On minors of the compound matrix of a Laplacian

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August 28, 2013

Abstract: Let L be an $n \times n$ matrix with zero row and column sums, $n \geq 3$. We obtain a formula for any minor of the $(n-2)$-th compound of L. An application to counting spanning trees extending a given forest is given.

Key words. Laplacian, Minors, Compound, Trees.

AMS Subject Classifications. 15A15, 05C05.

1 Introduction

We consider graphs with no loops but possibly with multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We usually take $V(G) = \{1, \ldots, n\}$. Let G be a graph with $V(G) = \{1, \ldots, n\}$. The adjacency matrix A of G is the $n \times n$ matrix with a_{ij} equal to the number of edges between i and j if $i \neq j$, and $a_{ii} = 0, i = 1, \ldots, n$. The Laplacian matrix L of G is defined as $D - A$, where D is the diagonal matrix of vertex degrees. Clearly, L is symmetric with zero row and column sums. (A row sum of a matrix is the sum of all the elements in a row. A column sum is defined similarly.) It is well-known that G is connected if and only if the rank of L is $n-1$. We refer to [1] for background concerning graphs and matrices.

The Matrix-Tree theorem asserts that any cofactor of the Laplacian equals the number of spanning trees of the graph. A combinatorial interpretation of all minors of the Laplacian matrix can also be given, see [2,3].

We introduce some further notation and definitions. The determinant of the square matrix A is denoted by $|A|$. Let A be an $m \times n$ matrix and let $1 \leq k \leq \min\{m, n\}$. We denote by $Q_{k,n}$, the set of increasing sequences of k elements from $\{1, \ldots, n\}$. For indices $I \subset \{1, \ldots, m\}, J \subset \{1, \ldots, n\}, A[I|J]$ will denote the submatrix of A formed by the rows indexed by I and the columns indexed by J. The k-th *compound* of A, denoted by $C_k(A)$, is

[1]The author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India.

an $\binom{m}{k} \times \binom{n}{k}$ matrix defined as follows. The rows and the columns of $C_k(A)$ are indexed by $Q_{k,m}, Q_{k,n}$, respectively, where the ordering is arbitrary but fixed. If $I \in Q_{k,m}, J \in Q_{k,n}$, then the (I, J)-entry of $C_k(A)$ is set to be $|A[I|J]|$.

If A and B are matrices of order $m \times n, n \times m$ respectively and if $1 \leq k \leq \min\{m, n, p\}$, then it follows from the Cauchy-Binet formula that $C_k(AB) = C_k(A)C_k(B)$.

If $S, T \subset \{1, \ldots, n\}$, then we denote by $A(S|T)$, the submatrix of A obtained by deleting the rows indexed by S and the columns indexed by T. If $S = \{i\}$ and $T = \{j\}$, then we denote $A(S|T)$ simply by $A(i|j)$. Similarly, if $S = \{i, j\}$ and $T = \{k, \ell\}$, then we denote $A(S|T)$ by $A(ij|k\ell)$ and so on. The notation $A(S| :)$ will be used to denote the matrix formed by deleting the rows corresponding to S (and keeping all the columns); $A(: |T)$ is defined similarly.

Let K_n be the complete graph on the vertices $\{1, \ldots, n\}$. We assume $n \geq 3$. Let $E(K_n)$ be the set of edges of K_n, which evidently is the set of unordered pairs of elements in $\{1, \ldots, n\}$. If G is a graph with vertex set $\{1, \ldots, n\}$ and if vertices i and j are adjacent, then we denote the edge joining i and j by (ij). We will be interested in $C_{n-2}(L)$ for a matrix L with zero row and column sums. (We do not impose any further conditions such as symmetry or the off-diagonal elements being non-positive.) The elements of $C_{n-2}(L)$ are indexed by subsets of $\{1, \ldots, n\}$ of cardinality $n - 2$. We prefer to index them instead by unordered pairs of elements from $\{1, \ldots, n\}$. Thus if $e = (ij)$ and $f = (k\ell)$ are in $E(K_n)$, then the (e, f)-entry of $C_{n-2}(L)$ is given by $|L(ij|k\ell)|$. The objective of the present paper is to provide a formula for any minor of $C_{n-2}(L)$. The motivation for our work is the paper by Burton and Pemantle[4], where a formula for a principal minor of $C_{n-2}(L)$ is given, in the special case when L is the Laplacian matrix of a graph. This result will be stated in Section 3. In Section 2 we prove several preliminary results and then obtain our main result.

2 Results

The following result is well-known. We include a proof for completeness.

Lemma 1 *Let $L = ((\ell_{ij}))$ be an $n \times n$ matrix with zero row and column sums. Then the cofactors of L are all equal.*

Proof: In the matrix $L(1|1)$, add all the columns to its first column. Then since L has zero row sums, the first column now becomes the negative of the first column of $L(1|2)$. Thus $|L(1|1)| = -|L(1|2)|$ and hence the cofactors of ℓ_{11} and ℓ_{12} are equal. We can prove similarly that all the cofactors of L are equal. ■

If L is an $n \times n$ matrix with zero row and column sums then we denote the common value of its cofactors by $\tau(L)$. Note that $C_{n-1}(L)$ has each element $\pm\tau(L)$. It turns out that there are intricate relationships among the $(n-2) \times (n-2)$ minors of such a matrix L. We begin by observing some such relationships in the next two results and these will form one of our main tools.

Lemma 2 *Let L be an $n \times n$ matrix with zero column sums. Let $1 \leq i < j < k \leq n$ and $1 \leq u < v \leq n$. Then*

$$|L(ij|uv)| = (-1)^{k-i-1}|L(jk|uv)| + (-1)^{k-j}|L(ik|uv)|. \tag{1}$$

Proof: In the matrix $L(ij|uv)$, add all the rows to the k-th row (i.e., the row indexed by k.) Let the resulting matrix be X. Since the column sums of L are zero, the row k of X equals the negative of the sum of the row i and the row j of $L(:|uv)$. Let Y be the matrix which is the same as $L(ij|uv)$, except that its row k is replaced by row i of $L(:|uv)$, and let Z be the matrix which is the same as $L(ij|uv)$, except that its row k is replaced by row j of $L(:|uv)$. By multilinearity of the determinant it follows that $|X| = -(|Y| + |Z|)$. In Y, make a series of exchanges of a pair of consecutive rows so that row k is taken to position i. This requires a total of $k - i - 2$ exchanges, since row indexed j is missing. Therefore $|Y| = (-1)^{k-i-2}|L(jk|uv)|$. Similarly in Z, make a series of exchanges of a pair of consecutive rows so that row k is taken to position j. This requires a total of $k-j-1$ exchanges. Therefore $|Z| = (-1)^{k-j-1}|L(jk|uv)|$. It follows that

$$\begin{aligned} |L(ij|uv)| &= |X| \\ &= -(|Y| + |Z|) \\ &= (-1)^{k-i-1}|L(jk|uv)| + (-1)^{k-j}|L(ik|uv)|, \end{aligned}$$

and the proof is complete. ∎

An analogue of Lemma 2 for a matrix with zero row sums is proved similarly and is stated next.

Lemma 3 *Let L be an $n \times n$ matrix with zero row sums. Let $1 \leq i < j \leq n$ and $1 \leq u < v < w \leq n$. Then*

$$|L(ij|uv)| = (-1)^{w-u-1}|L(ij|vw)| + (-1)^{w-v}|L(ij|uw)|. \tag{2}$$

We continue with more notation. A tree, together with a distinguished vertex called the root, is called a rooted tree. Let A be an $n \times n$ matrix. Let T be a rooted tree with $V(T) \subset \{1, \ldots, n\}$, with root v. In A, add all the rows indexed by $V(T) \setminus v$ to the row indexed by v. Then delete all the rows indexed by $V(T) \setminus v$. We say that the resulting matrix B is obtained from A by row condensation with respect to the rooted tree T. If T has a single vertex, then $B = A$. By row condensation with respect to the edge (ij) we mean row condensation with respect to the tree consisting of the single edge (ij). Column condensation is defined similarly.

An acyclic graph is a graph with no cycles and is also called a forest. Each component of a forest is a tree. By a rooted forest we mean a forest with each component being a rooted tree. If G is a rooted forest with $V(G) \subset \{1, \ldots, n\}$, then the matrix obtained from A by row (column) condensation with respect to G is defined to be the matrix obtained by successively applying row (column) condensation with respect to the components of G. It is easy to see that the order of components does not matter in this operation. If A has zero row sums, then this property is preserved by row condensation as well as by column condensation. Similarly, if A has zero column sums, then this property is preserved by row condensation as well as by column condensation. When we talk of row or column condensation with respect to a tree or a forest without specifying the root of the tree or the roots for the components of the forest, it will be assumed that the condensation is carried out after choosing the roots. In such cases the actual choice of the roots will not have any bearing on the subsequent discussion. When we talk of row or column condensation with respect to a set of edges, it is assumed that the set induces an acyclic graph, and we mean the row or column condensation with respect to the induced graph.

Example Consider the trees T_1 and T_2, rooted at 4 and 1 respectively.

T_1 :

$$\circ 2 \;\text{———}\; \bullet 4 \;\text{———}\; \circ 5$$

T_2 :

$$\bullet 1 \;\text{———}\; \circ 3 \;\text{———}\; \circ 4$$

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & -2 \\ 1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -1 & -1 \\ -4 & -1 & 2 & -1 & 4 \end{bmatrix}.$$

After row condensation with respect to T_1 (add rows $2, 5$ to row 4, then delete rows $2, 5$) and column condensation with respect to T_2 (add columns $3, 4$ to column 1, then delete columns $3, 4$) we get the following matrix.

$$B = \begin{bmatrix} 3 & -1 & -2 \\ 1 & -1 & 0 \\ -4 & 2 & 2 \end{bmatrix}.$$

Observe that A has zero row and column sums and so does B.

Lemma 4 *Let L be an $n \times n$ matrix. Let $1 \le i < j \le n, 1 \le u < v \le n$. Let the matrix X be obtained from L by row condensation with respect to (ij) with root j and by column* *condensation with respect to* (uv) *with root* v*. Then*

$$|X| = |L(i|u)| + (-1)^{j-i-1}|L(j|u)| + (-1)^{v-u-1}|L(i|v)| + (-1)^{v-u+j-i}|L(j|v)|. \quad (3)$$

Proof: Let X_1 be obtained from X by replacing row j of X by row j of $L(:|u)$ and by replacing column v of X by column v of $L(i|:)$. Let X_2 be obtained from X by replacing row j of X by row i of $L(:|u)$ and by replacing column v of X by column v of $L(i|:)$. Let X_3 be obtained from X by replacing row j of X by row j of $L(:|u)$ and by replacing column v of X by column u of $L(i|:)$. Let X_4 be obtained from X by replacing row j of X by row i of $L(:|u)$ and by replacing column v of X by column u of $L(i|:)$. By the multilinearity of the determinant we have

$$|X| = |X_1| + |X_2| + |X_3| + |X_4|. \quad (4)$$

Note that $X_1 = L(i|u)$. In X_2 if we shift row j to the place of row i then X_2 turns into $L(j|u)$ and this shift requires $j-i-1$ row exchanges. Therefore $|X_2| = (-1)^{j-i-1}|L(j|u)|$. Similarly, $|X_3| = (-1)^{v-u-1}|L(i|v)|$ and $|X_4| = (-1)^{v-u+j-i}|L(j|v)|$. Substituting in (4) the result is proved. ■

We will use the next result, see, for example, [5],p.4-5.

Lemma 5 (Sylvester's identity) *Let* A *be an* $n \times n$ *matrix and let* $1 \le i < j \le n, 1 \le k < \ell \le n$*. Then*

$$|A(i|k)||A(j|\ell)| - |A(i|\ell)||A(j|k)| = |A||A(ij|k\ell)|. \quad (5)$$

In the next three results (Lemmas 6-8) we consider 2×2 minors of $C_{n-2}(L)$ and obtain a formula. These results will form a basis for an induction proof of the main result, Theorem 13.

Lemma 6 *Let* L *be an* $n \times n$ *matrix with zero row and column sums. Let* i_1, j_1, i_2, j_2 *be distinct integers in* $\{1, \ldots, n\}$ *such that* $i_1 < j_1, i_2 < j_2$*. Let* u_1, v_1, u_2, v_2 *be distinct integers in* $\{1, \ldots, n\}$ *such that* $u_1 < v_1, u_2 < v_2$*. Let* X *be the* 2×2 *matrix*

$$\begin{bmatrix} |L(i_1j_1|u_1v_1)| & |L(i_1j_1|u_2v_2)| \\ |L(i_2j_2|u_1v_1)| & |L(i_2j_2|u_2v_2)| \end{bmatrix}.$$

Let Y *be the matrix obtained from* L *by row condensation with respect to the graph induced by the edges* $(i_1j_1), (i_2j_2)$ *with roots* j_1, j_2 *respectively and by column condensation with respect to the graph induced by the edges* $(u_1v_1), (u_2v_2)$ *with roots* v_1, v_2 *respectively. Let* $Z = Y(j_2|v_2)$*. Then*

$$|X| = (-1)^{j_1+v_1}\tau(L)|Z|.$$

Proof: We assume that $j_1 < i_2$ and $v_1 < u_2$. The proof in the remaining cases is similar. An application of Lemma 2 gives

$$|L(i_1 j_1|u_2 v_2)| = (-1)^{j_2-i_1-1}|L(j_1 j_2|u_2 v_2)| + (-1)^{j_2-j_1}|L(i_1 j_2|u_2 v_2)|. \tag{6}$$

By Lemma 3,

$$|L(i_2 j_2|u_1 v_1)| = (-1)^{u_2-u_1-1}|L(i_2 j_2|v_1 u_2)| + (-1)^{u_2-v_1}|L(i_2 j_2|u_1 u_2)|. \tag{7}$$

Similarly

$$\begin{aligned} |L(i_1 j_1|u_1 v_1)| &= (-1)^{j_2-i_1-1}|L(j_1 j_2|u_1 v_1)| + (-1)^{j_2-j_1}|L(i_1 j_2|u_1 v_1)| \\ &= (-1)^{j_2-i_1-1}((-1)^{u_2-u_1-1}|L(j_1 j_2|v_1 u_2)| + (-1)^{u_2-v_1}|L(j_1 j_2|u_1 u_2)|) \\ &+ (-1)^{j_2-j_1}((-1)^{u_2-u_1-1}|L(i_1 j_2|v_1 u_2)| + (-1)^{u_2-v_1}|L(i_1 j_2|u_1 u_2)|). \end{aligned} \tag{8}$$

It follows from (6),(7),(8) that

$$|X| = |L(i_1 j_1|u_1 v_1)||L(i_2 j_2|u_2 v_2)| - |L(i_1 j_1|u_2 v_2)||L(i_2 j_2|u_1 v_1)| \tag{9}$$

is the sum of the following four terms:

$$(-1)^{j_2+i_1+u_1+u_2}(|L(j_1 j_2|v_1 u_2)||L(i_2 j_2|u_2 v_2)| - |L(j_1 j_2|u_2 v_2)||L(i_2 j_2|v_1 u_2)|) \tag{10}$$

$$(-1)^{j_2+i_1+u_2+v_1+1}(|L(j_1 j_2|u_1 u_2)||L(i_2 j_2|u_2 v_2)| - |L(j_1 j_2|u_2 v_2)||L(i_2 j_2|u_1 u_2)|) \tag{11}$$

$$(-1)^{j_1+j_2+u_1+u_2+1}(|L(i_1 j_2|v_1 u_2)||L(i_2 j_2|u_2 v_2)| - |L(i_1 j_2|u_2 v_2)||L(i_2 j_2|v_1 u_2)|) \tag{12}$$

$$(-1)^{j_1+j_2+v_1+u_2}(|L(i_1 j_2|u_1 u_2)||L(i_2 j_2|u_2 v_2)| - |L(i_1 j_2|u_2 v_2)||L(i_2 j_2|u_1 u_2)|) \tag{13}$$

By Lemma 5 the terms in (10)-(13) are respectively equal to

$$(-1)^{j_2+i_1+u_1+u_2}|L(j_2|u_2)||L(j_1 i_2 j_2|v_1 u_2 v_2)| \tag{14}$$

$$(-1)^{j_2+i_1+u_2+v_1+1}L(j_2|u_2)||L(j_1 i_2 j_2|u_1 u_2 v_2)| \tag{15}$$

$$(-1)^{j_1+j_2+u_1+u_2+1}|L(j_2|u_2)||L(i_1 i_2 j_2|u_2 v_1 v_2)| \tag{16}$$

$$(-1)^{j_1+j_2+u_2+v_1}|L(j_2|u_2)||L(i_1 i_2 j_2|u_1 u_2 v_2)| \tag{17}$$

Let the matrices Y and Z be as in the statement of the present lemma. Taking the sum of (14)-(17) and using Lemma 4 it follows that

$$\begin{aligned} |X| &= (-1)^{j_2+u_2}|L(j_2|u_2)|(-1)^{j_1+v_1}\left\{|L(i_1 i_2 j_2|u_1 u_2 v_2)| + (-1)^{u_1+v_1+1}|L(i_1 i_2 j_2|u_2 v_1 v_2)|\right. \\ &+ \left.(-1)^{i_1+j_1+1}|L(j_1 i_2 j_2|u_1 u_2 v_2)| + (-1)^{i_1+j_1+u_1+v_1}|L(j_1 i_2 j_2|v_1 u_2 v_2)|\right\} \\ &= (-1)^{j_1+v_1}\tau(L)|Z| \end{aligned}$$

and the proof is complete. ■

Lemma 7 *Let L be an $n\times n$ matrix with zero row and column sums. Let $i<j<k$ be integers in $\{1,\ldots,n\}$. Let u_1,v_1,u_2,v_2 be distinct integers in $\{1,\ldots,n\}$ such that $u_1<v_1, u_2<v_2$. Let X be the 2×2 matrix*

$$\begin{bmatrix} |L(ik|u_1v_1)| & |L(ik|u_2v_2)| \\ |L(jk|u_1v_1)| & |L(jk|u_2v_2)| \end{bmatrix}.$$

Let Y be the matrix obtained from L by row condensation with respect to the tree formed by the edges $(ik),(jk)$ with root k and by column condensation with respect to the graph formed by the edges $(u_1v_1),(u_2v_2)$ with roots v_1,v_2 respectively. Let $Z=Y(k|u_2)$. Then

$$|X|=(-1)^{k+v_1}\tau(L)|Z|.$$

Proof: We assume that $v_1<u_2$. The proof in the remaining cases is similar. By Lemma 3 we have

$$|L(ik|u_1v_1)|=(-1)^{u_2-u_1-1}|L(ik|v_1u_2)|+(-1)^{u_2-v_1}|L(ik|u_1u_2)|. \tag{18}$$

$$|L(jk|u_1v_1)|=(-1)^{u_2-u_1-1}|L(jk|v_1u_2)|+(-1)^{u_2-v_1}|L(jk|u_1u_2)|. \tag{19}$$

It follows from (18),(19) that

$$\begin{aligned} |L(ik|u_1v_1)||L(jk|u_2v_2)| &= (-1)^{u_2-u_1-1}|L(ik|v_1u_2)||L(jk|u_2v_2)| \\ &+ (-1)^{u_2-v_1}|L(ik|u_1u_2)||L(jk|u_2v_2)| \end{aligned} \tag{20}$$

$$\begin{aligned} |L(ik|u_2v_2)||L(jk|u_1v_1)| &= (-1)^{u_2-u_1-1}|L(ik|u_2v_2)||L(jk|v_1u_2)| \\ &+ (-1)^{u_2-v_1}|L(ik|u_2v_2)||L(jk|u_1u_2)| \end{aligned} \tag{21}$$

Using (20),(21) and Lemma 5 we have

$$\begin{aligned} |X| &= |L(ik|u_1v_1)||L(jk|u_2v_2)|-|L(ik|u_2v_2)||L(jk|u_1v_1)| \\ &= (-1)^{u_2-u_1-1}|L(k|u_2)||L(ijk|v_1u_2v_2)|+(-1)^{u_2-v_1}|L(k|u_2)||L(ijk|u_1u_2v_2)| \\ &= (-1)^{k+u_2}((-1)^{k+u_1+1}|L(ijk|v_1u_2v_2)|+(-1)^{k+v_1}|L(ijk|u_1u_2v_2)|) \\ &= \tau(L)(-1)^{k+v_1}(|L(ijk|u_1u_2v_2)+(-1)^{u_1+v_1+1}|L(ijk|v_1u_2v_2)|). \end{aligned} \tag{22}$$

Using an argument similar to the one used in the proof of Lemma 4 it follows that the expression in (22) equals $(-1)^{k+v_1}\tau(L)|Z|$ and the proof is complete. ∎

Lemma 8 *Let L be an $n\times n$ matrix with zero row and column sums. Let $i<j<k, u<v<w$ be integers in $\{1,\ldots,n\}$. Let X be the 2×2 matrix*

$$\begin{bmatrix} |L(ik|uw)| & |L(ik|vw)| \\ |L(jk|uw)| & |L(jk|vw)| \end{bmatrix}.$$

Let Y be the matrix obtained from L by row condensation with respect to the tree formed by the edges $(ik),(jk)$ with root k and by column condensation with respect to the tree formed by the edges $(uw),(vw)$ with root w. Let $Z = Y(k|w)$. Then

$$|X| = (-1)^{k+w}\tau(L)|Z|.$$

Proof: By Lemma 5 we have

$$\begin{aligned} |X| &= |L(ik|uw)||L(jk|vw)| - |L(ik|vw)||L(jk|uw)| \\ &= |L(k|w)||L(ijk|uvw)|) \\ &= (-1)^{k+w}(-1)^{k+w}|L(k|w)||L(ijk|uvw)| \\ &= (-1)^{k+w}\tau(L)|Z| \end{aligned}$$

and the proof is complete. ■

In the next result we summarize the results in Lemmas 6-8 in a form that we will need later.

Lemma 9 *Let L be an $n \times n$ matrix with zero row and column sums. Let $1 < i_p < j_p \leq n, 1 < u_q < v_q \leq n$. Let X be the 2×2 matrix*

$$\begin{bmatrix} |L(1n|1n)| & |L(1n|u_qv_q)| \\ |L(i_pj_p|1n)| & |L(i_pj_p|u_qv_q)| \end{bmatrix}.$$

Let Y be the matrix obtained from L by row condensation with respect to $\{(1n),(i_pj_p)\}$ and by column condensation with respect to $\{(1n),(u_qv_q)\}$. For these row and column condensations the roots are chosen as follows. If $j_p \neq n$, then for row condensation with respect to $(1n)$, the root is n, while for row condensation with respect to (i_pj_p), the root is arbitrary. If $j_p = n$, then the root is n. Similar remarks apply to column condensation. In the process of these row and column condensations, precisely one row out of i_p, j_p and one column out of u_q, v_q will get deleted. From Y, delete the row and the column indexed by the remaining elements of the two pairs $(i_pj_p),(u_qv_q)$ and let the resulting matrix be Z. Then $|A| = \tau(L)|Z|$.

Lemma 10 *Let L be an $n \times n$ matrix with zero column sums. Let $S \subset E(K_n)$ be a set of edges of K_n. If the graph induced by S has a cycle, then the rows of $C_{n-2}(L)$ indexed by S are linearly dependent.*

Proof: For convenience, we will refer to a row indexed by the edge (ij) as simply the row (ij). Let the graph induced by S contain the edges $(i_1i_2),(i_2i_3),\ldots,(i_{k-1}i_k),(i_1i_k)$. We prove the result by induction on k. After relabeling the rows of L if necessary, we assume that $i_1 < i_2 < \cdots < i_k$. If $k = 3$, then by Lemma 2, for any edge (uv),

$$|L(i_1i_2|uv)| = (-1)^{i_3-i_1-1}|L(i_2i_3|uv)| + (-1)^{i_3-i_2}|L(i_1i_3|uv)|.$$

It follows that the rows $(i_1i_2), (i_2i_3), (i_1i_3)$ are linearly dependent. Assume that the result holds if the cycle induced by S has $k-1$ edges and proceed by induction.

By the induction hypothesis, the rows $(i_1i_2), (i_2i_3), \ldots, (i_{k-2}i_{k-1}), (i_1i_{k-1})$ are linearly dependent. If the rows $(i_1i_2), (i_2i_3), \ldots, (i_{k-2}i_{k-1})$ are linearly dependent then the result is proved. Otherwise, the row (i_1i_{k-1}) must be a linear combination of the rows $(i_1i_2), (i_2i_3), \ldots, (i_{k-2}i_{k-1})$. Also, by Lemma 2, the row (i_1i_{k-1}) is a linear combination of the rows $(i_{k-1}i_k)$ and (i_1i_k). It follows that there is a linear combination of the rows $(i_1i_2), (i_2i_3), \ldots, (i_{k-1}i_k), (i_1i_k)$ that equals zero and the proof is complete. ■

The following analogue of Lemma 10 is proved similarly.

Lemma 11 *Let L be an $n \times n$ matrix with zero row sums. Let $S \subset E(K_n)$ be a set of edges of K_n. If the graph induced by S has a cycle, then the columns of $C_{n-2}(L)$ indexed by S are linearly dependent.*

The next result is the well-know Dodgson condensation formula, see, for example, [5],p.4-4.

Lemma 12 *Let A be an $n \times n$ matrix, $n \geq 2$. Let B be the $(n-1) \times (n-1)$ matrix defined by $b_{ij} = |A[1(i+1)|1(j+1)]|, i, j = 1, \ldots, n-1$. Then $|A|a_{11}^{n-2} = |B|$.*

The following is the main result of this paper. The result provides a formula for any minor of $C_{n-2}(L)$, where L is an $n \times n$ matrix with zero row and column sums.

Theorem 13 *Let L be an $n \times n$ matrix with zero row and column sums. Let S, T be subsets of $E(K_n)$ of cardinality k and let X be the submatrix of $C_{n-2}(L)$ formed by the rows indexed by S and the columns indexed by T. Let G_S and G_T be the subgraphs of K_n induced by the edges in S and T respectively. If either G_S or G_T has a cycle, then $|X| = 0$. If both G_S and G_T are acyclic, then $|X| = \pm(\tau(L))^{k-1}\tau(M)$, where M is the matrix obtained from L by row condensation with respect to G_S, and by column condensation with respect to G_T.*

Proof: If either G_S or G_T has a cycle, then by Lemmas 10,11, the corresponding rows, or columns, of $C_{n-2}(L)$ are linearly dependent and hence $|X| = 0$. Therefore we assume that both G_S and G_T are forests. Let T_1 and T_2 be components of G_S and G_T respectively. By relabeling the rows and the columns of L we assume that 1 and n are both in $V(T_1)$ as well as in $V(T_2)$, and that $(1n)$ is an edge in both T_1 and T_2. We further assume that 1 is a pendant vertex in both T_1 and T_2. We also set n as the root of T_1 and T_2. We choose and fix a root for the remaining components of G_S and G_T.

We prove the result by induction on k. The case $k = 2$ is settled in Lemmas 6,7,8. Assume the result to be true for $k-1$ and proceed. Let U be the matrix obtained from L by row condensation with respect to $(1n)$ and by column condensation with respect to $(1n)$. We assume the first row and column of X to be indexed by $(1n)$.

Let the elements of S be $(1n), (i_2j_2), \ldots, (i_kj_k)$ and those of T be $(1n), (u_2v_2), \ldots, (u_kv_k)$. Consider a 2×2 submatrix of X that includes the first row and column. We may take the matrix to be

$$A = \begin{bmatrix} |L(1n|1n)| & |L(1n|u_qv_q)| \\ |L(i_pj_p|1n)| & |L(i_pj_p|u_qv_q)| \end{bmatrix}. \tag{23}$$

Let Y be the matrix obtained from L by row condensation with respect to $\{(1n), (i_pj_p)\}$ and by column condensation with respect to $\{(1n), (u_qv_q)\}$. Note that for these row and column condensations the roots have already been chosen. For example, if $(1n)$ and (i_pj_p) belong to the same component of G_S, then the root is n. If they belong to different components, then for row condensation with respect to $(1n)$ the root is n, while for row condensation with respect to (i_pj_p), the root is the root chosen for the corresponding component of G_S. Similar remarks apply to column condensation. In the process of these row and column condensations, precisely one row out of i_p, j_p and one column out of u_q, v_q will get deleted. From Y, delete the row and the column indexed by the remaining elements of the two pairs $(i_pj_p), (u_qv_q)$ and let the resulting matrix be Z. By Lemma 9, $|A| = \tau(L)|Z|$. (Although the matrices A, Y and Z depend on (i_pj_p) and (u_qv_q), we have suppressed the dependence in the notation.)

Let W be the $(k-1) \times (k-1)$ matrix defined as follows. The rows and the columns of W are indexed by $(i_2j_2), \ldots, (i_kj_k)$ and $(u_2v_2), \ldots, (u_kv_k)$ respectively. The element of W in the position $((i_pj_p), (u_qv_q))$ is $|A|$, where A is as in (23), $p = 2, \ldots, k; q = 2, \ldots, k$. By the discussion in the preceding paragraph we observe that W is $\tau(L)$ times a $(k-1) \times (k-1)$ submatrix of $C_{n-2}(U)$, which we denote by R. It follows from this observation and by Lemma 12 that

$$|X| = |L(1n|1n)|^{-(k-2)}|W| = |L(1n|1n)|^{-(k-2)}(\tau(L))^{k-1}|R|. \tag{24}$$

Note that if we apply row condensation with respect to $\{(i_2j_2), \ldots, (i_kj_k)\}$ and column condensation with respect to $\{(u_2v_2), \ldots, (u_kv_k)\}$ to the matrix U, then we obtain the matrix M defined in the statement of the present theorem. Hence by the induction hypothesis,

$$|R| = \pm(\tau(U))^{k-2}\tau(M). \tag{25}$$

Finally, observe that $\tau(U) = |L(1n|1n)|$ and hence from (24),(25) we have

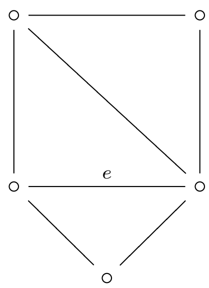
$$|X| = \pm(\tau(L))^{k-1}\tau(M),$$

and the proof is complete. ■

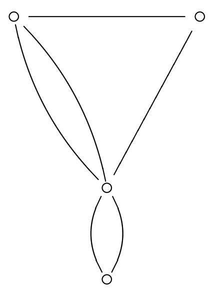
3 Application to graph Laplacians

In this section we interpret Theorem 13 in the special case when L is the Laplacian matrix of a graph. The operations of row and column condensation are related to the operation of contracting an edge as we describe now. Let G be a graph with $V(G) = \{1, \ldots, n\}$ and let $e = (ij)$ be an edge of G. The contraction of e is executed as follows, see, for example, [6], p.84. We replace i and j with a single vertex and corresponding to every edge incident to either i or j, with the exception of e, we create an edge incident to the new vertex, keeping the other end-vertex the same.

Example Consider the graph G



Contracting the edge e produces the following graph H.



The Laplacian matrices L and M of G and H respectively are as follows, where we have taken the end-vertices of e as the first two vertices of G. Note that M is obtained by row and column condensation of L with respect to e.

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 5 & -2 & -1 & -2 \\ -2 & 3 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}.$$

We now make the observation in the preceding example more precise. Let G be a graph with $V(G) = \{1, \ldots, n\}$ and let $e = (ij)$ be an edge of G. Let L be the Laplacian of G. Let H be the graph obtained from G by contracting e. Let the matrix M be obtained from L by row condensation with respect to (ij) and by column condensation with respect to (ij). Then it can be seen that the Laplacian of H is PMP' for a suitable permutation matrix P. (Here P' denotes the transpose of P.) The presence of P and P' is merely to take care of the label that we choose to give to the vertex of H that replaces the two vertices of G (or, the end-vertices of e). Note that $\tau(M)$ is precisely the number of spanning trees of G that include e. This observation extends to the case of contracting with respect to several edges and thus the following result is a consequence of Theorem 13.

Theorem 14 *Let G be a graph with $V(G) = \{1, \ldots, n\}$ and let L be the Laplacian of G. Let $S \subset E(G)$ be of cardinality k and let X be the $k \times k$ submatrix of $C_{n-2}(L)$ formed by the rows and the columns indexed by S. If the subgraph induced by S has a cycle, then $|X| = 0$. If the subgraph induced by S is a forest, then $|X|$ equals $(\tau(L))^{k-1}$ times the number of spanning trees of G that include all the edges in S.*

We remark that Theorem 14 is essentially proved in [4] (see Theorems 1.1 and 4.2) in the framework of random walks and electrical networks.

Acknowledgment I sincerely thank Souvik Dhara and Arbind K. Lal for a careful reading of the manuscript and for making several helpful suggestions.

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