Dual representations of cardinal preferences

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Abstract

Given a set of possible vector outcomes and the set of lotteries over it, we define sets of (a) von Neumann-Morgenstern representations of preferences over the lotteries, (b) mappings that yield the certainty equivalent outcomes corresponding to a lottery, (c) mappings that yield the risk premia corresponding to a lottery, (d) mappings that yield the acceptance set of lotteries corresponding to an outcome, and (e) vector-valued functions that yield generalized Arrow-Pratt coefficients corresponding to an outcome. Our main results establish bijections between these sets of mappings for very general specifications of outcome spaces, lotteries and preferences. As corollaries of these results, we derive analogous dual representations of risk averse preferences. Some applications to financial theory illustrate the potential uses of our results. Finally, we provide criteria for comparing the risk aversion of preferences in terms of the dual representations.

JEL classification: C02, D01, D81

Key words: von Neumann-Morgenstern utility, risk aversion, vector outcomes, certainty equivalence, risk premia, acceptance set, Arrow-Pratt coefficient, eikonal equation, Dirichlet problem, viscosity solution

1 Introduction

The classical theory of risk aversion (Arrow [1], Pratt [24], Yaari [28]) features real-valued outcomes and characterizes risk aversion and comparative risk aversion in terms of von Neumann-Morgenstern (henceforth, vN-M) utilities, certainty equivalents, risk premia, acceptance sets and Arrow-Pratt coefficients. Given these constructs, two natural questions arise. (1) Can a decision-maker's preference, usually represented by a vN-M utility, be represented equivalently in terms of the other constructs? (2) Can these dualities be established, not only in the real outcomes setting, but very generally in

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the vector outcomes setting after modifying the dual notions to account for vector outcomes in an economically meaningful way?

A positive answer to (1) would enable greater flexibility and precision in the specification of cardinal preferences, just as dual constructs such as expenditure functions and indirect utility functions have done in the case of ordinal preferences. This is because, in applications of vN-M utility theory, one is usually interested in objects such as risk premia, certainty equivalents, acceptance sets and Arrow-Pratt functions, while the vN-M utility is merely the means for systematically generating these objects. Duality results, such as the ones we report below, allow one to directly specify and work with the objects of interest, safe in the knowledge that, if these objects satisfy the properties we postulate, then they are indeed generated by a vN-M utility, and therefore are well-grounded in expected utility theory.

A positive answer to (2) would enable applications of the theory to situations where outcomes are properly modeled as vectors rather than scalars. For instance, consider the problem of choosing among financial assets whose returns are random processes. As random processes can be represented by lotteries over a set of sample paths, the decision problem is essentially one of choosing among lotteries over sample paths. These sample paths are the relevant vector outcomes that cannot be reduced to a scalar "wealth" outcome without some degree of *ad hoc* aggregation. Such problems, exemplified by Application 7.4, also motivate the generality of our formalism. As sample paths in financial economics are typically continuous functions of time or belong to an even more general vector space, a useful theory should strive to specify outcome spaces as generally as tractable and necessary.[1]

In this paper, we identify general classes of vector outcome spaces and preferences for which (1) and (2) can simultaneously be answered in the affirmative in the case of certainty equivalents, risk premia and acceptance sets. We also identify a class of vector outcome spaces and preferences for which analogous results hold in the case of generalized Arrow-Pratt functions, which are mappings that yield the appropriate generalized Arrow-Pratt coefficient at each outcome. Our results are general in two senses. First, they hold very generally in the vector outcomes case and not merely in the real outcomes case. Secondly, although the derivation of dual representations of risk averse preferences is a prime motivation for our work, our main results will characterize a larger set of preferences in terms of the above-mentioned constructs and the dual characterizations of risk averse preferences will be derived as corollaries of the main results.

The first novelty in this paper, namely the setting-up and analysis of the

[1]For instance, the Wiener measure on the space of continuous sample paths results in the coordinate process being the Wiener process, which generates Brownian motion and geometric Brownian motion *via* elementary transformations. Itô and McKean [14] is the classic reference for the mathematics of diffusions and Duffie [6] is a useful introduction to the financial theory applications.

duality problem with respect to cardinal preferences, seems to have received no attention in the literature.[2] The second novelty, namely the study of risk aversion in the vector outcomes setting, has received some attention. There are two distinct strands in this literature. One strand (Duncan [7], Karni [15], Kihlstrom and Mirman [16], [17], Levy and Levy [19], Shah [26]) studies risk aversion directly in the context of vector-valued risks that are given as primitive objects, as we do in this paper. The emphasis in these papers is to develop measures of risk aversion and notions of comparative risk aversion that are appropriate in the vector outcomes context. The other strand (Grant et al. [10], [11], Hanoch [12], Martinez-Legaz and Quah [20], Stiglitz [27]) studies the relationship between lotteries on commodity bundles and lotteries on wealth when they are linked by a consumer's budget constraint. It is natural in this setting to interpret the vN-M utility function on a real domain as the indirect utility function for a fixed price vector. For each price vector, this enables the application of the classical theory of risk aversion couched in terms of real outcomes. This context also permits a restricted indirect theory of choice among vector-valued risks since lotteries over wealth levels amount to lotteries over commodity bundles on the Engel curve corresponding to a given price vector. While Stiglitz [27] explores the implications of the purely indirect approach, Grant et al. ([10], [11]) and Martinez-Legaz and Quah [20] study the nature and extent of duality between the direct and indirect approaches.

The results

Consider an outcome set O that is a subset of an ordered vector space X and let $\Delta(O)$ be the set of lotteries on O; see Section 2 for the specification of X, O and $\Delta(O)$. Let $\mathcal{U}$ be the set of vN-M utility functions $u : O \to \Re$ that are continuous and increasing with respect to the given partial order on X. Let $\mathcal{F}$ be a set of mappings $F : \Delta(O) \Rightarrow O$, where $F(\mu)$ is interpreted as the set of certainty equivalent outcomes corresponding to a lottery μ.[3] Let $\mathcal{P}$ be a set of mappings $P : \Delta(O) \Rightarrow X$, where $P(\mu)$ is interpreted as the set of risk premia corresponding to a lottery μ. Unlike in the real outcomes setting, the notions of certainty equivalent outcomes and risk premia are necessarily set-valued in the vector outcomes setting. Finally, let $\mathcal{A}$ be a set of mappings $A : O \Rightarrow \Delta(O)$, where $A(x)$ is interpreted as the acceptance set of lotteries corresponding to an outcome x. We specify $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$ in Sections 3, 4 and 5 by imposing appropriate requirements on their elements. We also specify the sets $\mathcal{U}_a$, $\mathcal{F}_a$, $\mathcal{P}_a$ and $\mathcal{A}_a$ as subsets of $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$ respectively, where $\mathcal{U}_a$ consists of risk averse utilities.

[2] There is, of course, a considerable literature on the ordinal version of the problem; see Diewert [5] for a survey and Section 9 for a comparison of the two problems.

[3] We use $\Rightarrow$ to denote set-valued mappings as well as logical implication. The intended meaning should be clear from the context.

The first contribution of this paper is to show the existence of bijections $\phi : \mathcal{U} \to \mathcal{F}$, $\psi : \mathcal{F} \to \mathcal{P}$ and $\xi : \mathcal{U} \to \mathcal{A}$ (Theorems 3.6, 4.1 and 5.5); clearly, these bijections generate other bijections $\psi \circ \phi : \mathcal{U} \to \mathcal{P}$, $\xi \circ \phi^{-1} : \mathcal{F} \to \mathcal{A}$ and $\xi \circ \phi^{-1} \circ \psi^{-1} : \mathcal{P} \to \mathcal{A}$. We use these results to show that analogous results hold (Corollaries 3.7, 4.2 and 5.6) with $\mathcal{U}_a$, $\mathcal{F}_a$, $\mathcal{P}_a$ and $\mathcal{A}_a$ replacing $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$ respectively. We should mention here a minor conceptual complication in the interpretation of the injectiveness of ϕ, ξ and $\psi \circ \phi$. In all three cases, equivalent vN-M utilities in the domain $\mathcal{U}$ have identical images. However, these mappings are injective when we identify each function $u \in \mathcal{U}$ with the equivalence class of vN-M functions equivalent to it. This identification is sensible and legitimate as we are seeking dual representations of preferences, not of vN-M utilities; see Section 3 for a more formal statement of this point.

Our method for deriving these dualities may be illustrated by describing how the duality ϕ between $\mathcal{U}$ and $\mathcal{F}$ is established. Given $u \in \mathcal{U}$ and a lottery $\mu \in \Delta(O)$, the set of certainty equivalents $\phi(u)(\mu)$ is defined in the natural way by (1) and it is straightforward to confirm that the resulting mapping $\phi(u) : \Delta(O) \rightrightarrows O$ satisfies the properties that define the elements of $\mathcal{F}$, i.e., $\phi(u) \in \mathcal{F}$. Next, we show that ϕ is injective. The final step is to show that ϕ is surjective. Given $F \in \mathcal{F}$, we define a complete preordering $\succeq_F$ on $\Delta(O)$; let $\succ_F$ be the asymmetric factor of $\succeq_F$. The vN-M utility representation problem with respect to $\succ_F$ is to find $u_F : O \to \Re$ such that, for all $\mu, \lambda \in \Delta(O)$, $\mu \succ_F \lambda$ if and only if $\int_O \mu(dz)\, u_F(z) > \int_O \lambda(dz)\, u_F(z)$. We show that ϕ is surjective by showing that the expected utility representation problem with respect to $\succ_F$ has a solution $u_F : O \to \Re$ such that $u_F \in \mathcal{U}$ and $\phi(u_F) = F$.

When we study the duality between vN-M utilities and Arrow-Pratt functions, we start with the classical setting $X = \Re$. In this case, we define a set $\mathcal{U}^{1d}$ (resp. $\mathcal{U}^{1d}_a$) of vN-M (resp. risk averse vN-M) utilities $u : O \to \Re$ and a set $\mathcal{R}^1$ (resp. $\mathcal{R}^1_+$) of Arrow-Pratt functions $a : O \to \Re$. In the Euclidean setting $X = \Re^n$, we define a set $\mathcal{U}^{nd}$ of vN-M utilities $u : O \to \Re$, a set $\mathcal{R}^n$ of generalized Arrow-Pratt functions $a : O \to \Re^n$ and a set $\mathcal{G}$ of functions $g : \partial O \to \Re$ that specify the boundary values of utility functions. The sets $\mathcal{U}^{1d}$ and $\mathcal{U}^{nd}$ are more restrictive than the set $\mathcal{U}$ as utilities that are dual to Arrow-Pratt functions necessarily have to be sufficiently smooth for Arrow-Pratt functions to be derived from them.

The second contribution of this paper is to show the existence of bijections between vN-M utility functions and Arrow-Pratt functions. In the case $X = \Re$, we show the existence of bijections $\chi : \mathcal{U}^{1d} \to \mathcal{R}^1$ and $\chi : \mathcal{U}^{1d}_a \to \mathcal{R}^1_+$ (Theorem 6.3 and Corollary 6.5). In the Euclidean case $X = \Re^n$, we show the existence of a bijection $\Gamma : \mathcal{U}^{nd} \to \mathcal{R}^n \times \mathcal{G}$ (Theorem 6.13).

In addition to the two sets of duality results, we also illustrate their potential for applications by deriving some of their implications in a financial theory setting. For instance, we show in Theorem 7.1 that $F \in \mathcal{F}$ and $A \in \mathcal{A}$

are continuous mappings. We use these facts to characterize the value of financial assets to a risk averse investor when the assets are characterized by a known or random stream of dividends. This is done in a discrete time setting as well as in the continuous time setting.

An important aspect of preferences is their degree of risk aversion. So, can we compare the risk aversion of preferences in terms of the dual representations derived in this paper? We show in Section 8 that the results of Shah [26] facilitate this comparison in the case of acceptance set mappings in $\mathcal{A}_a$ and risk premia mappings in $\mathcal{P}_a$.

Before turning to the formalism, it is useful to foreshadow the salient implications of X being a general vector space, instead of $X = \Re$.

(1) If $X = \Re$, then the usual order $>$ on $\Re$ is complete. For a general vector space X, there is no natural complete analogue of $>$.

(2) If $X = \Re$ and the vN-M utility $u : O \to \Re$ is strictly increasing, then the risk premia and certainty equivalents are singletons ordered by $>$. If X is a general vector space, then these constructs cease to be singleton-valued. A conceptual problem created by this fact is the question: what meaning is to be ascribed to the relation "the risk premia generated by a lottery μ are larger than the risk premia generated by lottery λ"?

(3) The vector outcome setting forces one to define the mean of a lottery over quite general vector outcomes, which entails integrating vector-valued functions. In this regard, it is important to confirm that a unique mean exists for every lottery.

(4) We need to define an economically meaningful generalized Arrow-Pratt function for the vector outcomes setting.

(5) When dealing with Arrow-Pratt functions, the cases $X = \Re$ and $X = \Re^n$ are treated separately as they entail distinct technical problems. If $X = \Re$, then the key problem is to derive the unique utility $u \in \mathcal{U}^{1d}$ that solves an initial value problem for an ordinary differential equation (henceforth, ODE) generated by an Arrow-Pratt function $a \in \mathcal{R}^1$. If $X = \Re^n$, then the key problem is to derive the unique utility $u \in \mathcal{U}^{nd}$ that solves a Dirichlet problem for a system of second order partial differential equations (henceforth, PDEs) generated by a generalized Arrow-Pratt function $a \in \mathcal{R}^n$, which can be reduced to a Dirichlet problem for an eikonal PDE. Solutions of these two problems involve entirely different concepts, methods and levels of difficulty.

The rest of this paper is organized as follows. Section 2 contains definitions and technical preliminaries. In Sections 3, 4 and 5, we establish the dualities between the sets $\mathcal{U}$, $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{A}$ (resp. $\mathcal{U}_a$, $\mathcal{F}_a$, $\mathcal{P}_a$ and $\mathcal{A}_a$). Section 6 contains the dualities between $\mathcal{U}^{1d}$ (resp. $\mathcal{U}_a^{1d}$, $\mathcal{U}^{nd}$) and $\mathcal{R}^1$ (resp. $\mathcal{R}_+^1$, $\mathcal{R}^n$). Section 7 is devoted to some applications. Section 8 shows how the risk aversion of different preferences can be compared in terms of the dual representations. In Section 9, we compare the ordinal and cardinal utility representation problems. We conclude in Section 10. The proofs of Theorems 2.1, 7.1, 7.2 and 7.3 are relegated to the Appendix.

2 Formal setting

Let X be a real locally convex topological vector space ordered by a reflexive, transitive and antisymmetric binary relation $\geq$ such that (a) if $x, y, z \in X$ and $x \geq y$, then $x + z \geq y + z$, and (b) if $x, y \in X$, $t \in \Re_{++}$ and $x \geq y$, then $tx \geq ty$. Define the relation $>$ on X by: for $x, y \in X$, $x > y$ if and only if $x \geq y$ and $x \neq y$. Given nonempty sets $E, F \subset X$, we say that $E \geq^* F$ if $\neg y > x$ for all $x \in E$ and $y \in F$. Let $X_+ = \{x \in X \mid x \geq 0\}$.

Let O be a convex compact subset of X_+ such that $0 \in O$ and $\geq$ is latticial on O, i.e., for all $x, y \in O$, there exists $z \in O$ such that $z \geq x$ and $z \geq y$. O is given the subspace topology, which we require to be metrizable. Moreover, O is given the Borel σ-algebra $\mathcal{B}(O)$. As O is metrizable, every singleton subset of O is closed in O and so $\{x\} \in \mathcal{B}(O)$ for every $x \in O$.

For every positive integer n, the space $\Re^n$ will be given the Euclidean metric topology, with $\|x\|$ denoting the Euclidean norm of $x \in \Re^n$.

Let $\Delta(O)$ be the set of countably-additive probability measures (henceforth, lotteries) on $(O, \mathcal{B}(O))$. Let $\mathcal{C}(O, \Re)$ denote the set of continuous functions $g : O \to \Re$. As O is compact, every $g \in \mathcal{C}(O, \Re)$ is bounded. For every $g \in \mathcal{C}(O, \Re)$, the formula $L(\mu, g) = \int_O \mu(dz)\, g(z)$ defines the functional $L(., g) : \Delta(O) \to \Re$. $\Delta(O)$ is given the weak* topology, which is the projective topology generated on $\Delta(O)$ by the family $\{L(., g) \mid g \in \mathcal{C}(O, \Re)\}$.

We note some consequences of our assumptions. $\Delta(O)$ is compact and metrizable (Parthasarathy [22], Theorem II.6.4). As O is compact metric, it is separable, i.e., there is a countable set $E \subset O$ that is dense in O. Given $x \in O$, δ_x denotes the Dirac measure at x, i.e., for every $B \in \mathcal{B}(O)$, $\delta_x(B) = 1$ if $x \in B$ and $\delta_x(B) = 0$ otherwise. As $\{x\} \in \mathcal{B}(O)$ for every $x \in O$, $\delta_x \in \Delta(O)$ for every $x \in O$. Let $\Delta^0(E)$ denote the set of $\mu \in \Delta(O)$ with finite support in E, i.e., μ is a finite convex combination of Dirac measures in E. Then, $\Delta^0(E)$ is dense in $\Delta(O)$ (Parthasarathy [22], Theorem II.6.3). Given $\mu \in \Delta(O)$, $m_\mu = \int_O \mu(dz)\, z$ denotes the mean of μ, where the integral on the right-hand side is the Pettis integral; see Pettis [23] for details.

Theorem 2.1 *If O is nonempty, convex, compact and metrizable, and $\mu \in \Delta(O)$, then m_μ exists, is unique and $m_\mu \in O$.*

We say that $u : O \to \Re$ is risk averse if $u(m_\mu) \geq \int_O \mu(dz)\, u(z)$ for every $\mu \in \Delta(O)$. For every vN-M utility function $u : O \to \Re$, the set of functions $[u] = \{a + bu \mid a \in \Re \ \wedge \ b \in \Re_{++}\}$ is an equivalence class of functions that are vN-M representations of the same preference as u. Therefore, we denote by $[u]$ the preference over lotteries represented by the vN-M utility u.

3 Utilities and certainty equivalents

The preferences over $\Delta(O)$ that are admissible for our duality theory are those with a vN-M representation in the following set.

Definition 3.1 *$\mathcal{U}$ is the set of functions $u : O \to \Re$ such that*
(a) u is continuous,
(b) u is increasing with respect to $>$, and
(c) $u(0) = 0$.
$\mathcal{U}_a$ is the subset of $\mathcal{U}$ consisting of risk averse utility functions.

(a) is a regularity condition. (b) is a natural requirement in most economic contexts. As $O \subset X_+$, (b) and (c) imply that $u(x) \in \Re_+$ for every $x \in O$. Given (a) and (b), (c) does not further restrict the set of preferences on $\Delta(O)$ that have a vN-M representation in $\mathcal{U}$ (resp. $\mathcal{U}_a$) because, if u satisfies (a) and (b), then $u - u(0) \in [u] \cap \mathcal{U}$; moreover, if u is risk averse, then so is $u - u(0)$. We now define a set of multi-valued mappings $F : \Delta(O) \Rightarrow O$ with the interpretation that $F(\mu)$ is the set of certainty equivalent outcomes corresponding to the lottery μ.

Definition 3.2 *$\mathcal{F}$ is the set of mappings $F : \Delta(O) \Rightarrow O$ such that*
(A) F has nonempty values,
(B) $\geq^*$ is a complete and antisymmetric preordering on $\{F(\mu) \mid \mu \in \Delta(O)\}$,
(C) for all $\mu, \lambda, \gamma \in \Delta(O)$, $F(\mu) = F(\lambda)$ implies $F(\mu/2+\gamma/2) = F(\lambda/2+\gamma/2)$,
(D) for every $\lambda \in \Delta(O)$, $\{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\lambda)\}$ and $\{\mu \in \Delta(O) \mid F(\lambda) \geq^* F(\mu)\}$ are closed in $\Delta(O)$,
(E) $x \in F(\delta_x)$ for every $x \in O$,
(F) if $x, y \in O$ and $x > y$, then $F(\delta_x) \geq^* F(\delta_y)$ and $\neg F(\delta_y) \geq^* F(\delta_x)$, and
(G) $x \in F(\mu)$ implies $F(\mu) = F(\delta_x)$.
$\mathcal{F}_a \subset \mathcal{F}$ consists of mappings $F : \Delta(O) \Rightarrow O$ such that $F(\delta_{m_\mu}) \geq^* F(\mu)$ for every $\mu \in \Delta(O)$.

For $u \in \mathcal{U}$, define $\phi(u) : \Delta(O) \Rightarrow O$ by

$$\phi(u)(\mu) = \left\{ x \in O \,\middle|\, u(x) = \int_O \mu(dz)\, u(z) \right\} \tag{1}$$

Given a utility function u and a lottery μ, $\phi(u)(\mu)$ is the set of outcomes that yield the same utility as the expected utility derived from u and μ. In the case of scalar outcomes and an increasing utility function, the set of certainty equivalent outcomes is a singleton set; this is no longer the case when outcomes are vectors.

Note that, (a) if $u \in \mathcal{U}$, then $bu \in [u] \cap \mathcal{U}$ for every $b \in \Re_{++}$, i.e., if $\mathcal{U}$ contains a representation of some preference on $\Delta(O)$, then it contains multiple representations of that preference, and (b) if $u \in \mathcal{U}$ and $v \in [u] \cap \mathcal{U}$, then $\phi(u) = \phi(v)$. These observations mean that ϕ is not injective on $\mathcal{U}$ in the usual sense of the term. However, the appropriate notion of injectiveness for the dual representation of preferences is that, if $u, v \in \mathcal{U}$ are such that $\phi(u) = \phi(v)$, then u and v must represent the same preference on $\Delta(O)$, i.e., $[u] = [v]$. In other words, a function $u \in \mathcal{U}$ should be identified with the equivalence class $[u]$. This preference-based interpretation of the elements of $\mathcal{U}$ and associated notion of injectiveness will apply throughout this paper.

The main result of this section, Theorem 3.6, shows that ϕ is a bijection between $\mathcal{U}$ and $\mathcal{F}$. A corollary of this result is the fact that ϕ is also a bijection between $\mathcal{U}_a$ and $\mathcal{F}_a$. The proof is divided into three lemmas. In Lemma 3.3, we show that $\phi(u) \in \mathcal{F}$ for every $u \in \mathcal{U}$. In Lemma 3.4, we show that ϕ is injective. Finally, in Lemma 3.5, we show that ϕ is surjective by showing that $\phi^{-1}(\{F\}) \neq \emptyset$ for every $F \in \mathcal{F}$.

Lemma 3.3 *If $u \in \mathcal{U}$, then $\phi(u) \in \mathcal{F}$.*

Proof. Fix $u \in \mathcal{U}$ and denote $\phi(u)$ by F. (a) implies that u is measurable, and as O is compact, u is bounded. Therefore, the generalized Lebesgue integral $\int_O \mu(dz)\, u(z)$ exists for every $\mu \in \Delta(O)$. So, the function $U : \Delta(O) \to \Re$, given by $U(\mu) = \int_O \mu(dz)\, u(z)$, is well-defined. Thus,

$$F(\mu) = \{x \in O \mid u(x) = U(\mu)\} = \{x \in O \mid U(\delta_x) = U(\mu)\} \qquad (2)$$

As $\Delta(O)$ is given the weak* topology, (a) implies that U is continuous.

(A) As O is convex, it is connected. As O is nonempty and connected, (a) implies $u(O) \subset \Re$ is nonempty and connected. (b), (c) and the facts that O is compact and connected imply that $U(\mu) \in [0, \sup\{u(x) \mid x \in O\}] \subset u(O)$. Consequently, there exists $x \in O$ such that $u(x) = U(\mu)$, i.e., $x \in F(\mu)$.

Before demonstrating the other properties of F, we confirm that

$$F(\mu) \geq^* F(\lambda) \quad \Leftrightarrow \quad U(\mu) \geq U(\lambda) \qquad (3)$$

for all $\mu, \lambda \in \Delta(O)$.

Suppose $U(\mu) < U(\lambda)$. By (A), $F(\mu) \neq \emptyset$ and $F(\lambda) \neq \emptyset$. Let $x \in F(\mu)$ and $y \in F(\lambda)$. As $0 \in O$ and $O \subset X_+$, (b) and (c) imply $0 \leq u(x) = U(\mu) < U(\lambda) = u(y)$. By (c), $y > 0$. As O is convex and $0 \in O$, $ty \in O$ for every $t \in [0, 1)$. As $[0, 1)$ is connected and X is a topological vector space, $\{ty \mid t \in [0, 1)\}$ is connected. Then, (a) implies that $\{u(ty) \mid t \in [0, 1)\}$ is connected. (b) implies $\{u(ty) \mid t \in [0, 1)\} = [0, u(y))$. As $u(x) \in [0, u(y))$, there exists $t \in [0, 1)$ such that $u(ty) = u(x)$, i.e., $ty \in F(\mu)$. As $ty < y$, we have $\neg F(\mu) \geq^* F(\lambda)$.

Conversely, suppose $\mu, \lambda \in \Delta(O)$ and $\neg F(\mu) \geq^* F(\lambda)$. Then, there exists $x \in F(\mu)$ and $y \in F(\lambda)$ such that $y > x$. By (b), $u(y) > u(x)$. Thus, $U(\lambda) = u(y) > u(x) = U(\mu)$.

We now check that F satisfies (B) to (G).

(B) (3) implies that $\geq^*$ is a complete preordering on $\{F(\mu) \mid \mu \in \Delta(O)\}$. To see that $\geq^*$ is antisymmetric on $\{F(\mu) \mid \mu \in \Delta(O)\}$, suppose $\mu, \lambda \in \Delta(O)$ are such that $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. (3) implies that $U(\mu) = U(\lambda)$. It follows from (2) that $F(\mu) = F(\lambda)$.

(C) Suppose $\mu, \lambda, \gamma \in \Delta(O)$ and $F(\mu) = F(\lambda)$. As $\geq^*$ is reflexive on $\{F(\mu) \mid \mu \in \Delta(O)\}$, we have $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. By (3), $U(\mu) = U(\lambda)$. The linearity of U implies $U(\mu/2+\gamma/2) = U(\mu)/2+U(\gamma)/2 = U(\lambda)/2 + U(\gamma)/2 = U(\lambda/2 + \gamma/2)$. By (3) and the antisymmetry of $\geq^*$ on $\{F(\mu) \mid \mu \in \Delta(O)\}$, we have $F(\mu/2 + \gamma/2) = F(\lambda/2 + \gamma/2)$.

(D) Consider $\lambda \in \Delta(O)$. By (3) and the continuity of U, $\{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\lambda)\} = \{\mu \in \Delta(O) \mid U(\mu) \geq U(\lambda)\}$ is closed in $\Delta(O)$. Similarly, $\{\mu \in \Delta(O) \mid F(\lambda) \geq^* F(\mu)\}$ is closed in $\Delta(O)$.

(E) For every $x \in O$, $u(x) = U(\delta_x)$, and so $x \in F(\delta_x)$.

(F) Consider $x, y \in O$ such that $x > y$. Let $x' \in F(\delta_x)$ and $y' \in F(\delta_y)$. If $y' > x'$, then (b) implies $u(y) = u(y') > u(x') = u(x)$, a contradiction. So, $F(\delta_x) \geq^* F(\delta_y)$. As $x \in F(\delta_x)$, $y \in F(\delta_y)$ and $x > y$, it follows that $\neg F(\delta_y) \geq^* F(\delta_x)$.

(G) Let $x \in F(\mu)$. If $y \in F(\delta_x)$, then $u(y) = U(\delta_x) = u(x) = U(\mu)$. So, $y \in F(\mu)$. Thus, $F(\delta_x) \subset F(\mu)$. If $y \in F(\mu)$, then $u(y) = U(\mu) = u(x) = U(\delta_x)$. So, $y \in F(\delta_x)$. Thus, $F(\mu) \subset F(\delta_x)$. ∎

We now execute the second step in the proof of Theorem 3.6.

Lemma 3.4 *ϕ is injective.*

Proof. Consider $u, v \in \mathcal{U}$ such that $\phi(u) = \phi(v)$. We show that $[u] = [v]$.

(1) We first show that u and v are comonotonic, i.e., induce the same ordering on O. Suppose there exist $x, y \in O$ such that $u(x) \geq u(y)$ and $v(x) < v(y)$. Then there exists $t \in [0, 1)$ such that $ty \in O$ and $v(ty) = v(x)$. It follows that $ty \in \phi(v)(\delta_x)$. However, as (b) implies that $u(x) \geq u(y) > u(ty)$, we have $ty \notin \phi(u)(\delta_x)$, a contradiction.

(2) By Lemma 3.3 and (A), $\phi(u)$ and $\phi(v)$ have nonempty values. For $\nu \in \Delta(O)$, let $x_\nu \in \phi(u)(\nu)$. Given $\mu, \lambda \in \Delta(O)$, step (1) implies

$$U(\mu) \geq U(\lambda) \;\Leftrightarrow\; u(x_\mu) \geq u(x_\lambda) \;\Leftrightarrow\; v(x_\mu) \geq v(x_\lambda) \;\Leftrightarrow\; V(\mu) \geq V(\lambda)$$

Thus, U and V are comonotonic linear mappings on $\Delta(O)$.

(3) As $\Delta(O)$ is compact and U is continuous, there exist $\alpha, \beta \in \Delta(O)$ such that $U(\alpha) \leq U(\mu) \leq U(\beta)$ for every $\mu \in \Delta(O)$. If $U(\alpha) = U(\beta)$, then U is constant over $\Delta(O)$, say $U(\mu) = k_U$ for every $\mu \in \Delta(O)$. If U is constant over $\Delta(O)$, then by step (2) so is V. Let $V(\mu) = k_V$ for every $\mu \in \Delta(O)$. Setting $a = k_V - k_U$ and $b = 1$ implies $V = a + bU$.

Suppose $U(\beta) > U(\alpha)$. By step (2), $V(\beta) > V(\alpha)$. Define

$$a = V(\alpha) - U(\alpha)\left[\frac{V(\beta) - V(\alpha)}{U(\beta) - U(\alpha)}\right] \qquad \text{and} \qquad b = \frac{V(\beta) - V(\alpha)}{U(\beta) - U(\alpha)}$$

Clearly, $b > 0$. Now consider $\mu \in \Delta(O)$. We show that $V(\mu) = a + bU(\mu)$.

As $U(\mu) \in [U(\alpha), U(\beta)]$, there is a unique $t \in [0,1]$ such that $U(\mu) = tU(\beta) + (1-t)U(\alpha)$. As U is linear, $U(\mu) = U(t\beta + (1-t)\alpha)$. By step (2) and the linearity of V, we have $V(\mu) = V(t\beta + (1-t)\alpha) = tV(\beta) + (1-t)V(\alpha)$. Then, using the definitions of a and b, we have $a + bU(\mu) = V(\alpha) + b[U(\mu) - U(\alpha)]$. As $b[U(\mu) - U(\alpha)] = bt[U(\beta) - U(\alpha)] = t[V(\beta) - V(\alpha)]$, we have

$$a + bU(\mu) = V(\alpha) + t[V(\beta) - V(\alpha)] = tV(\beta) + (1-t)V(\alpha) = V(\mu)$$

It follows that $v(x) = V(\delta_x) = a + bU(\delta_x) = a + bu(x)$ for every $x \in O$. Thus, $[u] = [v]$. ■

The last step of the argument is the following.

Lemma 3.5 *If $F \in \mathcal{F}$, then $\phi^{-1}(\{F\}) \neq \emptyset$.*

Proof. Consider $F \in \mathcal{F}$. By (A), $F(\mu) \neq \emptyset$ for every $\mu \in \Delta(O)$. Define the relation $\succeq^*$ on $\Delta(O)$ by: $\mu \succeq^* \lambda$ if and only if $F(\mu) \geq^* F(\lambda)$. (B) implies that $\succeq^*$ is a complete preordering. Define the relation $\sim^*$ on $\Delta(O)$ by: $\mu \sim^* \lambda$ if and only if $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$.

If $\mu, \lambda \in \Delta(O)$ are such that $\mu \sim^* \lambda$, then $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$. Therefore, $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. So, the antisymmetry property in (B) implies $F(\mu) = F(\lambda)$. Conversely, if $F(\mu) = F(\lambda)$, then the reflexivity property in (B) implies $F(\mu) \geq^* F(\lambda)$ and $F(\lambda) \geq^* F(\mu)$. Thus, $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$, and consequently, $\mu \sim^* \lambda$. Thus, $\mu \sim^* \lambda$ if and only if $F(\mu) = F(\lambda)$.

Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $\mu \sim^* \lambda$. Then, $F(\mu) = F(\lambda)$ and (C) implies that $F(\mu/2 + \gamma/2) = F(\lambda/2 + \gamma/2)$. Thus, $\mu/2 + \gamma/2 \sim^* \lambda/2 + \gamma/2$.

Given $\gamma \in \Delta(O)$, (D) implies that $S = \{\mu \in \Delta(O) \mid \mu \succeq^* \gamma\} = \{\mu \in \Delta(O) \mid F(\mu) \geq^* F(\gamma)\}$ is closed in $\Delta(O)$. Consider $\mu, \lambda, \gamma \in \Delta(O)$ and the function $f : [0,1] \to \Delta(O)$ defined by $f(t) = t\mu + (1-t)\lambda$. Consider $g \in \mathcal{C}(O, \Re)$ and a sequence (t_n) in $[0,1]$ converging to $t \in [0,1]$. Then, $\int_O f(t_n)(dx)\, g(x) = t_n \int_O \mu(dx)\, g(x) + (1-t_n) \int_O \lambda(dx)\, g(x)$. Taking limits, we have $\lim_{n\uparrow\infty} \int_O f(t_n)(dx)\, g(x) = t \int_O \mu(dx)\, g(x) + (1-t) \int_O \lambda(dx)\, g(x) = \int_O f(t)(dx)\, g(x)$. Thus, f is continuous, and as S is closed in $\Delta(O)$,

$$\{t \in [0,1] \mid t\mu + (1-t)\lambda \succeq^* \gamma\} = \{t \in [0,1] \mid f(t) \succeq^* \gamma\} = f^{-1}(S)$$

is closed in $[0,1]$. By an analogous argument, $\{t \in [0,1] \mid \gamma \succeq^* t\mu + (1-t)\lambda\}$ is closed in $[0,1]$.

It follows (Herstein and Milnor [13], Theorem 8) that there exists a linear representation $V : \Delta(O) \to \Re$ of $\succeq^*$. Clearly, $U : \Delta(O) \to \Re$, defined by $U(\mu) = V(\mu) - V(\delta_0)$, is a linear representation of $\succeq^*$ and $U(\delta_0) = 0$. As for every $\lambda \in \Delta(O)$, the sets $\{\mu \in \Delta(O) \mid U(\mu) \geq U(\lambda)\} = \{\mu \in \Delta(O) \mid \mu \succeq^* \lambda\}$ and $\{\mu \in \Delta(O) \mid U(\mu) \leq U(\lambda)\} = \{\mu \in \Delta(O) \mid \lambda \succeq^* \mu\}$ are closed in $\Delta(O)$, U is continuous. Define $u : O \to \Re$ by $u(z) = U(\delta_z)$. It is straightforward to check that $z \mapsto \delta_z$ is continuous. As U is continuous, so is u. Also, $u(0) = U(\delta_0) = 0$.

Given $\mu \in \Delta(O)$, as $\Delta^0(E)$ is dense in $\Delta(O)$, there exists a sequence $(\mu_n) \subset \Delta^0(E)$ that converges to μ in the weak* topology. As each μ_n has finite support and U is linear, we have

$$U(\mu_n) = U\left(\sum_{z \in \operatorname{supp} \mu_n} \mu_n(\{z\})\delta_z\right) = \sum_{z \in \operatorname{supp} \mu_n} \mu_n(\{z\})U(\delta_z)$$

The definition of u implies

$$U(\mu_n) = \sum_{z \in \operatorname{supp} \mu_n} \mu_n(\{z\})u(z) = \int_O \mu_n(dz)\, u(z)$$

The continuity of U and u imply

$$U(\mu) = \lim_{n \uparrow \infty} U(\mu_n) = \lim_{n \uparrow \infty} \int_O \mu_n(dz)\, u(z) = \int_O \mu(dz)\, u(z)$$

We now verify that $u \in \mathcal{U}$. By construction, u satisfies (a) and (c). To check that (b) is satisfied, consider $x, y \in O$ such that $x > y$. Then, (F) implies $F(\delta_x) \geq^* F(\delta_y)$ and $\neg F(\delta_y) \geq^* F(\delta_x)$. Consequently, $\delta_x \succeq^* \delta_y$ and $\neg\delta_y \succeq^* \delta_x$. Therefore, $u(x) = U(\delta_x) > U(\delta_y) = u(y)$.

Finally, we show that $\phi(u) = F$. We need to show that

$$F(\mu) = \{x \in O \mid u(x) = U(\mu)\} = \{x \in O \mid U(\delta_x) = U(\mu)\}$$

for every $\mu \in \Delta(O)$. Observe that, for all $\mu, \lambda \in \Delta(O)$,

$$\begin{aligned} F(\mu) = F(\lambda) &\Leftrightarrow F(\mu) \geq^* F(\lambda) \;\wedge\; F(\lambda) \geq^* F(\mu) \\ &\Leftrightarrow \mu \succeq^* \lambda \;\wedge\; \lambda \succeq^* \mu \\ &\Leftrightarrow U(\mu) \geq U(\lambda) \;\wedge\; U(\lambda) \geq U(\mu) \\ &\Leftrightarrow U(\mu) = U(\lambda) \end{aligned}$$

The first equivalence follows from (B) as $\geq^*$ is reflexive and antisymmetric, while the second and third equivalences follow from the definitions of $\succeq^*$ and U.

Consider $x \in O$ such that $U(\delta_x) = U(\mu)$. It follows that $F(\delta_x) = F(\mu)$. By (E), $x \in F(\delta_x) = F(\mu)$. Conversely, consider $x \in F(\mu)$. By (G), $F(\mu) = F(\delta_x)$. Therefore, $U(\mu) = U(\delta_x)$. Thus, $\{x \in O \mid U(\delta_x) = U(\mu)\} = F(\mu)$. ■

As is evident from Theorem 8 in Herstein and Milnor [13] and the above proof, (D) is stronger than the "continuity" condition that is sufficient for the existence of a linear representation of $\succeq^*$. However, the extra power of (D) is useful for showing that the derived linear representation is continuous and admits an expected utility representation.

Lemmas 3.3, 3.4 and 3.5 yield

Theorem 3.6 *ϕ is a bijection from $\mathcal{U}$ to $\mathcal{F}$.*

The following result is a straightforward corollary.

Corollary 3.7 *ϕ is a bijection from $\mathcal{U}_a$ to $\mathcal{F}_a$.*

Proof. It follows from Theorem 3.6 that $\phi(\mathcal{U}_a) \subset \phi(\mathcal{U}) = \mathcal{F}$. Consider $u \in \mathcal{U}_a$. We check that $F \equiv \phi(u) \in \mathcal{F}_a$. Suppose there exists $\mu \in \Delta(O)$ such that $\neg F(\delta_{m_\mu}) \geq^* F(\mu)$; note that $m_\mu \in O$ by Theorem 2.1. Then, there exists $x \in F(\mu)$ and $y \in F(\delta_{m_\mu})$ such that $x > y$. By definition, $u(x) = U(\mu)$ and $u(y) = U(\delta_{m_\mu}) = u(m_\mu)$. As $x > y$, we have $u(m_\mu) = u(y) < u(x) = U(\mu)$, which contradicts the fact that $u \in \mathcal{U}_a$. Thus, $\phi(\mathcal{U}_a) \subset \mathcal{F}_a$.

Injectiveness of $\phi : \mathcal{U}_a \to \mathcal{F}_a$ follows from Theorem 3.6 as $\mathcal{U}_a \subset \mathcal{U}$.

To check that $\phi : \mathcal{U}_a \to \mathcal{F}_a$ is surjective, consider $F \in \mathcal{F}_a$. As $\mathcal{F}_a \subset \mathcal{F}$, we have $F \in \mathcal{F}$. By Theorem 3.6, there exists $u \in \mathcal{U}$ such that $\phi(u) = F$. We need to confirm that $u \in \mathcal{U}_a$. Consider $\mu \in \Delta(O)$. As $F \in \mathcal{F}_a$, we have $F(\delta_{m_\mu}) \geq^* F(\mu)$. So, $\delta_{m_\mu} \succeq^* \mu$. This implies $u(m_\mu) = U(\delta_{m_\mu}) \geq U(\mu) = \int_O \mu(dz)\, u(z)$, as required. ■

4 Certainty equivalents and risk premia

Another object of interest is the set of risk premia associated with a lottery. Given $F \in \mathcal{F}$, define $\psi(F) : \Delta(O) \rightrightarrows X$ by $\psi(F)(\mu) = \{m_\mu - x \in X \mid x \in F(\mu)\}$. Setting $u = \phi^{-1}(F)$, it follows that,

$$\begin{aligned}\psi(F)(\mu) &= \{y \in X \mid m_\mu - y \in F(\mu)\} \\ &= \left\{ y \in X \;\middle|\; m_\mu - y \in O \;\wedge\; u(m_\mu - y) = \int_O \mu(dz)\, u(z) \right\}\end{aligned}$$

As in the case of certainty equivalents, while the set of risk premia is a singleton set when outcomes are scalars and u is increasing, this is not the case when outcomes are vectors.

Define $T : \Delta(O) \times X \to X$ by $T(\mu, x) = m_\mu - x$. Clearly, given $\mu \in \Delta(O)$, $T(\mu, .)$ is a bijection. Define

$$\mathcal{P} = \{T(., F(.)) : \Delta(O) \rightrightarrows X \mid F \in \mathcal{F}\}$$

and let $\mathcal{P}_a$ be the subset of $\mathcal{P}$ consisting of mappings $P : \Delta(O) \rightrightarrows X$ such that $P(\mu) \geq^* P(\delta_{m_\mu})$. The following duality result follows.

Theorem 4.1 *ψ is a bijection from $\mathcal{F}$ to $\mathcal{P}$.*

Proof. Clearly, $\psi(F)(.) = \{T(., x) \mid x \in F(.)\} = T(., F(.)) \in \mathcal{P}$. By definition, ψ is surjective. To check injectiveness, suppose $F, G \in \mathcal{F}$ are such that $F \neq G$. Then, there exists $\mu \in \Delta(O)$ such that $F(\mu) \neq G(\mu)$. Without loss of generality, there exists $x \in F(\mu) - G(\mu)$. Thus, $m_\mu - x \in \psi(F)(\mu) - \psi(G)(\mu)$. Consequently, $\psi(F) \neq \psi(G)$. ∎

Corollary 4.2 *ψ is a bijection from $\mathcal{F}_a$ to $\mathcal{P}_a$.*

Proof. Theorem 4.1 implies that $\psi(\mathcal{F}_a) \subset \psi(\mathcal{F}) = \mathcal{P}$. Consider $F \in \mathcal{F}_a$. We check that $P \equiv \psi(F) \in \mathcal{P}_a$. Suppose there exists $\mu \in \Delta(O)$ such that $\neg P(\mu) \geq^* P(\delta_{m_\mu})$; note that $m_\mu \in O$ by Theorem 2.1. Then, there exists $x \in P(\mu)$ and $y \in P(\delta_{m_\mu})$ such that $y > x$. It follows that $m_\mu - x > m_\mu - y$. As $m_\mu - x \in F(\mu)$ and $m_\mu - y \in F(\delta_{m_\mu})$, we have $\neg F(\delta_{m_\mu}) \geq^* F(\mu)$, which contradicts the fact that $F \in \mathcal{F}_a$. Thus, $\psi(\mathcal{F}_a) \subset \mathcal{P}_a$.

Injectiveness of $\psi : \mathcal{F}_a \to \mathcal{P}_a$ follows from Theorem 4.1 as $\mathcal{F}_a \subset \mathcal{F}$.

To check that $\psi : \mathcal{F}_a \to \mathcal{P}_a$ is surjective, consider $P \in \mathcal{P}_a$. As $\mathcal{P}_a \subset \mathcal{P}$, we have $P \in \mathcal{P}$. By Theorem 4.1, there exists $F \in \mathcal{F}$ such that $\psi(F) = P$. We only need to confirm that $F(\delta_{m_\mu}) \geq^* F(\mu)$ for every $\mu \in \Delta(O)$. Suppose $\neg F(\delta_{m_\mu}) \geq^* F(\mu)$ for some $\mu \in \Delta(O)$. Then, there exists $x \in F(\mu)$ and $y \in F(\delta_{m_\mu})$ such that $x > y$. It follows that $m_\mu - y > m_\mu - x$. As $m_\mu - x \in P(\mu)$ and $m_\mu - y \in P(\delta_{m_\mu})$, we have $\neg P(\mu) \geq^* P(\delta_{m_\mu})$, which contradicts the fact that $P \in \mathcal{P}_a$. ∎

5 Utilities and acceptance sets

We now establish the duality between $\mathcal{U}$ and the set of mappings $A : O \Rightarrow \Delta(O)$ that yield the acceptance set $A(x) \subset \Delta(O)$ for every outcome $x \in O$. Given $A : O \Rightarrow \Delta(O)$, define the lower inverse mapping $A^- : \Delta(O) \Rightarrow O$ by $A^-(\mu) = \{x \in O \mid \mu \in A(x)\}$.

Definition 5.1 *$\mathcal{A}$ is the set of mappings $A : O \Rightarrow \Delta(O)$ such that*

(A) A^- has nonempty values,

(B) $\subset$ is complete on $\{A^-(\mu) \mid \mu \in \Delta(O)\}$,[4]

(C) for every $\lambda \in \Delta(O)$, $\{\mu \in \Delta(O) \mid A^-(\mu) \subset A^-(\lambda)\}$ and $\{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\lambda)\}$ are closed in $\Delta(O)$,

(D) for all $\mu, \lambda, \gamma \in \Delta(O)$, if $A^-(\lambda) = A^-(\mu)$, then $A^-(\lambda/2 + \gamma/2) = A^-(\mu/2 + \gamma/2)$,

(E) for all $x, y \in O$, $x > y$ implies $A^-(\delta_x) \supset A^-(\delta_y)$ and $A^-(\delta_x) \not\subset A^-(\delta_y)$, and

(F) for $\mu \in \Delta(O)$ and $x \in O$, $x \in A^-(\mu)$ if and only if $A^-(\delta_x) \subset A^-(\mu)$.

$\mathcal{A}_a$ is the subset of $\mathcal{A}$ consisting of mappings $A : O \Rightarrow \Delta(O)$ such that, for every $x \in O$, $\mu \in A(x)$ implies $\neg x > m_\mu$.

[4] That $\subset$ is an antisymmetric preordering comes for free.

Given $u \in \mathcal{U}$, define $\xi(u) : O \Rightarrow \Delta(O)$ by $\xi(u)(x) = \{\mu \in \Delta(O) \mid u(x) \leq \int_O \mu(dz)\, u(z)\}$. The main result of this section, Theorem 5.5, shows that ξ is a bijection between $\mathcal{U}$ and $\mathcal{A}$. The proof is divided into three lemmas. In Lemma 5.2, we show that $\xi(u) \in \mathcal{A}$ for every $u \in \mathcal{U}$. In Lemma 5.3, we show that ξ is injective. Finally, in Lemma 5.4, we show that ξ is surjective by showing that $\xi^{-1}(\{A\}) \neq \emptyset$ for every $A \in \mathcal{A}$.

Lemma 5.2 *If $u \in \mathcal{U}$, then $\xi(u) \in \mathcal{A}$.*

Proof. Fix $u \in \mathcal{U}$, denote $\xi(u)$ by A and define $U : \Delta(O) \to \Re$ by $U(\mu) = \int_O \mu(dz)\, u(z)$. By definition, for every $z \in O$ and $\mu \in \Delta(O)$,

$$z \in A^-(\mu) \quad \Leftrightarrow \quad \mu \in A(z) \quad \Leftrightarrow \quad U(\mu) \geq u(z) \tag{4}$$

Consider $\mu, \lambda \in \Delta(O)$. (4) implies that $A^-(\mu) \subset A^-(\lambda)$ is equivalent to $[U(\mu) \geq u(z) \Rightarrow U(\lambda) \geq u(z)]$. Also, $U(\lambda) \geq U(\mu)$ implies $[U(\mu) \geq u(z) \Rightarrow U(\lambda) \geq u(z)]$. Conversely, suppose $U(\lambda) < U(\mu)$. As O is connected and compact, (a) implies that $u(O) \subset \Re$ is a closed interval. Thus, $U(\lambda), U(\mu) \in u(O)$, and there exists $z \in O$ such that $U(\lambda) < u(z) < U(\mu)$. Thus, $U(\lambda) \geq U(\mu)$ is equivalent to $[U(\mu) \geq u(z) \Rightarrow U(\lambda) \geq u(z)]$. Consequently, for all $\mu, \lambda \in \Delta(O)$,

$$A^-(\mu) \subset A^-(\lambda) \quad \Leftrightarrow \quad U(\lambda) \geq U(\mu) \tag{5}$$

(A) (b) and (c) imply that $u(x) \geq 0$ for every $x \in O$. Therefore, for every $\mu \in \Delta(O)$, we have $U(\mu) \geq 0 = u(0)$. (4) implies that $0 \in A^-(\mu)$.

(B) Given $\mu, \lambda \in \Delta(O)$, we have either $U(\mu) \geq U(\lambda)$ or $U(\mu) \leq U(\lambda)$. Thus, (5) implies that either $A^-(\mu) \supset A^-(\lambda)$ or $A^-(\mu) \subset A^-(\lambda)$.

(C) (a) implies that U is continuous. Therefore, given $\lambda \in \Delta(O)$, (5) implies that $\{\mu \in \Delta(O) \mid A^-(\mu) \subset A^-(\lambda)\} = \{\mu \in \Delta(O) \mid U(\mu) \leq U(\lambda)\}$ is closed in $\Delta(O)$. Similarly, $\{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\lambda)\}$ is closed in $\Delta(O)$.

(D) Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $A^-(\lambda) = A^-(\mu)$. (5) implies $U(\mu) = U(\lambda)$. It follows that $U(\mu/2 + \gamma/2) = U(\mu)/2 + U(\gamma)/2 = U(\lambda)/2 + U(\gamma)/2 = U(\lambda/2 + \gamma/2)$. (5) implies $A^-(\lambda/2 + \gamma/2) = A^-(\mu/2 + \gamma/2)$.

(E) Suppose $x > y$. Let $z \in A^-(\delta_y)$. (4) implies that $u(y) = U(\delta_y) \geq u(z)$. (b) implies $u(x) > u(y)$. Therefore, $U(\delta_x) = u(x) > u(z)$. By (4), $z \in A^-(\delta_x)$. Thus, $A^-(\delta_y) \subset A^-(\delta_x)$.

Suppose $A^-(\delta_x) \subset A^-(\delta_y)$. By (4), $u(x) = U(\delta_x) \geq u(z)$ implies $u(y) = U(\delta_y) \geq u(z)$. This means $u(y) \geq u(x)$, a contradiction of (b).

(F) Combining (4) and (5) yields

$$x \in A^-(\mu) \quad \Leftrightarrow \quad U(\mu) \geq U(\delta_x) \quad \Leftrightarrow \quad A^-(\delta_x) \subset A^-(\mu)$$

as required. ■

We now show that $\xi : \mathcal{U} \to \mathcal{A}$ is an injection.

Lemma 5.3 *$\xi : \mathcal{U} \to \mathcal{A}$ is an injection.*

Proof. Consider $u, v \in \mathcal{U}$ such that $\xi(u) = \xi(v)$. Define the binary relations $\succeq^*_u$ and $\succeq^*_v$ on $\Delta(O)$ as follows: for all $\mu, \lambda \in \Delta(O)$, $\mu \succeq^*_u \lambda$ if and only if $U(\mu) \geq U(\lambda)$, and $\mu \succeq^*_v \lambda$ if and only if $V(\mu) \geq V(\lambda)$. Combining this with (5), we have

$$\mu \succeq^*_u \lambda \qquad \Leftrightarrow \qquad U(\mu) \geq U(\lambda) \qquad \Leftrightarrow \qquad \xi(u)^-(\lambda) \subset \xi(u)^-(\mu)$$

and

$$\mu \succeq^*_v \lambda \qquad \Leftrightarrow \qquad V(\mu) \geq V(\lambda) \qquad \Leftrightarrow \qquad \xi(v)^-(\lambda) \subset \xi(v)^-(\mu)$$

As $\xi(u) = \xi(v)$, $\mu \succeq^*_u \lambda$ if and only if $\mu \succeq^*_v \lambda$. It follows that

$$U(\mu) \geq U(\lambda) \quad \Leftrightarrow \quad \mu \succeq^*_u \lambda \quad \Leftrightarrow \quad \mu \succeq^*_v \lambda \quad \Leftrightarrow \quad V(\mu) \geq V(\lambda)$$

Copying the argument of Lemma 3.4, there exists $a \in \Re$ and $b \in \Re_{++}$ such that $V = a + bU$, i.e., $v = a + bu$. Thus, $[u] = [v]$. ■

Finally, we show that ξ is surjective.

Lemma 5.4 *If $A \in \mathcal{A}$, then $\xi^{-1}(\{A\}) \neq \emptyset$.*

Proof. Fix $A \in \mathcal{A}$. Define the relation $\succeq^*$ on $\Delta(O)$ by: $\mu \succeq^* \lambda$ if and only if $A^-(\mu) \supset A^-(\lambda)$. (B) implies that $\succeq^*$ is a complete preordering. Define the relation $\sim^*$ on $\Delta(O)$ by: $\mu \sim^* \lambda$ if and only if $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$.

If $\mu, \lambda \in \Delta(O)$ are such that $\mu \sim^* \lambda$, then $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$. Therefore, $A^-(\mu) \supset A^-(\lambda)$ and $A^-(\mu) \subset A^-(\lambda)$. It follows that $A^-(\mu) = A^-(\lambda)$. Conversely, if $A^-(\mu) = A^-(\lambda)$, then $A^-(\mu) \supset A^-(\lambda)$ and $A^-(\mu) \subset A^-(\lambda)$. Thus, $\mu \succeq^* \lambda$ and $\lambda \succeq^* \mu$, and consequently, $\mu \sim^* \lambda$. Thus, $\mu \sim^* \lambda$ if and only if $A^-(\mu) = A^-(\lambda)$.

Consider $\mu, \lambda, \gamma \in \Delta(O)$ such that $\mu \sim^* \lambda$. Then, $A^-(\mu) = A^-(\lambda)$, and (D) implies that $A^-(\mu/2 + \gamma/2) = A^-(\lambda/2 + \gamma/2)$. Thus, $\mu/2 + \gamma/2 \sim^* \lambda/2 + \gamma/2$.

Given $\gamma \in \Delta(O)$, (C) implies that $S = \{\mu \in \Delta(O) \mid \mu \succeq^* \gamma\} = \{\mu \in \Delta(O) \mid A^-(\mu) \supset A^-(\gamma)\}$ is closed in $\Delta(O)$. Consider $\mu, \lambda, \gamma \in \Delta(O)$ and the function $f : [0, 1] \to \Delta(O)$ defined by $f(t) = t\mu + (1 - t)\lambda$. Copying the argument of Lemma 3.5, f is continuous. So, as S is closed in $\Delta(O)$,

$$\{t \in [0, 1] \mid t\mu + (1 - t)\lambda \succeq^* \gamma\} = \{t \in [0, 1] \mid f(t) \succeq^* \gamma\} = f^{-1}(S)$$

is closed in $[0, 1]$. Analogously, $\{t \in [0, 1] \mid \gamma \succeq^* t\mu + (1 - t)\lambda\}$ is closed in $[0, 1]$.

It follows (Herstein and Milnor [13], Theorem 8) that $\succeq^*$ has a linear representation $V : \Delta(O) \to \Re$. Define $U : \Delta(O) \to \Re$ by $U(\mu) = V(\mu) - V(\delta_0)$. Clearly, U is a linear representation of $\succeq^*$ and $U(\delta_0) = 0$. Define

$u : O \to \Re$ by $u(x) = U(\delta_x)$. Copying the argument of Lemma 3.5, U and u are continuous, and for every $\mu \in \Delta(O)$, $U(\mu) = \int_O \mu(dz)\, u(z)$.

By definition, u satisfies (a) and (c). To check (b), let $x, y \in O$ such that $x > y$. By (E), $A^-(\delta_x) \supset A^-(\delta_y)$ and $A^-(\delta_x) \not\subset A^-(\delta_y)$. Then, $\delta_x \succeq^* \delta_y$ and $\neg \delta_y \succeq^* \delta_x$. It follows that $u(x) = U(\delta_x) > U(\delta_y) = u(y)$.

Finally, we show that $\xi(u) = A$. We need to show that, for every $x \in O$,

$$A(x) = \{\mu \in \Delta(O) \mid u(x) \leq U(\mu)\} = \{\mu \in \Delta(O) \mid U(\delta_x) \leq U(\mu)\}$$

Fix $x \in O$. As, for all $\mu, \lambda \in \Delta(O)$,

$$U(\mu) \geq U(\lambda) \qquad \Leftrightarrow \qquad \mu \succeq^* \lambda \qquad \Leftrightarrow \qquad A^-(\lambda) \subset A^-(\mu)$$

the problem reduces to showing that $A(x) = \{\mu \in \Delta(O) \mid A^-(\delta_x) \subset A^-(\mu)\}$. Using the definition of A^- and (F), we have

$$\mu \in A(x) \qquad \Leftrightarrow \qquad x \in A^-(\mu) \qquad \Leftrightarrow \qquad A^-(\delta_x) \subset A^-(\mu)$$

as required. ■

Combining Lemmas 5.2, 5.3 and 5.4, we have

Theorem 5.5 *ξ is a bijection from $\mathcal{U}$ to $\mathcal{A}$.*

The following result characterizes risk averse preferences.

Corollary 5.6 *ξ is a bijection from $\mathcal{U}_a$ to $\mathcal{A}_a$.*

Proof. Consider $u \in \mathcal{U}_a$. As Theorem 5.5 implies $\xi(\mathcal{U}_a) \subset \xi(\mathcal{U}) = \mathcal{A}$, we have $A \equiv \xi(u) \in \mathcal{A}$. If $\mu \in A(x)$ and $x > m_\mu$, then $m_\mu \in O$ by Theorem 2.1 and $U(\mu) \geq u(x) > u(m_\mu)$, which contradicts $u \in \mathcal{U}_a$. So, $A \in \mathcal{A}_a$ and $\xi(\mathcal{U}_a) \subset \mathcal{A}_a$.

Injectiveness of $\xi : \mathcal{U}_a \to \mathcal{A}_a$ follows from Theorem 5.5 as $\mathcal{U}_a \subset \mathcal{U}$.

To check that $\xi : \mathcal{U}_a \to \mathcal{A}_a$ is surjective, consider $A \in \mathcal{A}_a$. As $\mathcal{A}_a \subset \mathcal{A}$, we have $A \in \mathcal{A}$. By Theorem 5.5, there exists $u \in \mathcal{U}$ such that $\xi(u) = A$. It suffices to confirm that $u \in \mathcal{U}_a$. Suppose there exists $\mu \in \Delta(O)$ such that $U(\mu) > u(m_\mu)$. As O is convex, it is connected. As u is continuous and O is compact and connected, $u(O)$ is a closed interval in $\Re$. Thus, $U(\mu) \in u(O)$, i.e., there exists $x \in O$ such that $u(m_\mu) < U(\mu) = u(x)$. As u is continuous, there exists an open neighborhood V of m_μ such that $u(y) < u(x)$ for every $y \in V \cap O$. As $\geq$ is latticial on O, there exists $z \in O$ such that $z \geq m_\mu$ and $z \geq x$. If $z = m_\mu$, then $u(m_\mu) = u(z) \geq u(x)$, a contradiction. So, $z > m_\mu$. As O is convex, $m_\mu + t(z - m_\mu) \in O$ for every $t \in (0,1)$. So, for some $t_0 \in (0,1)$, $y \equiv m_\mu + t_0(z - m_\mu) \in V \cap O$. As $z > m_\mu$, we have $y > m_\mu$. As $y \in V \cap O$, we have $u(y) < u(x) = U(\mu)$, i.e., $U(\delta_y) < U(\mu)$. As U represents $\succeq^*$, this means $A^-(\delta_y) \subset A^-(\mu)$. By (F), we have $y \in A^-(\mu)$, i.e., $\mu \in A(y)$. As $y > m_\mu$ we have a contradiction of $A \in \mathcal{A}_a$. So, $u \in \mathcal{U}_a$. ■

6 Utilities and Arrow-Pratt functions

An Arrow-Pratt function refers to the mapping that yields at each outcome the Arrow-Pratt coefficient at that outcome. In this section we study the duality between vN-M utility functions and Arrow-Pratt functions in the setting of Euclidean outcome spaces. We start with elementary duality results in the setting $X = \Re$ and then go to the setting $X = \Re^n$.

The real outcomes case

We start by defining a set of vN-M utility functions in this setting.

Definition 6.1 *$\mathcal{U}^{1d}$ is the set of functions $u : \Re \to \Re$ that are twice differentiable, with $Du > 0$, $u(0) = 0$ and $Du(0) = 1$.*

Consider the preference $[v]$ represented by the twice differentiable vN-M utility $v : \Re \to \Re$ with $Dv > 0$. Then, $u \in \mathcal{U}^{1d} \cap [v]$ where $u(x) = [v(x) - v(0)]/Dv(0)$. Thus, the auxiliary conditions $u(0) = 0$ and $Du(0) = 1$ do not restrict the class of preferences represented by elements of $\mathcal{U}^{1d}$. Moreover, if $u \in \mathcal{U}^{1d}$, then $[u] \cap \mathcal{U}^{1d} = \{u\}$, i.e., $\mathcal{U}^{1d}$ contains only one vN-M representation of an admissible preference $[u]$. Next, we define a set of Arrow-Pratt functions.

Definition 6.2 *$\mathcal{R}^1$ is the set of functions $a : \Re \to \Re$ such that $a = Df$ for some differentiable function $f : \Re \to \Re$ with $f(0) = 0$.*

Clearly, for every $a \in \mathcal{R}^1$, the appropriate f is unique. For $u \in \mathcal{U}^{1d}$, define the Arrow-Pratt function $\chi(u) : \Re \to \Re$ by $\chi(u)(x) = -D^2u(x)/Du(x)$. As $\chi(u)(x) = -D \ln Du(x)$, we have the function $\chi : \mathcal{U}^{1d} \to \mathcal{R}^1$.

In order to confirm that χ is a bijection, it suffices to show that, for every $a \in \mathcal{R}^1$, the ODE $D^2u = -aDu$ has a unique solution $u \in \mathcal{U}^{1d}$. Existence of a solution shows that χ is surjective, while its uniqueness shows that χ is injective. It is straightforward to confirm that, given $a \in \mathcal{R}^1$ with $a = Df$, $x \mapsto \int_0^x dy\, e^{-f(y)}$ is the unique element of $\mathcal{U}^{1d}$ that solves the given equation. Therefore, we have the following duality result.

Theorem 6.3 *$\chi : \mathcal{U}^{1d} \to \mathcal{R}^1$ is a bijection.*

More specific dualities can also be derived. For instance, let $\mathcal{U}^{1cd}$ be the subset of $\mathcal{U}^{1d}$ consisting of functions with a continuous second derivative. Also, $\mathcal{C}(\Re, \Re) \subset \mathcal{R}^1$ because $D \int_0^x dy\, a(y) = a(x)$ for every $a \in \mathcal{C}(\Re, \Re)$.

Corollary 6.4 *$\chi : \mathcal{U}^{1cd} \to \mathcal{C}(\Re, \Re)$ is a bijection; given $a \in \mathcal{C}(\Re, \Re)$, we have $\chi^{-1}(a)(x) = \int_0^x dz\, \exp[-\int_0^z dy\, a(y)]$.*

Another duality characterizes risk averse utilities. Let $\mathcal{U}^{1d}_a$ be the subset of $\mathcal{U}^{1d}$ consisting of functions u such that $D^2u \leq 0$. Let $\mathcal{R}^1_+$ be the set of nonnegative-valued functions in $\mathcal{R}^1$.

Corollary 6.5 *$\chi : \mathcal{U}^{1d}_a \to \mathcal{R}^1_+$ is a bijection.*

In the above results, we have implicitly set $O = X = \Re$. Clearly, analogous results can be derived for proper subsets of $X = \Re$, e.g., $O = \Re_+$.

The vector outcomes case

We now consider the above duality problem in the setting of general Euclidean outcome spaces. While the generalized theory preserves many aspects of the elementary theory, technical considerations force significant deviations that we explain as we develop our formulation of the theory.

A version of Theorem 6.3 when $X = \Re^n$ requires the definition of an economically useful generalized Arrow-Pratt function in this setting. Towards this end, fix a compact and convex outcome space $O \subset \Re^n$ with boundary ∂O, interior $\operatorname{Int} O$ and $0 \in O$. For $u : O \to \Re$, we define the (generalized) Arrow-Pratt function $\Gamma_1(u) : O \to \Re^n$ by

$$\Gamma_1(u)(x) = \frac{-D^2u(x)Du(x)}{\|Du(x)\|^2}$$

if u is twice differentiable at x and $\|Du(x)\| > 0$, and $\Gamma_1(u)(x) = 0$ elsewhere. The restriction of $u : O \to \Re$ to ∂O is denoted by

$$\Gamma_2(u) = u_{\partial O}$$

We interpret $\Gamma_1(u)(x)$ as the Arrow-Pratt coefficient of u at x because (a) $\Gamma_1(u)(x) \in \Re^n$ reduces to the scalar Arrow-Pratt coefficient $\chi(u)(x)$ when $n = 1$, and (b) this definition has a compelling economic interpretation as it yields the same partial ordering of risk averse utility functions in the vector outcomes case as other definitions in terms of acceptance sets, risk premia and concave transformations (Shah [26], Theorems 4.5 and 5.5).

Our objective is to specify a set $\mathcal{U}^{nd}$ of utility functions $u : O \to \Re$, a set $\mathcal{R}^n$ of Arrow-Pratt functions $a : O \to \Re^n$ and a set $\mathcal{G}$ of functions $g : \partial O \to \Re$ such that (a) for every $u \in \mathcal{U}^{nd}$, $\Gamma(u) = (\Gamma_1(u), \Gamma_2(u)) \in \mathcal{R}^n \times \mathcal{G}$, (b) $\Gamma : \mathcal{U}^{nd} \to \mathcal{R}^n \times \mathcal{G}$ is injective, and (c) for every $(a, g) \in \mathcal{R}^n \times \mathcal{G}$, there exists $u \in \mathcal{U}^{nd}$ such that $\Gamma_1(u) = a$ and $\Gamma_2(u) \in [g]$. Some observations about this set-up are in order.

Given $a \in \mathcal{R}^n$, the system of PDEs $a = \Gamma_1(u)$ cannot have a unique solution $u \in \mathcal{U}^{nd}$. An obvious problem is that $\Gamma_1(u) = \Gamma_1(v)$ for every $v \in [u]$. We overcome this technical problem by defining $\mathcal{U}^{nd}$ such that it contains no more than one vN-M representation of a given preference. A less obvious problem noted in Corollary 6.14 is that representations in $\mathcal{U}^{nd}$ of distinct preferences can generate the same Arrow-Pratt function. As shown in Theorem 6.13, this problem is overcome by formulating the duality result in terms of Γ instead of Γ_1.

The key element of our problem is to show that, given an appropriate Arrow-Pratt function $a : O \to \Re^n$ and appropriate boundary data $g : \partial O \to \Re$, there exists a unique utility $u : O \to \Re$ such that

$$\Gamma_1(u) = a \qquad \text{and} \qquad \Gamma_2(u) = g \tag{6}$$

$\Gamma_1(u) = a$ is equivalent to the system of PDEs $a(.) = -D \ln \|Du(.)\|$. Therefore, if $a = Df$ for some $f : O \to \Re$, then a solution of the Dirichlet problem

$$\|Du(.)\| = e^{-f(.)} \qquad \text{and} \qquad u_{\partial O} = g \tag{7}$$

is a solution of (6) and a solution of (6) is unique only if (7) does not have multiple solutions.

The fortunate aspect of (7) is that the PDE belongs to the widely studied class of eikonal PDEs.[5] The unfortunate aspect of (7) is that it is ill-posed in the classical setting; as simple examples bear out, it does not generally admit differentiable solutions. So, our problem requires a weaker solution concept in two respects. First, we need a larger class of solutions than the class of differentiable functions. Secondly, we need a weaker notion of "solving the PDE" than the classical one of satisfying the PDE everywhere on $\operatorname{Int} O$.

Of the many weaker solution concepts, we shall consider the notion of a "generalized solution" proposed in Krŭzkov [18] and the notion of a "viscosity solution" proposed in Crandall and Lions [4]. While the former notion is considered directly because of the ease of exposition, the latter is considered only indirectly *via* well-known connections between the two solution concepts. We will note in Theorem 6.12 that the two concepts yield identical solutions of (7) in the context of our problem.

We start our analysis of problems (6) and (7) with some definitions. Let L (resp. L_{loc}) be the set of Lipschitz (resp. locally Lipschitz) continuous real-valued functions on O. Let $\mathcal{C}^{2,\alpha}$ denote the set of real-valued functions on O whose derivatives up to second order are α-Hölder continuous for some $\alpha \in (0, 1)$.

$u : O \to \Re$ is said to be semiconcave if there exists $C \geq 0$ such that the function $x \mapsto u(x) - C\|x\|^2/2$ is concave on O; u is called semiconvex if $-u$ is semiconcave. Clearly, a concave u is semiconcave. We note the following facts for future reference.

[5] Apart from pure mathematics, Hamilton-Jacobi PDEs generally and eikonal PDEs specifically are studied as model problems in computation theory, as characterizations of value functions in control theory (Bardi and Capuzzo-Dolcetta [2], Fleming and Soner [9]), and are widely studied in physics, especially in the area of geometric optics (Luneburg [21]). To the best of our knowledge, this is their first sighting in economics.

Remark 6.6 *Consider a semiconcave function* $u : O \to \Re$.

(A) As O *is convex, a generalization of the classical Alexandrov's theorem (Fleming and Soner [9]) implies that* u *is twice differentiable a.e.*[6] *Thus,* u *determines* $\Gamma_1(u)$ *a.e.*

(B) By Proposition 4.6 in Chap. II of Bardi and Capuzzo-Dolcetta [2] (henceforth, B-C [2]), $u \in L_{\text{loc}}$.

(C) There exists a risk averse preference $[v]$ *such that, for every* $\epsilon > 0$*, there exist* $u' \in [u]$ *and* $v' \in [v]$ *such that* $\sup\{|u'(x) - v'(x)| \mid x \in O\} < \epsilon$*. Thus, given appropriate representations, a preference with semiconcave representations is arbitrarily and uniformly close to a risk averse preference.*

By definition, there is some $C \geq 0$ *such that* $v(.) \equiv u(.) - C\|.\|^2/2$ *is concave, i.e.,* $[v]$ *is risk averse. If* $C = 0$*, the result is obvious. Suppose* $C > 0$*. For every* $b > 0$*,* $bv \in [v]$ *and* $bu \in [u]$*. Picking* $b > 0$ *such that* $bC \sup\{\|x\|^2/2 \mid x \in O\} < \epsilon$*, we have* $\sup\{|bu(x) - bv(x)| \mid x \in O\} < \epsilon$*.*

A generalized solution of (7) is a function $u \in L_{\text{loc}}$ such that

$$\|Du(.)\| = e^{-f(.)} \text{ a.e.} \qquad \text{and} \qquad u_{\partial O} = g \tag{8}$$

L_{loc} is an appropriate class of potential solutions of (8) as Rademacher's theorem (Ziemer [29]) implies that $u \in L_{\text{loc}}$ is differentiable a.e. With this weaker solution concept, (6) is weakened to

$$\Gamma_1(u) = a \text{ a.e.} \qquad \text{and} \qquad \Gamma_2(u) = g \tag{9}$$

We now specify the set of vN-M utility functions.

Definition 6.7 $\mathcal{U}^{nd}$ *consists of semiconcave functions* $u \in L$ *such that*

(a) $u(0) = 0$*, and*

(b) $\|Du(.)\| > 0$ *a.e. and* $\|Du(.)\|$ *has an extension* $e^{-f(.)} \in \mathcal{C}^{2,\alpha}$ *for some* $f : O \to \Re$ *with* $f(0) = 0$*.*

Conditions (a) and (b) echo analogous restrictions in the scalar case considered in Definition 6.1. Consider the preference $[v]$ represented by the vN-M utility $v : O \to \Re$ that satisfies all the properties listed in Definition 6.7 other than the auxiliary conditions $v(0) = 0$ and $f(0) = 0$. It is easy to check that $u \in \mathcal{U}^{nd} \cap [v]$ where $u(x) = [v(x) - v(0)]e^{f(0)}$. Thus, the auxiliary conditions $u(0) = 0$ and $f(0) = 0$ do not restrict the set of preferences represented by elements of $\mathcal{U}^{nd}$. Also, if $u \in \mathcal{U}^{nd}$, then $[u] \cap \mathcal{U}^{nd} = \{u\}$, i.e., $\mathcal{U}^{nd}$ contains only one vN-M representation of an admissible preference $[u]$. Next, we specify the set of admissible generalized Arrow-Pratt functions.

Definition 6.8 $\mathcal{R}^n$ *consists of functions* $a : O \to \Re^n$ *with the representation* $a = Df$ *a.e. for some* $f : O \to \Re$ *such that* $e^{-f} \in \mathcal{C}^{2,\alpha}$ *and* $f(0) = 0$*.*

[6] "P holds a.e." means that property P holds everywhere on O with the exception of a subset of Lebesgue measure zero.

Finally, we have the functions that determine $\Gamma_2(u) = u_{\partial O}$ in (6) to (9).

Definition 6.9 *$\mathcal{G}$ consists of functions $g : \partial O \to \Re$ where $g = G_{\partial O}$ for some $G \in L$.*

We note the following facts about $\mathcal{G}$.

Remark 6.10 *(A) As $\mathcal{U}^{nd} \subset L$, we have $\{u_{\partial O} \mid u \in \mathcal{U}^{nd}\} \subset \mathcal{G}$.*

(B) $\mathcal{G}$ is identical to the set of functions $g : \partial O \to \Re$ that are Lipschitz continuous on ∂O. Clearly, every $g \in \mathcal{G}$ is Lipschitz continuous on ∂O. Conversely, by Kirszbraun's theorem (Schwartz [25]), if $g : \partial O \to \Re$ is Lipschitz continuous on ∂O, then it has an extension $G \in L$, and so $g \in \mathcal{G}$.

(C) If $G \in L$, then Rademacher's theorem (Ziemer [29]) implies that it is differentiable a.e.

(D) Suppose $G \in L$ has Lipschitz constant $k \geq 0$ and is differentiable at $x \in O$. If $\|DG(x)\| = 0$, then $\|DG(x)\| \leq k$. If $\|DG(x)\| > 0$, then for every $c > 0$ such that $x + cDG(x) \in O$, we have

$$\begin{aligned} \|DG(x)\| &\leq \frac{|G(x + cDG(x)) - G(x)|}{\|cDG(x)\|} + |R(x, cDG(x))| \\ &\leq k + |R(x, cDG(x))| \end{aligned}$$

where $\lim_{h \to 0} R(x, h) = 0$. Letting $c \downarrow 0$, we have $\|DG(x)\| \leq k$.

(A) means that the condition $\Gamma_2(u) \in \mathcal{G}$ does not constitute an additional restriction on $\mathcal{U}^{nd}$. The following lemma is the key to our duality result.

Lemma 6.11 *Consider $f : O \to \Re$ such that $f(0) = 0$ and $e^{-f} \in \mathcal{C}^{2,\alpha}$ and let $g = G_{\partial O}$ for some $G \in L$.*

(A) If $u : O \to \Re$ and $v : O \to \Re$ are semiconcave and solve (8), then $u = v$.

(B) If G has Lipschitz constant $k \leq \inf\{e^{-f(y)} \mid y \in O\}$, then (8) has a semiconcave solution $u : O \to \Re$.

Proof. (A) Suppose u and v are semiconcave and solve (8). Remark 6.6(B) implies that $u, v \in L_{\text{loc}}$. Therefore, $-u, -v \in L_{\text{loc}}$ are semiconvex and satisfy the conditions $-u_{\partial O} = -g = -v_{\partial O}$ and $\| - Du(.)\| = e^{-f(.)} = \| - Dv(.)\|$ a.e. Theorem 2.1 in Krŭzkov [18] implies that $-u = -v$, and so, $u = v$.

(B) Clearly, $-G \in L$ with Lipschitz constant k and $-g = -G_{\partial O}$. By Remark 6.10(C), $-G$ is differentiable a.e. Therefore, using Remark 6.10(D), $\| - DG(.)\| \leq k \leq \inf\{e^{-f(y)} \mid y \in O\} \leq e^{-f(.)}$ a.e. Given this estimate, Theorem 3.1 in Krŭzkov [18] implies the existence of a semiconvex function $v \in L_{\text{loc}}$ such that $v_{\partial O} = -g$ and $\|Dv(.)\| = e^{-f(.)}$ a.e. Then, $u = -v \in L_{\text{loc}}$ is semiconcave and solves (8). ■

Before proving the main duality result, we note some useful connections between generalized solutions and viscosity solutions; as the relevant definitions and motivation require considerable space, we refer the reader to B-C [2] for a lucid account of viscosity theory.[7]

Theorem 6.12 *Let $f : O \to \Re$ be continuous and $u : O \to \Re$ semiconcave.*

(A) If u is a generalized solution of (7), then $u \in L$ and u is the unique viscosity solution of (7).

(B) If u is a viscosity solution of (7), then $u \in L$ and u is the unique generalized solution of (7).

Proof. Define $H : O \times \Re^n \to \Re$ by $H(x,p) = \|p\|^2 - e^{-2f(x)}$. The PDE in (7) is equivalent to $H(., Du(.)) = 0$.

(A) Suppose u is a generalized solution of (7). By Corollary 5.2 in Chap. II of B-C [2], u is a viscosity solution of $H(., Du(.)) = 0$ in O. By assumption, $u_{\partial O} = g$. Thus, u is a viscosity solution of (7). As f is continuous and O is compact, $\sup\{e^{-2f(y)} \mid y \in O\} \in \Re$ and $H(x,p) \geq \|p\|^2 - \sup\{e^{-2f(y)} \mid y \in O\}$ for every $x \in O$. Then, $H(x,p) \to +\infty$ uniformly in $x \in O$ as $\|p\| \to +\infty$. Given this coercivity property of H, Proposition 4.1 in Chap. II of B-C [2] implies that $u \in L$. Using the facts that O is compact and H is convex in p, a comparison principle (B-C [2], Theorem 5.9 in Chap. II) implies that u is the only viscosity solution of (7).

(B) Suppose u is a viscosity solution of (7). By the arguments made above, $u \in L$ and u is the unique viscosity solution of (7). By Proposition 1.9 in Chap. II of B-C [2], $H(., Du(.)) = 0$ a.e. Thus, u is a generalized solution of (7). Uniqueness follows from (A). ■

We finally use Lemma 6.11 and Theorem 6.12 to prove the following duality theorem.

Theorem 6.13 *If $u \in \mathcal{U}^{nd}$, then $\Gamma(u) \in \mathcal{R}^n \times \mathcal{G}$.*

(A) $\Gamma : \mathcal{U}^{nd} \to \mathcal{R}^n \times \mathcal{G}$ is injective.

(B) For every $(a, g) \in \mathcal{R}^n \times \mathcal{G}$, there exists $u \in \mathcal{U}^{nd}$ such that $\Gamma_1(u) = a$ and $\Gamma_2(u) \in [g]$.

Proof. Consider $u \in \mathcal{U}^{nd}$. Then $\|Du(.)\|$ has an extension $e^{-f(.)} \in \mathcal{C}^{2,\alpha}$ for some $f : O \to \Re$ such that $f(0) = 0$. Therefore, $\Gamma_1(u)(.) = -D \ln \|Du(.)\| = Df(.)$ a.e. Thus, $\Gamma_1(u) \in \mathcal{R}^n$, and as $u \in L$, we have $\Gamma_2(u) = u_{\partial O} \in \mathcal{G}$.

(A) Suppose $u, v \in \mathcal{U}^{nd}$ are such that $\Gamma_1(u) = a = \Gamma_1(v)$ a.e. and $\Gamma_2(u) = g = \Gamma_2(v)$ for some $(a, g) \in \mathcal{R}^n \times \mathcal{G}$. So, $\|Du(.)\|$ and $\|Dv(.)\|$ have $\mathcal{C}^{2,\alpha}$ extensions $e^{-f^1(.)}$ and $e^{-f^2(.)}$ respectively, where $f^1 : O \to \Re$ and $f^2 : O \to \Re$ are such that $f^1(0) = 0 = f^2(0)$ and

$$Df^1(.) = -D \ln \|Du(.)\| = \Gamma_1(u) = \Gamma_1(v) = -D \ln \|Dv(.)\| = Df^2(.)$$

[7] The wide scope of applicability of viscosity solutions in the context of Hamilton-Jacobi PDEs can be gauged from the fact that this solution concept expands the class of potential solutions of such equations to all continuous functions.

a.e. It follows that $f^1 = f^2 + c$ for some $c \in \Re$. As $f^1(0) = 0 = f^2(0)$, we have $c = 0$. Thus, $f^1 = f^2$. This means u and v solve (8) with $f = f^1 = f^2$. Lemma 6.11(A) implies that $u = v$.

(B) Consider $(a, g) \in \mathcal{R}^n \times \mathcal{G}$. As $a \in \mathcal{R}^n$, there exists $f : O \to \Re$ such that $e^{-f} \in \mathcal{C}^{2,\alpha}$, $f(0) = 0$ and $a = Df$ a.e. As $g \in \mathcal{G}$, there exists $G \in L$ with Lipschitz constant $k \geq 0$ such that $g = G_{\partial O}$. Since $\inf\{e^{-f(y)} \mid y \in O\} > 0$, there exists $b > 0$ such that $k/b \leq \inf\{e^{-f(y)} \mid y \in O\}$.

Note that $G/b \in L$ with Lipschitz constant k/b and $g/b = (G/b)_{\partial O}$. Lemma 6.11(B) implies that there is a semiconcave function $v \in L_{\text{loc}}$ such that $v_{\partial O} = g/b$ and $\|Dv(.)\| = e^{-f(.)} > 0$ a.e. By Theorem 6.12(A), $v \in L$. Clearly, $e^{-f(.)}$ is a $\mathcal{C}^{2,\alpha}$ extension of $\|Dv(.)\|$. Thus, $\Gamma_2(v) = g/b \in [g]$ and $\Gamma_1(v)(.) = -D \ln \|Dv(.)\| = Df(.) = a(.)$ a.e. Set $u = v - v(0)$. Then, $u \in \mathcal{U}^{nd}$, $\Gamma_1(u) = \Gamma_1(v) = a$ and $\Gamma_2(u) = \Gamma_2(v) - v(0) \in [g]$. ■

An obvious implication of this result is that two distinct preferences can generate the same Arrow-Pratt function.

Corollary 6.14 *$\Gamma_1 : \mathcal{U}^{nd} \to \mathcal{R}^n$ is not injective.*

Proof. Consider $a \in \mathcal{R}^n$ and $g, g' \in \mathcal{G}$ such that $g' \notin [g]$. By Theorem 6.13(B), there exist $u, v \in \mathcal{U}^{nd}$ such that $\Gamma_1(u) = a = \Gamma_1(v)$ a.e., $\Gamma_2(u) \in [g]$ and $\Gamma_2(v) \in [g']$. If $v \in [u]$, then $v_{\partial O} \in [u_{\partial O}]$. So, $g' \in [g]$, a contradiction. ■

7 Applications

Duality theory has two competing aspects. On the one hand, when choosing a dual representation such as $A \in \mathcal{A}$ to specify a preference, we want the definition of $\mathcal{A}$ to be minimal and easily verifiable. On the other hand, when using a dual representation A, we are free to use not only the properties defining the elements of $\mathcal{A}$, but also other properties implied by the definition of $\mathcal{A}$. So, an important aspect of duality theory is to derive various non-definitional properties possessed by dual representations. Our first application of the above duality results will derive such non-definitional properties of the elements of $\mathcal{A}$ and $\mathcal{F}_a$.

Theorem 7.1 *If $u \in \mathcal{U}$, then*
(A) $\xi(u)$ is continuous, and
(B) if $u \in \mathcal{U}_a$, then $\phi(u)$ is continuous.
Moreover, every $A \in \mathcal{A}$ and $F \in \mathcal{F}_a$ is continuous.

Combining this result with Berge's Maximum theorem yields the following facts that will be used below.

Theorem 7.2 *If $P : O \to \Re$ is continuous and $F \in \mathcal{F}_a$, then the mapping $V : \Delta(O) \to \Re$ defined by $V(\mu) = \min\{P(x) \mid x \in F(\mu)\}$ is continuous and* *the mapping $M : \Delta(O) \Rightarrow O$ defined by $M(\mu) = \{x \in F(\mu) \mid P(x) = V(\mu)\}$ is upper hemicontinuous with nonempty and compact values.*

A dual result is the following.

Theorem 7.3 *If $p : O \to \Re$ is continuous and $A \in \mathcal{A}$, then $v : O \to \Re$ defined by $v(x) = \min\{\int_O \mu(dz)\, p(z) \mid \mu \in A(x)\}$ is continuous; moreover, $B : O \Rightarrow \Delta(O)$ defined by $B(x) = \{\mu \in A(x) \mid \int_O \mu(dz)\, p(z) = v(x)\}$ is upper hemicontinuous with nonempty and compact values.*

As an application of these results, consider the following problem.

Application 7.4 *Let $\{1, \ldots, n\}$ be the set of future dates. Let $X = \Re^n$ and let O satisfy the requirements stated in Section 2. $x \in O$ is interpreted as an asset dividend path, with x_t being the dividend paid at date $t \in \{1, \ldots, n\}$. An asset is denoted by $\mu \in \Delta(O)$. Asset μ is said to be riskless if $\mu = \delta_x$ for some $x \in O$, and risky otherwise. Let asset prices be given by $P : \Delta(O) \to \Re$, where $P(\mu)$ is the price of asset μ and P is continuous when $\Delta(O)$ is given the weak* topology. A portfolio of assets is a function $\theta : \Delta(O) \to \Re$ with finite support, i.e., $\operatorname{supp}\theta \equiv \overline{\theta^{-1}(\Re - \{0\})}$ is finite. Suppose P is arbitrage-free, meaning that there is no portfolio θ of assets such that $\sum_{\mu \in \operatorname{supp}\theta} \theta(\mu)P(\mu) < 0$ and $\sum_{\mu \in \operatorname{supp}\theta} \theta(\mu) \int_O \mu(dz)\, z \geq 0$, i.e., a portfolio with a negative acquisition cost and non-negative expected dividends. Given this set-up, what is the value to a risk averse investor of asset $\mu \in \Delta(O)$? How does this value vary with μ?*

If P permits an arbitrage in the above sense, then a risk neutral investor would like to acquire an unboundedly large portfolio. Assuming the existence of a risk neutral investor, the above notion of "arbitrage-free" asset prices is a necessary property of equilibrium prices. The functional $\pi : O \to \Re$, defined by $\pi(x) = P(\delta_x)$, yields the prices of riskless assets. As O is separable, $\Delta^0(O)$ is dense in $\Delta(O)$ (Parthasarathy [22], Theorem II.6.3). We note some elementary facts about P and π.

Lemma 7.5 *Consider Application 7.4.*
(A) π is linear on O, $\pi(0) = 0$ and $\pi(x) \geq 0$ for every $x \in O$.
(B) π is continuous.
(C) If every unit vector $e_t \in O$, then there exists $(\pi_1, \ldots, \pi_n) \in \Re^n_+$ such that $\pi(x) = \sum_{t=1}^n \pi_t x_t$.
(D) For every $\mu \in \Delta(O)$, $P(\mu) = \pi(m_\mu)$.

Proof. (A) If π is not linear on O, then there exist $x, y \in O$ and $\alpha, \beta \in \Re$ such that $\alpha x + \beta y \in O$ and $\pi(\alpha x + \beta y) < \alpha\pi(x) + \beta\pi(y)$; an analogous argument holds if $\pi(\alpha x + \beta y) > \alpha\pi(x) + \beta\pi(y)$. Consider the portfolio $\theta = 1_{\delta_{\alpha x + \beta y}} - \alpha 1_{\delta_x} - \beta 1_{\delta_y}$. Then, $\sum_{\mu \in \operatorname{supp}\theta} \theta(\mu)P(\mu) = P(\delta_{\alpha x+\beta y}) - \alpha P(\delta_x) -$

$\beta P(\delta_y) = \pi(\alpha x + \beta y) - \alpha\pi(x) - \beta\pi(y) < 0$ and $\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu) \int_O \mu(dz)\, z = \alpha x + \beta y - \alpha x - \beta y = 0$, which violates the no-arbitrage condition.

$\pi(0) = 0$ follows from the linearity of π.

Suppose $\pi(x) < 0$ for some $x \in O$. Consider the portfolio $\theta = 1_{\delta_x}$. Then, $\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu)P(\mu) = P(\delta_x) = \pi(x) < 0$ and $\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu) \int_X \mu(dz)\, z = x \geq 0$, which violates the no-arbitrage condition.

(B) follows as π is linear.

(C) follows by setting $\pi_t = \pi(e_t)$.

(D) Consider $\mu \in \Delta^0(O)$. If $P(\mu) > \sum_{z\in\operatorname{supp}\mu} \mu(\{z\})\pi(z)$, then consider the portfolio $\theta = \sum_{z\in\operatorname{supp}\mu} \mu(\{z\})1_{\delta_z} - 1_\mu$. Given portfolio θ, we have $\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu)P(\mu) < 0$ and $\sum_{\mu\in\operatorname{supp}\theta} \theta(\mu) \int_O \mu(dz)\, z = 0$, which contradicts the arbitrage-free property. An analogous argument holds if $P(\mu) < \sum_{z\in\operatorname{supp}\mu} \mu(\{z\})\pi(z)$. Thus, $P(\mu) = \sum_{z\in\operatorname{supp}\mu} \mu(\{z\})\pi(z) = \int_O \mu(dz)\, \pi(z)$ for every $\mu \in \Delta^0(O)$.

Consider $\mu \in \Delta(O)$. As $\Delta^0(O)$ is dense in $\Delta(O)$, there exists a net $(\mu_n) \subset \Delta^0(O)$ converging to μ. As P and π are continuous, we have $P(\mu) = \lim_n P(\mu_n) = \lim_n \int_O \mu_n(dz)\, \pi(z) = \int_O \mu(dz)\, \pi(z)$. As π is linear, we have $\int_O \mu(dz)\, \pi(z) = \pi(\int_O \mu(dz)\, z) = \pi(m_\mu)$. Thus, $P(\mu) = \pi(m_\mu)$. ■

In this result, π_t is the price of an asset that delivers \$1 with certainty at date t and delivers nothing at other dates.

We define an asset's value to an investor as the maximum amount that the investor would be willing to pay for it. By Corollaries 3.7 and 5.6, a risk averse investor's preference on $\Delta(O)$ can be specified equivalently by $u \in \mathcal{U}_a$, $F \equiv \phi(u) \in \mathcal{F}_a$, or $A \equiv \xi(u) \in \mathcal{A}_a$. If the preference is represented by $u \in \mathcal{U}_a$, then the value of asset $\mu \in \Delta(O)$ to the given investor is $V(\mu) = \min\{P(\lambda) \mid \lambda \in \Delta(O) \ \wedge\ U(\lambda) \geq U(\mu)\}$. The next result provides dual characterizations of V and notes some properties.

Theorem 7.6 *Consider Application 7.4. Let $u \in \mathcal{U}_a$, $F \equiv \phi(u) \in \mathcal{F}_a$ and $A \equiv \xi(u) \in \mathcal{A}_a$.*

(A) If $\mu \in \Delta(O)$, then $V(\mu) = \min \pi \circ F(\mu)$.

(B) If $x \in O$, then $V(\delta_x) = \min \pi \circ F(\delta_x) = \min P \circ A(x)$.

(C) $P(\mu) \geq V(\mu)$ for every $\mu \in \Delta(O)$.

(D) V is continuous, and the mappings $\mu \rightrightarrows \{x \in O \mid \pi(x) = V(\mu)\}$ and $x \rightrightarrows \{\lambda \in \Delta(O) \mid P(\lambda) = V(\delta_x)\}$ are upper hemicontinuous with nonempty and compact values.

Proof. (A) If $x \in F(\mu)$, then $\delta_x \in \Delta(O)$ and $U(\delta_x) = u(x) = U(\mu)$. By definition, $V(\mu) \leq P(\delta_x) = \pi(x)$. It follows that $V(\mu) \leq \min \pi \circ F(\mu)$. Let $V(\mu) = P(\lambda)$ for some $\lambda \in \Delta(O)$ such that $U(\lambda) \geq U(\mu)$. As u is risk averse, $u(m_\lambda) \geq U(\lambda) \geq U(\mu) \geq 0$. Consequently, there exists $t \in [0, 1]$ such that $tm_\lambda \in F(\mu)$. As $m_\lambda \in O$, Lemma 7.5(A) implies that $\pi(m_\lambda) \geq 0$ and $\pi(tm_\lambda) = t\pi(m_\lambda) \leq \pi(m_\lambda)$. Using Lemma 7.5(D), we have $\min \pi \circ F(\mu) \leq \pi(tm_\lambda) \leq \pi(m_\lambda) = P(\lambda) = V(\mu)$. Thus, $V(\mu) = \min \pi \circ F(\mu)$.

(B) Specializing (A), we have $\min \pi \circ F(\delta_x) = V(\delta_x) = \min\{P(\lambda) \mid \lambda \in \Delta(O) \ \wedge \ U(\lambda) \geq U(\delta_x)\} = \min P \circ A(x)$.

(C) Consider $\mu \in \Delta(O)$. As u is risk averse, $u(m_\mu) \geq U(\mu)$. So, there exists $t \in [0,1]$ such that $u(tm_\mu) = U(\mu)$, i.e., $tm_\mu \in F(\mu)$. If $P(\mu) < V(\mu)$, then using Lemma 7.5 and (A), we have $\pi(m_\mu) = P(\mu) < V(\mu) = \min \pi \circ F(\mu) \leq \pi(tm_\mu) = t\pi(m_\mu) \leq \pi(m_\mu)$, a contradiction.

(D) follows from (A), (B) and Theorems 7.2 and 7.3. ■

Now consider the following continuous-time analogue of Application 7.4. Let $[0,1]$ be the set of dates and let $X = \mathcal{C}([0,1], \Re)$ be the set of continuous real-valued functions with domain $[0,1]$. Let P and π be as in Application 7.4. Parts (A) and (D) of Lemma 7.5 hold in this setting *via* unchanged arguments.

Lemma 7.5(B) is now proved as follows. Consider a sequence $(x_n) \subset O$ converging to x. If $f : O \to \Re$ is continuous, then $\int_O \delta_{x_n}(dz)\, f(z) = f(x_n) \to f(x) = \int_O \delta_x(dz)\, f(z)$. So, $(\delta_{x_n}) \subset \Delta(O)$ converges to δ_x. As P is continuous, $\pi(x_n) = P(\delta_{x_n}) \to P(\delta_x) = \pi(x)$. Thus, π is continuous.

The analogue of Lemma 7.5(C) is established as follows. Suppose π has a continuous linear extension to X. By the Riesz representation theorem (Dunford and Schwartz [8], Theorem IV.6.3), there exists a unique, non-negative, regular countably-additive measure Q on $[0,1]$ such that $\pi(x) = \int_{[0,1]} Q(dt)\, x(t)$ for every $x \in X$. As π is real-valued, Q is finite. If Q is absolutely continuous with respect to the Lebesgue measure on $[0,1]$, then by the Radon-Nikodym theorem (Dunford and Schwartz [8], Theorem III.10.2), there exists a unique (upto equivalence) Lebesgue integrable function $q : [0,1] \to \Re$ such that $Q(E) = \int_E dt\, q(t)$ for every $E \in \mathcal{B}([0,1])$. Therefore, $\pi(x) = \int_{[0,1]} dt\, q(t)x(t)$ for every $x \in X$. As Q is non-negative, q is non-negative on $[0,1]$, except possibly over a set of Lebesgue measure 0. Just like π_t in Lemma 7.5, $q(t)$ is interpreted as the price of an asset that delivers \$1 at time t and nothing at all other times.

8 Extensions

The results of Shah [26] complement the results in this paper by providing criteria for comparing the risk aversion of preferences directly in terms of the dual representations derived above. For example, consider the risk averse preferences represented by $A_1, A_2 \in \mathcal{A}_a$. A natural definition of the relation "the preference represented by A_1 is more risk averse than that represented by A_2" is that $A_1(x) \subset A_2(x)$ for every $x \in O$. This, however, raises the question: is this definition consistent with various familiar vN-M utility-based notions of comparative risk aversion (e.g., Arrow [1], Pratt [24], Yaari [28])? As we show below, this is indeed so once the classical criteria for real outcomes are modified to account for vector outcomes in Shah [26].

Corollary 5.6 implies that A_1 and A_2 represent the same risk averse preferences as utilities $\xi^{-1}(A_1) \in \mathcal{U}_a$ and $\xi^{-1}(A_2) \in \mathcal{U}_a$ respectively. Suppose $\xi \circ \xi^{-1}(A_1)(x) = A_1(x) \subset A_2(x) = \xi \circ \xi^{-1}(A_2)(x)$ for every $x \in O$. Then,

(a) $\xi^{-1}(A_1)$ and $\xi^{-1}(A_2)$ are comonotonic (Shah [26], Theorem 4.2);

(b) $\psi \circ \phi \circ \xi^{-1}(A_1)(\mu) \geq^* \psi \circ \phi \circ \xi^{-1}(A_2)(\mu)$ for every $\mu \in \Delta(O)$ (Shah [26], Theorem 4.5), i.e., the set of risk premia dual to A_1 is always larger in terms of $\geq^*$ than the set of risk premia dual to A_2;

(c) $\xi^{-1}(A_1) = f \circ \xi^{-1}(A_2)$ for an increasing and concave function $f : \xi^{-1}(A_2)(O) \to \Re$ (Shah [26], Theorem 4.5), i.e., the vN-M utility dual to A_1 is an increasing concave transformation of the vN-M utility dual to A_2; and

(d) if $(X, \geq)$ is an ordered real Hilbert space, and $\xi^{-1}(A_1)$ and $\xi^{-1}(A_2)$ are twice differentiable on $\operatorname{Int} O$, then $\Gamma_1 \circ \xi^{-1}(A_1)(x) \geq \Gamma_1 \circ \xi^{-1}(A_2)(x)$ for every $x \in \operatorname{Int} O$ (Shah [26], Theorem 5.5), i.e., the Arrow-Pratt function generated by $\xi^{-1}(A_1)$ is always larger than the Arrow-Pratt function generated by $\xi^{-1}(A_2)$. We should clarify that, although Theorem 6.13 restricts attention to generalized Arrow-Pratt functions with Euclidean domains for the purpose of deriving a duality result, the definition in Shah [26] of a generalized Arrow-Pratt function associated with a utility u, stated here as $\Gamma_1(u)$, applies generally to domains in ordered Hilbert spaces; for an exact description of $(X, \geq)$, we refer the reader to Assumption 5.1 in Shah [26].

Analogous results hold if we start with $P_1, P_2 \in \mathcal{P}_a$ such that $P_1(\mu) \geq^* P_2(\mu)$ for every $\mu \in \Delta(O)$.

9 Ordinal and cardinal representation problems

Our dual characterizations of vN-M utilities and the more familiar dual characterizations of ordinal utilities are entirely different in aims, techniques and the objects being studied. However, the set $\mathcal{U}$ remains a seemingly common element since a function $u \in \mathcal{U}$ can be interpreted as an ordinal utility as well as a vN-M utility. We clarify the distinction between the theories with two observations.

The first observation relates to the quotient sets of $\mathcal{U}$ generated by the ordinal and the vN-M interpretations of the elements of $\mathcal{U}$.[8] If the elements of $\mathcal{U}$ are interpreted as ordinal utility functions, then elements $u, v \in \mathcal{U}$ are considered to be equivalent, denoted by $u \equiv^1 v$, if they are increasing transforms of each other. This notion of equivalence generates the quotient set $\mathcal{U}/\equiv^1$. On the other hand, if the elements of $\mathcal{U}$ are interpreted as vN-M utility functions, then elements $u, v \in \mathcal{U}$ are considered to be equivalent, denoted by $u \equiv^2 v$, if they are increasing affine transforms of each other. This notion of equivalence generates the quotient set $\mathcal{U}/\equiv^2$. It is easy to see that $\mathcal{U}/\equiv^2$ is a sub-partition of $\mathcal{U}/\equiv^1$, i.e., if $[u]_1 \in \mathcal{U}/\equiv^1$ and

[8]The quotient set of a set S with respect to an equivalence relation $\equiv$ on S, denoted by $S/\equiv$, refers to the partition of S generated by $\equiv$.

$[u]_2 \in \mathcal{U}/\equiv^2$ are the equivalence classes to which $u \in \mathcal{U}$ belongs, then $[u]_2 \subset [u]_1$ and $[u]_2 \neq [u]_1$.

The second observation relates to the representation problems underlying the two theories. Let $\succeq^*$ be a complete preordering on $\Delta(O)$ and let $\succeq$ be induced on O *via* the definition: $x \succeq y$ if and only if $\delta_x \succeq^* \delta_y$; let $\sim$ be the symmetric factor (indifference) of $\succeq$ and $\succ$ the asymmetric factor (strict preference). Define the function $o : O \to O/\sim$ by the formula: given $x \in O$, $z \in o(x)$ if and only if $z \sim x$; $o(x) \in O/\sim$ is the indifference curve containing x. If $o_1, o_2 \in O/\sim$ and $o_1 \neq o_2$, then either $x \succ y$ for all $x \in o_1$ and $y \in o_2$, or $y \succ x$ for all $x \in o_1$ and $y \in o_2$. Therefore, by identifying equivalent elements of O with the equivalence class to which they belong, we may say that $\succ$ orders the elements of $O/\sim$. The ordinal representation problem is to find $u : O/\sim \to \Re$ such that, for all $o_1, o_2 \in O/\sim$, $o_1 \succ o_2$ if and only if $u(o_1) > u(o_2)$, i.e., real numbers are assigned to the indifference curves in $O/\sim$ in a manner consistent with $\succ$. Subject to this ordinal requirement on the chosen real numbers, each indifference curve may be assigned a number in isolation from the numbers assigned to the other indifference curves. The vN-M representation problem is to find $u : O/\sim \to \Re$ such that, for all $\mu, \lambda \in \Delta(O)$, $\mu \succ^* \lambda$ if and only if $\int_O \mu(dz)\, u \circ o(z) > \int_O \lambda(dz)\, u \circ o(z)$. A solution u of this problem also solves the ordinal representation problem since $x \succ y$ is equivalent to $\delta_x \succ^* \delta_y$, which is equivalent to $u \circ o(x) = \int_O \delta_x(dz)\, u \circ o(z) > \int_O \delta_y(dz)\, u \circ o(z) = u \circ o(y)$. Unlike in the ordinal representation problem, the indifference curves in $O/\sim$ cannot be assigned values in isolation as the expected utility function aggregates these numbers *via* integration. Thus, the assigned values have cardinal (up to increasing affine transformations), and not merely ordinal, significance. While the ordinal representation problem can be solved locally with respect to $O/\sim$, the vN-M representation problem has to be solved globally.

10 Conclusions

Our duality results are derived in two different environments. The first set of results concern an outcome space O that is a convex, compact and metrizable subset of the positive cone of an ordered, real locally convex topological vector space X with $0 \in O$. Given this setting, we defined a set $\mathcal{U}$ of vN-M utility functions, a set $\mathcal{F}$ of multi-valued mappings that yield the certainty equivalent outcomes in O corresponding to a lottery in $\Delta(O)$, a set $\mathcal{P}$ of multi-valued mappings that yield the risk premia in X corresponding to a lottery in $\Delta(O)$, and a set $\mathcal{A}$ of multi-valued mappings that yield the acceptance set of lotteries in $\Delta(O)$ corresponding to an outcome in O. We also define subsets $\mathcal{U}_a \subset \mathcal{U}$, $\mathcal{F}_a \subset \mathcal{F}$, $\mathcal{P}_a \subset \mathcal{P}$ and $\mathcal{A}_a \subset \mathcal{A}$, where $\mathcal{U}_a$ consists of all risk averse preferences in $\mathcal{U}$. We show that the usual definitions of the set of certainty equivalents, the set of risk premia and the acceptance set generate mappings $\phi : \mathcal{U} \to \mathcal{F}$, $\psi : \mathcal{F} \to \mathcal{P}$ and $\xi : \mathcal{U} \to \mathcal{A}$ respectively. Our main results (Theorems 3.6, 4.1 and 5.5) are that these mappings are bijective. As corollaries of these results, we show that $\phi : \mathcal{U}_a \to \mathcal{F}_a$, $\psi : \mathcal{F}_a \to \mathcal{P}_a$ and $\xi : \mathcal{U}_a \to \mathcal{A}_a$ are bijections too. These results provide very general dual representations of risk averse preferences.

The second environment that we are concerned with is more restrictive as we set X equal to the Euclidean space $\Re^n$. For $n = 1$, we define a set of utilities $\mathcal{U}^{1d}$ and a set of Arrow-Pratt functions $\mathcal{R}^1$; we also define a subset $\mathcal{U}_a^{1d}$ of risk averse utilities and a subset of Arrow-Pratt functions $\mathcal{R}_+^1$. We show in Theorem 6.3 that the usual definition of Arrow-Pratt coefficients yields a bijection $\chi : \mathcal{U}^{1d} \to \mathcal{R}^1$, and as a corollary, we have a bijection $\chi : \mathcal{U}_a^{1d} \to \mathcal{R}_+^1$. For $n > 1$, we define a set of utilities $\mathcal{U}^{nd}$, a set of generalized Arrow-Pratt functions $\mathcal{R}^n$ and a set $\mathcal{G}$ of boundary data. We show in Theorem 6.13 a bijection between $\mathcal{U}^{nd}$ and $\mathcal{R}^n \times \mathcal{G}$. There remain two potential areas for future work here. One is to extend this duality result beyond Euclidean spaces; as shown in Shah [26], our definition of a generalized Arrow-Pratt coefficient is meaningful even in the setting of Hilbert spaces. Unfortunately, the duality result requires the unique solvability of Dirichlet problems for eikonal PDEs and we are unaware whether such problems can be handled in settings more general than Euclidean spaces. A second potentially fruitful area of work is to derive a more special duality result than ours by characterizing the subset of risk averse, i.e., concave, functions in $\mathcal{U}^{nd}$; note that, functions in $\mathcal{U}^{nd}$ are required to be semiconcave, but not necessarily concave.

In Section 7, we present some illustrative applications of our results. We show in Theorem 7.1 that $\phi(u)$ and $\xi(u)$ are continuous mappings for every $u \in \mathcal{U}$. Consequently, every $F \in \mathcal{F}$ and every $A \in \mathcal{A}$ is continuous. We use these facts to derive a risk averse investor's valuation of financial assets that are characterized by known or randomly determined dividend paths. The first application derives such an investor's valuation of a risky asset and the second application derives the investor's valuation of a riskless asset. We reduce these problems to optimization problems and use our results to show that the value functions generated by these problems are continuous and the underlying optimal choice mappings are upper hemicontinuous.

In Section 8, we have shown that the risk aversion of cardinal preferences can be compared in terms of vN-M representations as well as dual representations such as the risk premia mappings and the acceptance set mappings.

Appendix

Proof of Theorem 2.1. Let X^* be the topological dual of X, i.e., the set of all continuous linear functionals $h : X \to \Re$. Local convexity of X

ensures that, if $x \in X$ is such that $h(x) = 0$ for every $h \in X^*$, then $x = 0$ (Dunford and Schwartz [8], Corollary V.2.13). Define $H : X \to \Re^{X^*}$ by $H(x) = (h(x))_{h \in X^*}$. Give $\Re^{X^*}$ the product topology. Consequently, H is continuous as every component function $H_h = h$ is continuous. Moreover, H is injective; if $H(x) = H(y)$ for some $x, y \in X$, then $h(x-y) = h(x) - h(y) = 0$ for every $h \in X^*$, which implies $x - y = 0$. As O is compact and $\Re^{X^*}$ is Hausdorff, H imbeds O in $\Re^{X^*}$. This implies $H(O)$ is closed in $\Re^{X^*}$ and metrizable.

First, consider $\mu \in \Delta(O)$ with $|\operatorname{supp}\mu| < \infty$. For every $h \in H$, the linearity of h implies

$$\int_O \mu(dz)\, h(z) = \sum_{z \in \operatorname{supp}\mu} \mu(\{z\}) h(z) = h\left(\sum_{z \in \operatorname{supp}\mu} \mu(\{z\}) z\right) \tag{A.1}$$

Setting $m_\mu = \sum_{z \in \operatorname{supp}\mu} \mu(\{z\}) z$, we have $m_\mu \in O$ as O is convex and $\operatorname{supp}\mu \subset O$. Thus, $H(m_\mu) \in H(O)$ for every $\mu \in \Delta(O)$ with $|\operatorname{supp}\mu| < \infty$.

Consider $\mu \in \Delta(O)$. As O is compact and metric, it is separable. Consequently, there exists a sequence $(\mu_n) \subset \Delta(O)$ converging to μ such that $|\operatorname{supp}\mu_n| < \infty$ for every $n \in \mathcal{N}$ (Parthasarathy [22], Theorem II.6.3). By the above argument, m_{μ_n} exists, $m_{\mu_n} \in O$ and $H(m_{\mu_n}) \in H(O)$ for every $n \in \mathcal{N}$. Using (A.1) and the definition of weak* convergence, we have

$$\lim_{n \uparrow \infty} h(m_{\mu_n}) = \lim_{n \uparrow \infty} \int_O \mu_n(dz)\, h(z) = \int_O \mu(dz)\, h(z)$$

for every $h \in X^*$. Thus, $\lim_{n \uparrow \infty} H(m_{\mu_n}) = (\int_O \mu(dz)\, h(z))_{h \in X^*}$. As the sequence $(H(m_{\mu_n})) \subset H(O)$ and $H(O)$ is closed in $\Re^{X^*}$ and metrizable, we have $(\int_O \mu(dz)\, h(z))_{h \in X^*} \in H(O)$. As H imbeds O in $\Re^{X^*}$, there exists a unique $x \in O$ such that $H(x) = (\int_O \mu(dz)\, h(z))_{h \in X^*}$. By the definition of H, we have $h(x) = \int_O \mu(dz)\, h(z)$ for every $h \in X^*$. Set $m_\mu = x$. ■

Proof of Theorem 7.1. Suppose (A) and (B) hold. Consider $A \in \mathcal{A}$ and $F \in \mathcal{F}_a$. By Theorem 5.5, $\xi^{-1}(A) \in \mathcal{U}$ and $A = \xi \circ \xi^{-1}(A)$. Therefore (A) implies that A is continuous. Similarly, by Corollary 3.7, $\phi^{-1}(F) \in \mathcal{U}_a$ and $F = \phi \circ \phi^{-1}(F)$. Thus, (B) implies that F is continuous. We now show (A) and (B).

Given $u \in \mathcal{U}$, denote the mapping $\mu \mapsto \int_O \mu(dz)\, u(z)$ by U. As u is continuous and $\Delta(O)$ is given the weak* topology, U is continuous. Therefore, $G : \Delta(O) \times O \to \Re$, defined by $G(\mu, x) = U(\mu) - u(x)$, is continuous.

(A) Consider $u \in \mathcal{U}$ and set $A \equiv \xi(u)$. As projections are continuous, the mapping $\pi : O \times \Delta(O) \to \Delta(O) \times O$, given by $\pi(x, \mu) = (\mu, x)$, is continuous. Then, $\operatorname{Gr} A = \{(x, \mu) \in O \times \Delta(O) \mid \mu \in A(x)\} = \{(x, \mu) \in O \times \Delta(O) \mid G \circ \pi(x, \mu) \geq 0\} = \pi^{-1} \circ G^{-1}(\Re_+)$ is closed in $O \times \Delta(O)$. Therefore, as $\Delta(O)$ is compact metric, A is upper hemicontinuous.

We now show that A is lower hemicontinuous at $x \in O$. As O and $\Delta(O)$ are metrizable, it is sufficient to show that, for every sequence $(x_n) \subset O$ converging to x and $\mu \in A(x)$, there exists a sequence $(\mu_n) \subset \Delta(O)$ converging to μ such that $\mu_n \in A(x_n)$ for every $n \in \mathcal{N}$. So, consider a sequence $(x_n) \subset O$ converging to $x \in O$ and let $\mu \in A(x)$. It follows that, if (x_m) is a subsequence of (x_n), then (x_m) converges to x, and as u is continuous, the subsequence $(u(x_m))$ converges to $u(x)$. As U is continuous and $\Delta(O)$ is compact, there exists $\nu \in \Delta(O)$ such that $U(\nu) \geq U(\mu)$ for every $\mu \in \Delta(O)$.

Suppose $U(\mu) = U(\nu)$. Then, $u(x_n) = U(\delta_{x_n}) \leq U(\mu)$ for every $n \in \mathcal{N}$. Set $\mu_n = \mu$ for every $n \in \mathcal{N}$. Then, $\mu_n \in A(x_n)$ for every $n \in \mathcal{N}$ and (μ_n) converges to μ.

Now suppose $U(\mu) < U(\nu)$. If $n \in \mathcal{N}$ is such that $u(x_n) \leq U(\mu)$, then set $t_n = 1$ and $\mu_n = t_n\mu$. Clearly, $\mu_n \in A(x_n)$. Now consider $n \in \mathcal{N}$ such that $u(x_n) > U(\mu)$. Then, $U(\nu) \geq U(\delta_{x_n}) = u(x_n) > U(\mu)$. Set $t_n \in [0,1]$ such that $t_nU(\mu) + (1-t_n)U(\nu) = u(x_n)$. Setting $\mu_n = t_n\mu + (1-t_n)\nu$, we have $\mu_n \in A(x_n)$. In both cases, $U(\mu_n) - U(\mu) \leq |u(x_n) - u(x)|$.

It suffices to show that (t_n) goes to 1. Suppose not. Then, there exists $r \in [0,1)$ and a subsequence of (t_n) in $[0,r]$. As this subsequence has a convergent subsequence, there exists a subsequence (t_m) of (t_n) such that $(t_m) \subset [0,r]$ and $(t_m) \to t \in [0,r]$. As $t_m < 1$ for every m, we have $\mu_m = t_m\mu + (1-t_m)\nu$. As U is continuous, we have $U(t\mu + (1-t)\nu) = U(\lim_{m\uparrow\infty}\mu_m) = \lim_{m\uparrow\infty} U(\mu_m)$. As $t \leq r < 1$ and $U(\nu) > U(\mu)$, we have $0 < rU(\mu) + (1-r)U(\nu) - U(\mu) \leq tU(\mu) + (1-t)U(\nu) - U(\mu) = U(t\mu+(1-t)\nu) - U(\mu) = \lim_{m\uparrow\infty} U(\mu_m) - U(\mu) \leq \lim_{m\uparrow\infty} |u(x_m) - u(x)| = 0$, a contradiction.

(B) Consider $u \in \mathcal{U}_a$ and set $F \equiv \phi(u)$. As u is risk averse, u is concave. As G is continuous, $\operatorname{Gr} F = \{(\mu, x) \in \Delta(O) \times O \mid x \in F(\mu)\} = \{(\mu, x) \in \Delta(O) \times O \mid G(\mu, x) = 0\} = G^{-1}(\{0\})$ is closed in $\Delta(O) \times O$. As O is compact metric, F is upper hemicontinuous.

We now show that F is lower hemicontinuous at $\mu \in \Delta(O)$. Consider a sequence $(\mu_n) \subset \Delta(O)$ converging to μ and $x \in F(\mu)$. It follows that, if (μ_m) is a subsequence of (μ_n), then (μ_m) converges to μ, and as U is continuous, the subsequence $(U(\mu_m))$ converges to $U(\mu)$.

As u is continuous and O is compact, there exists $y \in O$ such that $u(y) \geq u(z)$ for every $z \in O$. As u is increasing and $\geq$ is latticial on O, we may assume without loss of generality that $y \geq x$. If $u(y) = 0$, then $O = \{0\}$ and lower hemicontinuity is trivial. Suppose $u(y) > 0$. Then, $y > 0$. We consider three cases: (1) $u(x) \in (0, u(y))$, (2) $u(x) = u(y)$, and (3) $u(x) = 0$.

(1) Let $u(x) \in (0, u(y))$ and $n \in \mathcal{N}$.

Suppose $U(\mu_n) \geq U(\mu)$. Let $A = \{t \in [0,1] \mid u(tx + (1-t)y) \geq U(\mu_n)\}$ and $B = \{t \in [0,1] \mid u(tx + (1-t)y) \leq U(\mu_n)\}$. Clearly, $0 \in A$ and $A \cup B = [0,1]$. As $u(x) = U(\mu) \leq U(\mu_n)$, we have $1 \in B$. As u is continuous, both A and B are closed in $[0,1]$. As $[0,1]$ is connected, $A \cap B \neq \emptyset$. Let

$t_n \in A \cap B$ and set $x_n = t_n x + (1 - t_n)y$. Note that, if $r \in [t_n, 1)$, then $y \geq x$ implies that $t_n x + (1 - t_n)y \geq rx + (1 - r)y$ and $U(\mu_n) = u(x_n) = u(t_n x + (1 - t_n)y) \geq u(rx + (1 - r)y) \geq ru(x) + (1 - r)u(y) > u(x) = U(\mu)$ because u is increasing and concave, $u(y) > u(x)$ and $r < 1$.

Suppose $U(\mu_n) < U(\mu)$. Let $A = \{t \in [0, 1] \mid u(tx) \geq U(\mu_n)\}$ and $B = \{t \in [0, 1] \mid u(tx) \leq U(\mu_n)\}$. Clearly, $0 \in B$ and $A \cup B = [0, 1]$. As $u(x) = U(\mu) > U(\mu_n)$, we have $1 \in A$. As u is continuous, both A and B are closed in $[0, 1]$. As $[0, 1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = t_n x$. Note that, if $r \in [t_n, 1)$, then $x \geq 0$ implies $t_n x \leq rx$ and $U(\mu_n) = u(x_n) = u(t_n x) \leq u(rx) < u(x) = U(\mu)$ because u is increasing, $u(x) > 0$ and $r < 1$.

By construction, $x_n \in F(\mu_n)$ for every $n \in \mathcal{N}$. It suffices to show that (t_n) converges to 1. Suppose not. Then, there exists $r \in [0, 1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0, r]$. For every m, either $U(\mu_m) \geq u(rx + (1 - r)y) > U(\mu)$ or $U(\mu_m) \leq u(rx) < U(\mu)$. Therefore, the subsequence $(U(\mu_m))$ does not converge to $U(\mu)$, a contradiction.

(2) As $u(x) = u(y) > 0$, we have $x > 0$ and $U(\mu_n) \leq u(x)$ for every $n \in \mathcal{N}$. Given $n \in \mathcal{N}$, let $A = \{t \in [0, 1] \mid u(tx) \geq U(\mu_n)\}$ and $B = \{t \in [0, 1] \mid u(tx) \leq U(\mu_n)\}$. Clearly, $1 \in A$, $0 \in B$ and $A \cup B = [0, 1]$. As u is continuous, both A and B are closed in $[0, 1]$. As $[0, 1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = t_n x$. Clearly, $x_n \in F(\mu_n)$. It suffices to show that (t_n) converges to 1. Suppose not. Then, there exists $r \in [0, 1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0, r]$. Therefore, $U(\mu_m) = u(x_m) = u(t_m x) \leq u(rx) < u(x) = U(\mu)$ for every m. Thus, the subsequence $(U(\mu_m))$ does not converge to $U(\mu)$, a contradiction.

(3) As $u(x) = 0$, we have $x = 0$ and $U(\mu_n) \geq 0 = u(x)$ for every $n \in \mathcal{N}$. Given $n \in \mathcal{N}$, let $A = \{t \in [0, 1] \mid u((1 - t)y) \geq U(\mu_n)\}$ and $B = \{t \in [0, 1] \mid u((1 - t)y) \leq U(\mu_n)\}$. Clearly, $0 \in A$, $1 \in B$ and $A \cup B = [0, 1]$. As u is continuous, both A and B are closed in $[0, 1]$. As $[0, 1]$ is connected, $A \cap B \neq \emptyset$. Let $t_n \in A \cap B$ and set $x_n = (1 - t_n)y$. Clearly, $x_n \in F(\mu_n)$. It suffices to show that (t_n) converges to 1. Suppose not. Then, there exists $r \in [0, 1)$ and a subsequence (t_m) of (t_n) such that $(t_m) \subset [0, r]$. Therefore, $U(\mu_m) = u(x_m) = u((1 - t_m)y) \geq u((1 - r)y) > 0 = u(x) = U(\mu)$. Thus, $(U(\mu_m))$ does not converge to $U(\mu)$, a contradiction. ∎

Proof of Theorem 7.2. Consider $F \in \mathcal{F}_a$. By assumption, F has nonempty values. By Theorem 7.1, F is continuous. By Theorem 3.6, there exists $u \in \mathcal{U}$ such that $F = \phi(u)$. As u is continuous, $F(\mu) = \phi(u)(\mu)$ is closed in O for every $\mu \in \Delta(O)$. As O is compact, F has compact values. The result follows from the Maximum theorem (Berge [3], Section VI.3). ∎

Proof of Theorem 7.3. Consider $A \in \mathcal{A}$. By Theorem 5.5, $A = \xi(u)$ for some $u \in \mathcal{U}$. As $\delta_x \in \xi(u)(x)$ for every $x \in O$, we have $A(x) \neq \emptyset$ for every $x \in O$. By Theorem 7.1, A is continuous. As u is continuous, so is

U. It follows that $A(x) = \xi(u)(x)$ is closed in $\Delta(O)$ for every $x \in O$. As $\Delta(O)$ is compact, this means A has compact values. As p is continuous, the mapping $\mu \mapsto \int_O \mu(dz)\, p(z)$ is continuous. The result follows from the Maximum theorem (Berge [3], Section VI.3). ■

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