \geq 1} B_n$$

is the event that the gambling does not terminate in finite time. Thus, we have proved the first part of:

Theorem 1 *(i) The probability of the set of investor's paths which hit* 0 *or* N *at some finite time is one, so that with probability one the investor will, in finite amount of time, either go bankrupt or reach her goal* N*.*
(ii) The probability that the investor reaches N *in finite time is* $\frac{a}{N}$*, and the probability that she reaches* 0 *in finite time is* $1 - \frac{a}{N}$*.*

Next we prove *(ii)*. Let us be more mathematical. Write X_n for the amount the investor receives or has at time n. Given $X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}$, we know that X_n assumes at most two values and the probabilities with which these values are assumed satisfy the martingale condition (4) with n replaced by $n-1$. We now write this as a conditional expectation:

$$E(X_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}) = x_{n-1}.$$

We abbreviate $E(X_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1})$ as $E(X_n \mid x_1, x_2, \cdots, x_{n-1})$. Writing $E(X)$ for the expected value of a random variable X, we see that

$$E(X_n) = \sum_C E(X_n \mid x_1, x_2, \cdots, x_{n-1}) p(x_1, x_2, \cdots, x_{n-1})$$

$$= \sum_C x_{n-1} p(x_1, x_2, \cdots, x_{n-1}) = \sum_{x_{n-1} \in D} x_{n-1} p(X_{n-1} = x_{n-1}) = E(X_{n-1})$$

where C is the set of investors paths up to time $n-1$, while D is the the range of the random variable X_{n-1}. We thus have

$$E(X_n) = E(X_{n-1}) = \cdots = E(X_1) = a \qquad (7).$$

We have seen in *(i)* that $\lim_{n\to\infty} X_n = X_\infty$ exists with probability one and X_∞ assumes only two values 0 and N. Also

$$\mid EX_n - EX_\infty \mid \leq E \mid X_n - X_\infty \mid \leq 2NP(B_n).$$

By first part of the theorem $P(B_n) \to 0$ as $n \to \infty$, and since, by (7) $E(X_n) = a$ for all n, we see that

$$E(X_\infty) = a.$$

If s and t denote the probabilities with which X_∞ assumes values 0 and N respectively, then we have

$$E(X_\infty) = 0 \cdot s + t \cdot N = a,$$

so that

$$t = p(X_\infty = N) = \frac{a}{N}, s = p(X_\infty = 0) = 1 - \frac{a}{N}.$$

This proves the second part and completes the proof of the theorem.

A reader familiar with advanced probability will note that the above theorem follows from Doob's Martingale convergence theorem , since the process $X_n, n = 1, 2, \cdots$ is a uniformly bounded Martingale (Doob [**4**], Athreya and Lahiri [**1**]).

A special case of investor's walk, namely, symmetric random walk with absorbing barriers and its implications for a gambler are well discussed in the probability literature, even when the probabilities are not symmetric. (see Feller [**5**], Chapter XIV). Our model above, though based on a martingale building block, is much more general than simple symmetric random walk, and leads to same conclusions in so far as the investor is concerned. Since the transition probabilities at each stage are completely arbitrary (but for martingale requirement), they need not be Markovian, so the investor's walk we have discussed is not a random walk in strict sense of the term. Hence we call it a *Martingale walk*.

It is important to note that the probability $\frac{a}{N}$ of reaching N depends only on a and not on what strategy (bold or conservative) the investor adopts.

In the context of financial market, the upper bound N could be interpreted as existence of regulations which do not permit attempts at unlimited profit, while non-existence of absorbing barrier at the upper end, i.e., $N = \infty$ can be interpreted to mean no regulations (deregulation ?), i.e., permission to attmpt to make unlimited profit. Consequences in such a situation are very grave. Indeed if the investor is greedy and not contented with receiving the amount N, and executes her martingale walk without an absorbing barrier at the upper end, i.e., if $N = \infty$, then she will eventually hit zero with probability one, no matter how rich she is to begin with. To see this we note that $p(X_\infty = 0) = 1 - \frac{a}{N} \to 0$ as $N \to \infty$ no matter what the starting capital a. Does this explain the recent failure of some big financial institutions in U. S., triggering the global financial crises ? No such categorical claim can be made without investigation, but it must be mentioned the both presidential and the vice-presidential candidates of even the republican party have blamed this failure on the greed of these Wall Street institutions.
Thus, attempts to make unlimited profit is sure cause of failure in the long run, although there could be other causes failure of large institutions as well. (The term 'regulation' is interpreted here in a simple minded way and a professional economist may probably not agree with this.).

Martingale convergence theorem, even for the simplest of martingale discussed in this paper, is an absolute theorem, and no matter how good one is at risk management, if one looks for profit beyond limit, one will hit the bottom in the long run with probability one. This maxim, which should be at the beginning of any book on financial mathematics, is rarely mentioned any where at all.

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Even with regulations, i.e., with N finite (which will be very large in a rich economy), but with no other intervensions and totally subsidy free economy, the theorem tells that an investor with a larger capital has better chances of succeeding than a marginal or a medium investor, leading in the long run to unaceptable income disparity resulting in pervasive social discontent. In the context of U. S. Presidential election 2008 this was constantly mentioned by the democratic presidential candidate. Next section deals with this some more.

2. Implications for the population as a whole

So far we have discussed implications of our conclusions for an individual investor, locally, as they say in mathematics. Are there implications globally, or for society at large ? Surely there are. A constant refrain of sensitive and observant individuals, whether in India or in an advanced western country, is that *"rich are getting richer and poor are getting poorer"* [**7**],[**8**]. These individuals are not necessarily left leaning, and these are not views expressed out of ideological considerations, but rather out of concern for what they see. Indeed the growing disparity between the rich and the middle class is the constant theme of Obama's campaign Can one justify these views, especially when one sees an individual poor person doing well by sheer hard work, and a well to do person getting poor ? It seems we can. For the refrain *' rich getting richer and poor getting poorer'* is only an imperfect articulation of an obvious statistical consequence of theorem 1 together with the *law of large numbers.* Indeed, we see that the collective invested wealth of the investors does not change much, but only gets redistributed more lopsidedly.

Imagine that the market has two hundred investors, 100 of them well to do and the remaining hundred not so well to do. Assume that the maximum possible receivable amount is 10 units, i.e., $N = 10$. (A unit could be thousand, 10 thousand, 100 thousand, or a million or more.) Assume that each of the well to do investor invests 7 units, while those not so well to do invest 3 units each. (We will assume that the 'fair game' condition holds and the process $(X_n)_{n=1}^{\infty}$ of investor's earnings is a Martingale.) According to our theorem, the probability of a well to do investor reaching 10 units in the long run is $\frac{7}{10}$, and the probability that a well to do investor hits 0 is $\frac{3}{10}$. Assume that the investors act independently. Let W_1 and L_1 be the number of winners and losers among the well to do investors. Then by the law of large numbers W_1 is approximately 70 , and they will reach 10, while L_1 is approximately 30, and they will hit 0. In contrast, if W_2 and L_2 are the number of winners and losers among the not so well to do investors, then $L_2 =$ approximately 70, they will hit 0, and, $W_2 =$

approximately 30, and they will hit 10. Thus originally there were no paupers among the investors, now there are nearly 100 paupers among them, nearly 70 of them are those who were not well to do to begin with. Nearly 30 among them are previously well to do investors. Also there are now nearly 100 very well to do and contented investors, nearly 30 among them were not so well to do in the beginning. We also note that the total amount of investment at time zero is $7 \times 100 + 3 \times 100 = 1000$ which remains nearly the same since according to the new distribution of earned or lost wealth the total wealth remains approximately $10 \times 100 + 0 \times 100 = 1000$.

One can refine these assertions further by using what is known as the central limit theorem. (see Athreya and Lahiri [**1**])

Martingale coonvergence theorem not only identifies the problem, but together with the fundamental theorem of asset price theory it gives a solution at least at a theoretical level.

Section 2 An Application of the Fundamental Theorem of Asset Price Theory.

In this section we show how the martingale convergence theorem together with the fundamental theorem of asset price theory shows a a way out of the predicament discusssed in the last section. First we explain the fundamental theorem of asset price theory in its simplest form.

Assume that at time n the investor has an amount $x_n, 0 < x_n < N$, and that he invests it hoping to receive an amount $x_{n+1} > x_n$ but may end up receiving an amount $x'_{n+1} < x_n$. We have assumed in section 1 that probabilities

$$P(X_{n+1} = x_{n+1} \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n),$$

$$P(X_{n+1} = x'_{n+1} \mid X_1 = x_1, X_2 = x_2, \cdots X_n = x_n)$$

are given by the Martingale condition

$$x_{n+1} P(X_{n+1} = x_{n+1} \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$

$$+$$

$$x'_{n+1} P(X_{n+1} = x_{n+1} \mid x_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$

$$= x_n$$

However, in reality these probabilities are determined by the market conditions, and need not satisfy the Martingale condition. It is therefore not reasonable to assume that the discrete process $X_1, X_2, X_3, \cdots$ is a Martingale. Let us fix a large positive integer T and consider the the vector random variable $(X_0 = a, X_1, X_2, X_3, \cdots, X_T)$. Let Q_a denote the probability distribution of $(X_0 = a, X_1, X_2, \cdots, X_T)$ determined by the market forces, and P_a the probability distribution under which $(X_0, X_1, X_2, \cdots, X_T)$ is a Martingale. Note that P_a is uniquely determined by the strategy the investor adopts, as explained in section 1. We will assume that we are dealing with an ideal market, so that one can not have positive probability of making money without making any investment, i.e., in the language of business mathematics, there are no arbitrage opportunities in the market. Write $[0, N]$ for the segment $\{0, 1, 2, \cdots, N\}$ of non-negative integers..

The next theorem is a simple version of the fundamental theorem of the asset price theory [**3**,**5**,**9**], and has interesting implications:

Theorem 2: *P_a and Q_a are equivalent measures, i.e.,a sequence $(a, z_1, z_2, \cdots, z_T)$ in $[0, N]^T$ receives positive probability under P_a if and only if it receives positive probability under Q_a.*

Proof: We give a simple proof for the case on hand. Fix an n, $0 < n < T$, and let

$$(x_0 = a, x_1, x_2, \cdots, x_n)$$

denote a path of investor's fortune up to time n. Here fortune on a given date means the the value of his portfolio on that date. Surely, at time n the investor will *not* recast his portfolio (with value of the new portfolio also $= x_n$ at time n) in such a way that

$$Q_a(X_{n+1} \leq x_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) = 1$$

On the other hand, since the market does not allow arbitrage opportunity, he *can not* recast his portfolio (with value of the new portfolio $= x_n$ at time n) so that

$$Q_a(X_{n+1} > x_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) = 1.$$

Clearly then for each n, $0 < n < T$, and for each path $(a = x_0, x_1, x_2, \cdots, x_n)$, of the values of investor's portfolios up to time n,

$$0 < Q_a(X_{n+1} < x_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) < 1,$$

$$0 < Q_a(X_{n+1} > x_n \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) < 1.$$

This implies that for each i, $0 < i < T$, and for each $(z_0 = a, z_1, z_2, \cdots, z_T) \in [0, N]^T$

$$P_a(X_i = z_i \mid X_1 = z_1, X_2 = z_2, \cdots, X_{i-1} = z_{i-1}) > 0$$

if and only if

$$Q_a(X_i = z_i \mid X_1 = z_1, X_2 = z_2, \cdots, X_{i-1} = z_{i-1}) > 0$$

Since

$$P_a(X_1 = z_1, X_2 = z_2, \cdots, X_T = z_T) = \Pi_{i=1}^T P_a(X_i = z_i \mid X_j = z_j, 1 \leq j \leq i-1),$$

$$Q_a(X_1 = z_1, X_2 = z_2, \cdots, X_T = z_T) = \Pi_{i=1}^T Q_a(X_i = z_i \mid X_j = z_j, 1 \leq j \leq i-1)$$

the theorem follows.

Let A_T and B_T denote the events that the investor reaches N or 0 respectively at time T, i.e.,

$$A_T = \{(a, z_1, z_2, \cdots, z_T) : z_T = N\},$$
$$B_T = \{(a, z_1, z_2, \cdots, z_T) : z_T = 0\}.$$

By theorem 1 we can choose a large T such that $P_a(A_T) > 0, P_a(B_T) > 0$ for all a. For such a T, By theorem 2 we see that if the market does not admit arbitrage opportunities, then $Q_a(A_T) > 0, Q_a(B_T) > 0$. So, under the hypothesis that there are no arbitrage opportunities, we have:

(1) No matter how 'investor friendly' the market forces, and no matter how large the initial investment a, there is a positive probability that an investor starting with capital a will hit 0 at time T or before, i.e., $Q_a(B_T) > 0$, and (2) no matter how 'investor unfriendly' the market forces and no matter how small the initial investment a, there is a positive probability that an investor will hit N at time T or before, i.e., $Q_a(A_T) > 0$.

Although we have not defined here the terms 'investor friendly' and 'investor unfriendly' markets forces, most readers would agree to this statement as being obvious from observation.

Unlike the molecular forces of a fluid which propel pollen particles in a fluid, the forces that govern market are governed in large measure by decisions taken by human being, or groups of human beings, and should therefore be for the collective welfare of human beings. In our setting this means that the probability measure Q_a determined by the market forces should assign small probability to the set B_T and large probability to set of sequences which stay away from 0,and are more in the middle or near the top. Small positive probability which B_T receives may then be attributable to forces beyond human control, such as bad mansoon, or an earthquake, and the inherent minimum necessary conflict in the decisions taken by different groups of human beings. Thus the purpose of regulations (and subsidies if need be) and the self regulations of other institutions should not be to hamper market but to ensure that $Q_a(B_T)$ remains small even for small a, i.e.,small entreprenuers have optimally high probabability of success.Indeed the function $Q_a(B_T)$ as a function of a seems a good measure of economic welfare of a society as a whole.

Thus we see that the ideas dealing with gambler's ruin from the chapter on 'Random walk and the ruin problems' from Feller's very influential book [**5**] and Doob's Martingale convergence theorem combine very well with the concept of 'no arbitrage opportunities', one of the key axiom of financial mathematics .

This paper continues the theme of reference 2 [**2**] with some additional inputs.

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