Problems in Geometry

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Problem 1 [BMOTC]

Prove that the medians from the vertices A and B of triangle ABC are mutually perpendicular if and only if $|BC|^2 + |AC|^2 = 5|AB|^2$.

Problem 2 [BMOTC]

Suppose that $\angle A$ is the smallest of the three angles of triangle ABC. Let D be a point on the arc BC of the circumcircle of ABC which does not contain A. Let the perpendicular bisectors of AB, AC intersect AD at M and N respectively. Let BM and CN meet at T. Prove that $BT + CT \leq 2R$ where R is the circumradius of triangle ABC.

Problem 3 [BMOTC]

Let triangle ABC have side lengths a, b and c as usual. Points P and Q lie inside this triangle and have the properties that $\angle BPC = \angle CPA = \angle APB = 120°$ and $\angle BQC = 60° + \angle A$, $\angle CQA = 60° + \angle B$, $\angle AQB = 60° + \angle C$. Prove that

$$(|AP| + |BP| + |CP|)^3.|AQ|.|BQ|.|CQ| = (abc)^2.$$

Problem 4 [BMOTC]

The points M and N are the points of tangency of the incircle of the isosceles triangle ABC which are on the sides AC and BC. The sides of equal length are AC and BC. A tangent line t is drawn to the minor arc MN. Suppose that t intersects AC and BC at Q and P respectively. Suppose that the lines AP and BQ meet at T.

(a) Prove that T lies on the line segment MN.

(b) Prove that the sum of the areas of triangles ATQ and BTP is minimized when t is parallel to AB.

Problem 5 [BMOTC]

In a hexagon with equal angles, the lengths of four consecutive edges are 5, 3, 6 and 7 (in that order). Find the lengths of the remaining two edges.

Problem 6 [BMOTC]

The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T, and let S be the intersection of the line through C and D with the side AB. Show that $|AT| = |SB|$.

Problem 7 [BMOTC]

Let S and r be the area and the inradius of the triangle ABC. Let r_A denote the radius of the circle touching the incircle, AB and AC. Define r_B and r_C similarly. The common tangent of the circles with radii r and r_A cuts a little triangle from ABC with area S_A. Quantities S_B and S_C are defined in a similar fashion. Prove that

$$\tfrac{S_A}{r_A} + \tfrac{S_B}{r_B} + \tfrac{S_C}{r_C} = \tfrac{S}{r}$$

Problem 8 [BMOTC]

Triangle ABC in the plane Π is said to be *good* if it has the following property: for any point D in space, out of the plane Π, it is possible to construct a triangle with sides of lengths $|AD|$, $|BD|$ and $|CD|$. Find all *good* triangles.

Problem 9 [BMO]

Circle γ lies inside circle θ and touches it at A. From a point P (distinct from A) on θ, chords PQ and PR of θ are drawn touching γ at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

Problem 10 [BMO]

AP, AQ, AR, AS are chords of a given circle with the property that

$$\angle PAQ = \angle QAR = \angle RAS.$$

Prove that

$$AR(AP + AR) = AQ(AQ + AS).$$

Problem 11 [BMO]

The points Q, R lie on the circle γ, and P is a point such that PQ, PR are tangents to γ. A is a point on the extension of PQ and γ' is the circumcircle of triangle PAR. The circle γ' cuts γ again at B and AR cuts γ at the point C. Prove that $\angle PAR = \angle ABC$.

Problem 12 [BMO]

In the acute-angled triangle ABC, CF is an altitude, with F on AB and BM is a median with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

Problem 13 [BMO]

A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC = \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$. Prove that the circumcircle of ABC touches the circumcircle of PQR.

Problem 14 [BMO]

ABP is an isosceles triangle with AB=AP and $\angle PAB$ acute. PC is the line through P perpendicular to BP and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB). D completes the parallelogram $ABCD$. PC meets DA at M. Prove that M is the midpoint of DA.

Problem 15 [BMO]

In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

Problem 16 [BMO]

$ABCD$ is a rectangle, P is the midpoint of AB and Q is the point on PD such that CQ is perpendicular to PD. Prove that BQC is isosceles.

Problem 17 [BMO]

Let ABC be an equilateral triangle and D an internal point of the side BC. A circle, tangent to BC at D, cuts AB internally at M and N and AC internally at P and Q. Show that $BD + AM + AN = CD + AP + AQ$.

Problem 18 [BMO]

Let ABC be an acute-angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC and CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $CP = CQ$.

Problem 19 [BMO]

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is closer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

Problem 20 [BMO]

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is closer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects $\angle PMR$.

Problem 21 [BMO]

Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that $AP+BP+CP$ is minimized.

Problem 22 [BMO]

Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA respectively. Prove that PY, QZ, RX are concurrent.

Problem 23 [BMO]

The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P. Given that $CD = CP = DQ$, prove that $\angle CAD = 60°$.

Problem 24 [BMO]

The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations

$$\begin{aligned} u(v+w-u) &= a^2 \\ v(w+u-v) &= b^2 \\ w(u+v-w) &= c^2 \end{aligned}$$

Prove that triangle ABC is acute-angled and express the angles U, V, W in terms of A, B, C.

Problem 25 [BMO]

Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2 respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1 and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

Problem 26 [BMO]

Let ABC be an acute-angled triangle and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.

Problem 27 [BMO]

Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$.

Problem 28 [BMO]

ABC is a triangle, right-angled at C. The internal bisectors of $\angle BAC$ and $\angle ABC$ meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find the measure of $\angle MCN$.

Problem 29 [BMO]

The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the segment AC and BX meets S again at Q. Show that $BX = CX$ if and only if PQ is a diameter of S.

Problem 30 [BMO]

Let ABC be a triangle and let D be a point on AB such that $4AD = AB$. The half-line l is drawn on the same side of AB as C, starting from D and making an angle of θ with DA where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line l at P, show that $PB = 2PD$.

Problem 31 [BMO]

Let BE and CF be the altitudes of an acute triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

Problem 32 [BMO]

In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

Problem 33 [Putnam]

Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$ where T ranges over all triangles and R, S over all rectangles as above.

Problem 34 [Putnam]

A rectangle, $HOMF$, has sides HO=11 and OM=5. A triangle ABC has H as the *orthocentre*, O as the *circumcentre*, M the midpoint of BC and F the foot of the altitude from A. What is the length of BC?

Problem 35 [Putnam]

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem 36 [Putnam]

Let A, B and C denote distinct points with integer coordinates in R^2. Prove that if $(|AB| + |BC|)^2 < 8[ABC] + 1$ then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

Problem 37 [Putnam]

Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

Problem 38 [BMO]

Let ABC be a triangle and D, E, F be the midpoints of BC, CA, AB respectively. Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

Problem 39 [BMO]

The altitude from one of the vertex of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

Problem 40

Two cyclists ride round two intersecting circles, each moving with a constant speed. Having started simultaneously from a point at which the circles intersect, the cyclists meet once again at this point after one circuit. Prove that there is a fixed point such that the distances from it to the cyclists are equal all the time if they ride: (a) in the same direction (clockwise); (b) in opposite direction.

Problem 41

Prove that four circles circumscribed about four triangles formed by four intersecting straight lines in the plane have a common point. (*Michell's Point*).

Problem 42

Given an equilateral triangle ABC. Find the locus of points M inside the triangle such that $\angle MAB + \angle MBC + \angle MCA = \frac{\pi}{2}$.

Problem 43

In a triangle ABC, on the sides AC and BC, points M and N are taken, respectively and a point L on the line segment MN. Let the areas of the triangles ABC, AML and BNL be equal to S, P and Q, respectively. Prove that

$$S^{\frac{1}{3}} \geq P^{\frac{1}{3}} + Q^{\frac{1}{3}}.$$

Problem 44

For an arbitrary triangle, prove the inequality $\frac{bc\cos A}{b+c} + a < p < \frac{bc+a^2}{a}$, where a, b and c are the sides of the triangle and p its semiperimeter.

Problem 45

Given in a triangle are two sides: a and b ($a > b$). Find the third side if it is known that $a + h_a \leq b + h_b$, where h_a and h_b are the altitudes dropped on these sides (h_a the altitude drawn to the side a).

Problem 46

One of the sides in a triangle ABC is twice the length of the other and $\angle B = 2\angle C$. Find the angles of the triangle.

Problem 47

In a parallelogram whose area is S, the bisectors of its interior angles are drawn to intersect one another. The area of the quadrilateral thus obtained is equal to Q. Find the ratio of the sides of the parallelogram.

Problem 48

Prove that if one angle of a triangle is equal to $120°$, then the triangle formed by the feet of its angle bisectors is right-angled.

Problem 49

Given a rectangle $ABCD$ where $|AB| = 2a$, $|BC| = a\sqrt{2}$. With AB is diameter a semicircle is constructed externally. Let M be an arbitrary point on the semicircle, the line MD intersect AB at N, and the line MC at L. Find $|AL|^2 + |BN|^2$.

Problem 50

Let A, B and C be three points lying on the same line. Constructed on AB, BC and AC as diameters are three semicircles located on the same side of the line. The centre of a circle touching the three semicircles is found at a distance d from the line AC. Find the radius of this circle.

Problem 51

In an isosceles triangle ABC, $|AC| = |BC|$, BD is an angle bisector, $BDEF$ is a rectangle. Find $\angle BAF$ if $\angle BAE = 120°$.

Problem 52

Let M_1 be a point on the incircle of triangle ABC. The perpendiculars to the sides through M_1 meet the incircle again at M_2, M_3, M_4. Prove that the geometric mean of the six lengths M_iM_j, $1 \leq i \leq j \leq 4$, is less than or equal to $r\sqrt[3]{4}$, where r denotes the inradius. When does the equality hold?

Problem 53 [AMM]

Let ABC be a triangle and let I be the incircle of ABC and let r be the radius of I. Let K_1, K_2 and K_3 be the three circles outside I and tangent to I and to two of the three sides of ABC. Let r_i be the radius of K_i for $1 \leq i \leq 3$. Show that

$$r = \sqrt{r_1 r_2} + \sqrt{r_2 r_3} + \sqrt{r_3 r_1}$$

Problem 54 [Prithwijit's Inequality]

In triangle ABC suppose the lengths of the medians are m_a, m_b and m_c respectively. Prove that

$$\tfrac{am_a+bm_b+cm_c}{(a+b+c)(m_a+m_b+m_c)} \leq \tfrac{1}{3}$$

Problem 55 [Loney]

The base a of a triangle and the ratio $r(< 1)$ of the sides are given. Show that the altitude h of the triangle cannot exceed $\frac{ar}{1-r^2}$ and that when h has this value the vertical angle of the triangle is $\frac{\pi}{2} - 2\tan^{-1} r$.

Problem 56 [Loney]

The internal bisectors of the angles of a triangle ABC meet the sides in D, E and F. Show that the area of the triangle DEF is equal to $\frac{2\Delta abc}{(a+b)(b+c)(c+a)}$.

Problem 57 [Loney]

If a, b, c are the sides of a triangle, λa, λb, λc the sides of a similar triangle inscribed in the former and θ the angle between the sides a and λa, prove that $2\lambda \cos\theta = 1$.

Problem 58

Let a, b and c denote the sides of a triangle and $a+b+c=2p$. Let G be the median point of the triangle and O, I and I_a the centres of the circumscribed, inscribed and escribed circles, respectively (the escribed circle touches the side BC and the extensions of the sides AB and AC), R, r and r_a being their radii, respectively. Prove that the following relationships are valid:

(a) $a^2+b^2+c^2 = 2p^2 - 2r^2 - 8Rr$
(b) $|OG|^2 = R^2 - \frac{a^2+b^2+c^2}{9}$
(c) $|IG|^2 = \frac{p^2+5r^2-16Rr}{9}$
(d) $|OI|^2 = R^2 - 2Rr$
(e) $|OI_a|^2 = R^2 + 2Rr_a$
(f) $|II_a|^2 = 4R(r_a - r)$

Problem 59

MN is a diameter of a circle, $|MN| = 1$, A and B are points on the circles situated on one side of MN, C is a point on the other semicircle. Given: A is the midpoint of semicircle, $MB = \frac{3}{5}$, the length of the line segment formed by the intersection of the diameter MN with the chords AC and BC is equal to a. What is the greatest value of a?

Problem 60

Given a parallelogram $ABCD$. A straight line passing through the vertex C intersects the lines AB and AD at points K and L, respectively. The areas of the triangles KBC and CDL are equal to p and q, respectively. Find the area of the parallelogram $ABCD$.

Problem 61 [Loney]

Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point; prove that the distance of this point from either of their points of contact is $\sqrt{\frac{abc}{a+b+c}}$.

Problem 62 [Loney]

If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side BC externally, prove that its radius is $\frac{\Delta}{a}\tan^2\frac{A}{2}$.

Problem 63

Characterize all triangles ABC such that

$$AI_a : BI_b : CI_c = BC : CA : AB$$

where I_a; I_b, I_c are the vertices of the excentres corresponding to A, B, C respectively.

Problem 64

On the sides AB and BC of triangle ABC, points K and M are chosen such that the quadrilaterals $AKMC$ and $KBMN$ are cyclic, where
$N = AM \cap CK$. If these quadrilaterals have the same circumradii then find $\angle ABC$.

Problem 65 [AMM]

Let B' and C' be points on the sides AB and AC, respectively, of a given triangle ABC, and let P be a point on the segment $B'C'$. Determine the maximum value of

$$\frac{\min([BPB'],[CPC'])}{[ABC]}$$

where $[F]$ denotes the area of F.

Problem 66 [AMM]

For each point O on diameter AB of a circle, perform the following construction. Let the perpendicular to AB at O meet the circle at point P. Inscribe circles in the figures bounded by the circle and the lines AB and OP. Let R and S be the points at which the two incircles to the curvilinear triangles AOP and BOP are tangent to the diameter AB. Show that $\angle RPS$ is independent of the position of O.

Problem 67

Let E be a point inside the triangle ABC such that $\angle ABE = \angle ACE$. Let F and G be the feet of the perpendiculars from E to the internal and external bisectors, respectively, of angle BAC. Prove that the line FG passes through the mid-point of BC.

Problem 68

Let A, B, C and D be points on a circle with centre O and let P be the point of intersection of AC and BD. Let U and V be the circumcentres of triangles APB and CPD, respectively. Determine conditions on A, B, C and D that make O, U, P and V collinear and prove that, otherwise, quadrilateral $OUPV$ is a parallelogram.

Problem 69 [AMM]

Let R and r be the circumradius and inradius, respectively of triangle ABC.
(a) Show that ABC has a median whose length is at most $2R - r$.
(b) Show that ABC has an altitude whose length is at least $2R - r$.

Problem 70 [AMM]

Let $ABCD$ be a convex quadrilateral. Prove that if there is point P in the interior of $ABCD$ such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = 45^\circ$$

then $ABCD$ is a square.

Problem 71 [AMM]

Let M be any point in the interior of triangle ABC and let D, E and F be points on the perimeter such that AD, BE and CF are concurrent at M. Show that if triangles BMD, CME and AMF all have equal areas and equal perimeters then triangle ABC is equilateral.

Problem 72

The perpendiculars AD, BE, CF are produced to meet the circumscribed circle in X, Y, Z prove that

$$\tfrac{AX}{AD} + \tfrac{BY}{BE} + \tfrac{CZ}{CF} = 4$$

Problem 73 [AMM]

Given an odd positive integer n, let A_1, A_2,...,A_n be a regular polygon with circumcircle Γ. A circle O_i with radius r is drawn externally tangent to Γ at A_i for $i = 1, 2, \cdots, n$. Let P be any point on Γ between A_n and A_1. A circle C (with any radius) is drawn externally tangent to Γ at P. Let t_i be the length of the common external tangent between the circles C and O_i. Prove that $\sum_{i=1}^{n}(-1)^i t_i = 0$.

Problem 74 [INMO]

The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

Problem 75 [INMO]

In an acute-angled triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that

$$\tfrac{CD}{CE} = \tfrac{CA}{CB}, \tfrac{AE}{AF} = \tfrac{AB}{AC}, \tfrac{BF}{BD} = \tfrac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ABC.

Problem 76

In trapezoid $ABCD$, AB is parallel to CD and let E be the mid-point of BC. Suppose we can inscribe a circle in $ABED$ and also in $AECD$. Then if we denote $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$ prove that:

$$a + c = \tfrac{b}{3} + d \ , \ \tfrac{1}{a} + \tfrac{1}{c} = \tfrac{3}{b}.$$

Problem 77 [BMO]

Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that

$$\angle BPC + \angle BAC = 180^\circ$$

Problem 78 [Loney]

If D, E, F are the points of contact of the inscribed circle with the sides BC, CA, AB of a triangle, show that if the squares of AD, BE, CF are in arithmetic progression, then the sides of the triangle are in harmonic progression.

Problem 79 [Loney]

Through the angular points of a triangle straight lines making the same angle α with the opposite sides are drawn. Prove that the area of the triangle formed by them is to the area of the original triangle as $4\cos^2\alpha : 1$.

Problem 80 [Loney]

If D, E, F be the feet of the perpendiculars from ABC on the opposite sides and ρ, ρ_1, ρ_2, ρ_3 be the radii of the circles inscribed in the triangles DEF, AEF, BFD, CDE, prove that $r^3\rho = 2R\rho_1\rho_2\rho_3$.

Problem 81 [Loney]

A point O is situated on a circle of radius R and with centre O another circle of radius $\frac{3R}{2}$ is described. Inside the crescent-shaped area intercepted between these circles a circle of radius $\frac{R}{8}$ is placed. Show that if the small circle moves in contact with the original circle of radius R, the length of arc described by its centre in moving from one extreme position to the other is $\frac{7}{12}\pi R$.

Problem 82 [Crux]

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

In an integer triangle ABC, prove that the Gergonne point Γ bisects the Gergonne cevian AD if and only if b, c, $\frac{|3a-b-c|}{2}$ form a triangle where the measure of the angle between b and c is $\frac{\pi}{3}$.

Problem 83

Prove that the line which divides the perimeter and the area of a triangle in the same ratio passes through the centre of the incircle.

Problem 84

Let m_a, m_b, m_c and w_a, w_b, w_c denote, respectively, the lengths of the medians and angle bisectors of a triangle. Prove that

$$\sqrt{m_a} + \sqrt{m_b} + \sqrt{m_c} \geq \sqrt{w_a} + \sqrt{w_b} + \sqrt{w_c}.$$

Problem 85

A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths a, b, c and d of the sides of the quadrilateral satisfy the inequalities

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$

Problem 86 [Purdue Problem of the Week]

Given a triangle ABC, find a triangle $A_1B_1C_1$ so that

(1) $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$
(2) the centroids of triangles ABC and $A_1B_1C_1$ coincide

and subject to (1) and (2) triangle $A_1B_1C_1$ has minimal area.

Problem 87

Prove that if the perpendiculars dropped from the points A_1, B_1 and C_1 on the sides BC, CA and AB of the triangle ABC, respectively, intersect at the same point, then the perpendiculars dropped from the points A, B and C on the lines B_1C_1, C_1A_1 and A_1B_1 also intersect at one point.

Problem 88

Drawn through the intersection point M of medians of a triangle ABC is a straight line intersecting the sides AB and AC at points K and L, respectively, and the extension of the side BC at a point P (C lying between P and B). Prove that

$$\tfrac{1}{|MK|} = \tfrac{1}{|ML|} + \tfrac{1}{|MP|}$$

Problem 89

Prove that the area of the octagon formed by the lines joining the vertices of a parallelogram to the midpoints of the opposite sides is 1/6 of the area of the parallelogram.

Problem 90

Prove that if the altitude of a triangle is $\sqrt{2}$ times the radius of the circumscribed circle, then the straight line joining the feet of the perpendiculars dropped from the foot of this altitude on the sides enclosing it passes through the centre of the circumscribed circle.

Problem 91

Prove that the projections of the foot of the altitude of a triangle on the sides enclosing this altitude and on the two other altitudes lie on one straight line.

Problem 92

Let a, b, c and d be the sides of an inscribed quadrilateral (a is opposite to c), h_a, h_b, h_c and h_d the distances from the centre of the circumscribed circle to the corresponding sides. Prove that if the centre of the circle is inside the quadrilateral, then

$$ah_c + ch_a = bh_d + dh_b$$

Problem 93

Prove that three lines passing through the vertices of a triangle and bisecting its perimeter intersect at one point (called *Nagell's point*). Let M denote the centre of mass of the triangle, I the centre of the inscribed circle, S the centre of the circle inscribed in the triangle with vertices at the midpoints of the sides of the given triangle. Prove that the points N, M, I and S lie on a straight line and $|MN| = 2|IM|$, $|IS| = |SN|$.

Problem 94 [Loney]

If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle whose area is Δ and Δ_1, Δ_2 and Δ_3 the corresponding areas for the escribed circles prove that

$$\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta$$

Problem 95

Prove that the radius of the circle circumscribed about the triangle formed by the medians of an acute-angled triangle is greater than 5/6 of the radius of the circle circumscribed about the original triangle.

Problem 96

Let K denote the intersection point of the diagonals of a convex quadrilateral $ABCD$, L a point on the side AD, N a point on the side BC, M a point on the diagonal AC, KL and MN being parallel to AB, LM parallel to DC. Prove that $KLMN$ is a parallelogram and its area is less than 8/27 of the area of the quadrilateral $ABCD$ (*Hattori's Theorem*).

Problem 97

Two triangles have a common side. Prove that the distance between the centres of the circles inscribed in them is less than the distance between their non-coincident vertices (*Zalgaller's problem*).

Problem 98

Prove that the sum of the distances from a point inside a triangle to its vertices is not less than $6r$, where r is the radius of the inscribed circle.

Problem 99

Given a triangle. The triangle formed by the feet of its angle bisectors is isosceles. Is the given triangle isosceles?

Problem 100

Prove that the perpendicular bisectors of the line segments joining the intersection points of the altitudes to the centres of the circumscribed circles of the four triangles formed by four arbitrary straight lines in the plane intersect at one point (*Herwey's point*).

Problem 101 [Crux]

Given triangle ABC with $AB < AC$. Let I be the incentre and M be the mid-point of BC. The line MI meets AB and AC at P and Q respectively. A tangent to the incircle meets sides AB and AC at D and E respectively. Prove that

$$\tfrac{AP}{BD} + \tfrac{AQ}{CE} = \tfrac{PQ}{2MI}$$

Problem 102 [Crux]

Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

Problem 103

Prove that the sum of the squares of the distances from an arbitrary point in the plane to the sides of a triangle takes on the least value for such a point inside the triangle whose distances to the corresponding sides are proportional to these sides. Prove also that this point is the intersection point of the *symmedians* of the given triangle (*Lemoine's Point*).

Problem 104

Given a triangle ABC. AA_1, BB_1 and CC_1 are its altitudes. Prove that Euler's lines of the triangles AB_1C_1, A_1BC_1 and A_1B_1C intersect at a point P of the nine-point circles such that one of the line segments PA_1, PB_1, PC_1 is equal to sum of the other two (*Thebault's problem*).

Problem 105

Let M be an arbitrary point in the plane and G, the centroid of triangle ABC. Prove that

$$3|MG|^2 = |MA|^2 + |MB|^2 + |MC|^2 - \tfrac{1}{3}(|AB|^2 + |BC|^2 + |CA|^2)$$

(*Leibnitz's Theorem*)

Problem 106

Let ABC be a regular triangle with side a and M some point in the plane found at a distance d from the centre of the triangle ABC. Prove that the area of the triangle whose sides are equal to the line segments MA, MB and MC can be expressed by the formula

$$S = \tfrac{\sqrt{3}}{12}|a^2 - 3d^2|$$

Problem 107 [Todhunter]

If Q be any point in the plane of a triangle and R_1, R_2, R_3 the radii of the circles about QBC, QCA, QAB prove that

$$(\tfrac{a}{R_1} + \tfrac{b}{R_2} + \tfrac{c}{R_3})(-\tfrac{a}{R_1} + \tfrac{b}{R_2} + \tfrac{c}{R_3})(\tfrac{a}{R_1} - \tfrac{b}{R_2} + \tfrac{c}{R_3})(\tfrac{a}{R_1} + \tfrac{b}{R_2} - \tfrac{c}{R_3}) = \tfrac{a^2b^2c^2}{R_1^2R_2^2R_3^2}$$

Problem 108 [Mathematical Gazette]

$PQRS$ is a quadrilateral inscribed in a circle with centre O. E is the intersection of the diagonals PR and QS. Let F be theintersection of PQ and RS and G the intersection of PS and QR. The circle on FG as diameter meets OE at X. The perpendicular bisectors of SX and PX meet at A and B, C, D are defined similarly by cyclic change of letters.

(i) Prove that the tangents at P and Q and the line OB are concurrent.
(ii) Prove that PQ, AC, SR, FG are concurrent at F.
(iii)Prove that AD, BC, FG are concurrent.

Problem 109 [AMM]

Let X, Y and Z be three distinct points in the interior of an equilateral triangle ABC. Let α, β and γ be positive numbers adding up to $\frac{\pi}{3}$ with the property that $\angle XBA = \angle YAB{=}\alpha$, $\angle YCB = \angle ZBC = \beta$ and $\angle ZAC = \angle XCA = \gamma$. Find the angles of triangle XYZ in terms of α, β and γ.

Problem 110 [Todhunter]

If O be the centre of the circle inscribed in a triangle ABC and r_a, r_b, r_c the radii of the circles inscribed in the triangles OBC, OCA, OAB, show that

$$\tfrac{a}{r_a} + \tfrac{b}{r_b} + \tfrac{c}{r_c} = 2(\cot(\tfrac{A}{4}) + \cot(\tfrac{B}{4}) + \cot(\tfrac{C}{4}))$$

Problem 111 [BMO]

Let P be an internal point of triangle ABC and let α, β, γ be defined by

$$\begin{aligned} \alpha &= \angle BPC - \angle BAC \\ \beta &= \angle CPA - \angle CBA \\ \gamma &= \angle APB - \angle ACB \end{aligned}$$

Prove that

$$PA\tfrac{\sin(\angle BAC)}{\sin(\alpha)} = PB\tfrac{\sin(\angle CBA)}{\sin(\beta)} = PC\tfrac{\sin(\angle ACB)}{\sin(\gamma)}$$

Problem 112

Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, prove that

$$\tfrac{r_1}{r-r_1} + \tfrac{r_2}{r-r_2} + \tfrac{r_3}{r-r_3} = \tfrac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$$

Problem 113 [Loney]

Given the product p of the sines of the angles of a triangle and the product q of the cosines, show that the tangents of the angles are the roots of the equation

$$qx^3 - px^2 + (1+q)x - p = 0$$

Problem 114

The altitude of a right triangle drawn to the hypotenuse is equal to h. Prove that the vertices of the acute angles of the triangle and the projections of the foot of the altitude on the legs all lie on the same circle. Determine the length of the chord cut by this circle on the line containing the altitude and the segments of the chord into which it is divided by the hypotenuse.

Problem 115

Four villages are situated at the vertices of a square of side 2 Km. The villages are connected by roads so that each village is joined to any other. Is it possible for the total length of the roads to be less than 5.5 Km?

Problem 116

Prove that if the lengths of the internal angle bisectors of a triangle are less than 1, then its area is less than $\frac{\sqrt{3}}{3}$.

Problem 117

Given a convex quadrilateral ABCD circumscribed about a circle of diameter 1. Inside ABCD, there is a point M such that

$$|MA|^2 + |MB|^2 + |MC|^2 + |MD|^2 = 2.$$

Find the area of ABCD.

Problem 118

The circle inscribed in a triangle ABC divides the median BM into three equal parts. Find the ratio $|BC| : |CA| : |AB|$.

Problem 119

Prove that if the centres of the squares constructed externally on the sides of a given triangle serve as the vertices of the triangle whose area is twice the area of the given triangle, then the centres of the squares constructed internally on the sides of the triangle lie on a straight line.

Problem 120

Prove that the median drawn to the largest side of a triangle forms with the sides enclosing this median angles each of which is not less than half the smallest angle of the triangle.

Problem 121

Three squares $BCDE$, $ACFG$ and $BAHK$ are constructed externally on the sides BC, CA and AB of a triangle ABC. Let $FCDQ$ and $EBKP$ be parallelograms. Prove that the triangle APQ is a right-angled isosceles triangle.

Problem 122

Three points are given in a plane. Through these points three lines are drawn forming a regular triangle. Find the locus of centres of those triangles.

Problem 123

Drawn in an inscribed polygon are non-intersecting diagonals separating the polygon into triangles. Prove that the sum of the radii of the circles inscribed in those triangles is independent of the way the diagonals are drawn.

Problem 124

A polygon is circumscribed about a circle. Let l be an arbitrary line touching the circle and coinciding with no side of the polygon. Prove that the ratio of the product of the distances from the verices of the polygon to the line l to the product of the distances from the points of tangency of the sides of the polygon with the circle to l is independent of the position of the line l.

Problem 125 [Loney]

If $2\phi_1$, $2\phi_2$, $2\phi_3$ are the angles subtended by the circle escribed to the side a of a triangle at the centres of the inscribed circle and the other two escribed circles, prove that

$$\sin(\phi_1)\sin(\phi_2)\sin(\phi_3) = \frac{r_1^2}{16R^2}$$

Problem 126

If from any point in the plane of a regular polygon perpendiculars are drawn on the sides, show that the sum of the squares of these perpendiculars is equal to the sum of the squares on the lines joining the feet of the perpendiculars with the centre of the polygon.

Problem 127 [Loney]

The three medians of a triangle ABC make angles α, β, γ with each other. Prove that

$$\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0$$

Problem 128 [Loney]

A railway curve, in the shape of a quadrant of a circle, has n telegraph posts at its ends and at equal distances along the curve. A man stationed at a point on one of the extreme radii produced sees the pth and qth posts from the end nearest him in a straight line. Show that the radius of the curve is $\frac{a\cos(p+q)\phi}{2\sin(p\phi)\sin(q\phi)}$, where $\phi = \frac{\pi}{4(n-1)}$, and a is the distance from the man to the nearest end of the curve.

Problem 129

Let D be an arbitrary point on the side BC of a triangle ABC. Let E and F be points on the sides AC and AB such that DE is parallel to AB and DF is parallel to AC. A circle passing through D, E and F intersects for the second time BC, CA and AB at points D_1, E_1 and F_1, respectively. Let M and N denote the intersection points of DE and F_1D_1, DF and D_1E_1, respectively. Prove that M and N lie on the *symedian* emanating from the vertex A. If D coincides with the foot of the *symedian*, then the circle passing through D, E and F touches the side BC.(This circle is called *Tucker's Circle.*)

Problem 130

Let $ABCD$ be a cyclic quadrilateral. The diagonal AC is equal to a and forms angles α and β with the sides AB and AD, respectively. Prove that the magnitude of the area of the quadrilateral lies between $\frac{a^2\sin(\alpha+\beta)\sin\beta}{2\sin\alpha}$ and $\frac{a^2\sin(\alpha+\beta)\sin\alpha}{2\sin\beta}$

Problem 131

A triangle has sides of lengths a, b, c and respective altitudes of lengths h_a,h_b, h_c. If $a \geq b \geq c$ show that $a + h_a \geq b + h_b \geq c + h_c$.

Problem 132 [Crux]

Given a right-angled triangle ABC with $\angle BAC = 90°$. Let I be the incentre and let D and E be the intersections of BI and CI with AC and AB respectively. Prove that

$$\frac{|BI|^2+||ID|^2}{|IC|^2+|IE|^2} = \frac{|AB|^2}{|AC|^2}$$

Problem 133 [Hobson]

Straight lines whose lengths are successively proportional to $1, 2, 3, \cdots, n$ form a rectilineal figure whose exterior angles are each equal to $\frac{2\pi}{n}$; if a polygon be formed by joining the extremities of the first and last lines, show that its area is

$$\frac{n(n+1)(2n+1)}{24}\cot(\tfrac{\pi}{n}) + \frac{n}{16}\cot(\tfrac{\pi}{n})\csc^2(\tfrac{\pi}{n})$$

Problem 134

An arc AB of a circle is divided into three equal parts by the points C and D (C is nearest to A). When rotated about the point A through an angle of $\frac{\pi}{3}$, the points B, C and D go into points B_1, C_1 and D_1. F is the point of intersection of the straight lines AB_1 and DC_1; E is a point on the bisector of the angle B_1BA such that $|BD| = |DE|$. Prove that the triangle CEF is regular (*Finlay's theorem*).

Problem 135

In a triangle ABC, a point D is taken on the side AC. Let O_1 be the centre of the circle touching the line segments AD, BD and the circle circumscribed about the triangle ABC and let O_2 be the centre of the circle touching the line segments CD, BD and the circumscribed circle. Prove that the line O_1O_2 passes through the centre O of the circle inscribed in the triangle ABC and $|O_1O| : |OO_2| = \tan^2(\phi/2)$, where $\phi = \angle BDA$ (*Thebault's theorem*).

Problem 136

Prove the following statement. If there is an n-gon inscribed in a circle α and circumscribed about another circle β, then there are infinitely many n-gons inscribed in the circle α and circumscribed about the circle β and any point of the circle can be taken as one of the vertices of such an n-gon (*Poncelet's theorem*).

Problem 137 [Loney]

A point is taken in the plane of a regular polygon of n sides at a distance c from the centre and on the line joining the centre to a vertex, and the radius of the inscribed circle is r. Show that the product of the distances of the point from the sides of the polygon is

$$\frac{c^n}{2^{n-2}} \cos^2(\tfrac{n}{2} \cos^{-1} \tfrac{r}{c}) \text{ if } c > r \text{ and}$$
$$\frac{c^n}{2^{n-2}} \cosh^2(\tfrac{n}{2} \cosh^{-1} \tfrac{r}{c}) \text{ if } c < r$$

Problem 138 [Loney]

An infinite straight line is divided by an infinite number of points into portions each of length a. Prove that the sum of the fourth powers of the reciprocals of the distances of a point O on the line from all the points of division is

$$\frac{\pi^4}{3a^4}(3 \csc^4 \tfrac{\pi b}{a} - 2 \csc^2 \tfrac{\pi b}{a})$$

Problem 139 [Loney]

If $\rho_1, \rho_2, \cdots, \rho_n$ be the distances of the vertices of a regular polygon of n sides from any point P in its plane, prove that

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \cdots + \frac{1}{\rho_n^2} = \frac{n}{r^2-a^2} \frac{r^{2n}-a^{2n}}{r^{2n}-2a^n r^n \cos(n\theta)+a^{2n}}$$

where a is the radius of the circumcircle of the polygon, r is the distance of P from its centre O and θ is the angle that OP makes with the radius to any angular point of the polygon.

Problem 140

Given an angle with vertex A and a circle inscribed in it. An arbitrary straight line touching the given circle intersects the sides of the angle at points B and C. Prove that the circle circumscribed about the triangle ABC touches the circle inscribed in the given angle.

Problem 141

Let $ABCDEF$ be an inscribed hexagon in which $|AB| = |CD| = |EF| = R$, where R is the radius of the circumscribed circle, O its centre. Prove that the points of pairwise intersections of the circles circumscribed about the triangles BOC, DOE, FOA, distinct from O, serve as the vertices of an equilateral triangle with side R.

Problem 142

The diagonals of an inscribed quadrilateral are mutually perpendicular. Prove that the midpoints of its sides and the feet of the perpendiculars dropped from the point of intersection of the diagonals on the sides lie on a circle. Find the radius of that circle if the radius of the given circle is R and the distance from its centre to the point of intersection of the diagonals of the quadrilateral is d.

Problem 143

Prove that if a quadrialateral is both inscribed in a circle and circumscribed about a circle of radius r, the distance between the centres of those circles being d, then the relationship

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}$$

is true.

Problem 144

Let $ABCD$ be a convex quadrilateral. Consider four circles each of which touches three sides of this quadrilateral.

(a) Prove that the centres of these circles lie on one circle.
(b) Let r_1, r_2,r_3 and r_4 denote the radii of these circles (r_1 does not touch the side DC, r_2 the side DA, r_3 the side AB and r_4 the side BC). Prove that

$$\frac{|AB|}{r_1} + \frac{|CD|}{r_3} = \frac{|BC|}{r_2} + \frac{|AD|}{r_4}$$

Problem 145

The sides of a square is equal to a and the products of the distances from the opposite vertices to a line l are equal to each other. Find the distance from the centre of the square to the line l if it is known that neither of the sides of the square is parallel to l.

Problem 146

Find the angles of a triangle if the distance between the centre of the circumcircle and the intersection point of the altitudes is one-half the length of the largest side and equals the smallest side.

Problem 147

Prove that for the perpendiculars dropped from the points A_1, B_1 and C_1 on the sides BC, CA and AB of a triangle ABC to intersect at the same point, it is necessary and sufficient that

$$|A_1B|^2 - |BC_1|^2 + |C_1A|^2 - |AB_1|^2 + |B_1C|^2 - |CA_1|^2 = 0.$$

Problem 148

Each of the sides of a convex quadrilateral is divided into $(2n+1)$ equal parts. The division points on the opposite sides are joined correspondingly. Prove that the area of the central quadrilateral amounts to $1/(2n+1)^2$ of the area of the entire quadrilateral.

Problem 149

A straight line intersects the sides AB, BC and the extension of the side AC of a triangle ABC at points D, E and F, respectively. Prove that the midpoints of the line segments DC, AE and BF lie on a straight line (*Gaussian line*).

Problem 150

Given two intersecting circles. Find the locus of centres of rectangles with vertices lying on these circles.

Problem 151

An equilateral triangle is inscribed in a circle. Find the locus of intersection points of the altitudes of all possible triangles inscribed in the circle if two sides of the triangles are parallel to those of the given one.

Problem 152

Given two circles touching each other internally at a point A. A tangent to the smaller circle intersects the larger one at points B and C. Find the locus of centres of circles inscribed in triangles ABC.

Problem 153 [Loney]

Two circles, the sum of whose radii is a, are placed in the same plane with their centres at a distance $2a$ and an endless string is fully stretched so as partly to surround the circles and to cross between them. Show that the length of the string is $(\frac{4\pi}{3} + 2\sqrt{3})a$.

Problem 154 [Loney]

If p, q, r are the perpendiculars from the vertices of a triangle upon any straight line meeting the sides externally in D, E, F, prove that

$$a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)=4\Delta^2.$$

Problem 155 [Loney]

A regular polygon is inscribed in a circle; show that the arithmetic mean of the squares of the distances of its corners from any point (not necessarily in its plane) is equal to the arithmetic mean of the sum of the squares of the longest and shortest distances of the point from the circle.

Problem 156

In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE=CA$ if and only if $DE=AB$.

Problem 157 [BMO]

Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.

Problem 158

Given a triangle ABC and a point M. A straight line passing through the point M intersects the lines AB, BC and CA at points C_1, A_1 and B_1, respectively. The lines AM, BM and CM intersect the circle circumscribed about the triangle ABC at points A_2, B_2 and C_2, respectively. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 intersect at a point situated on the circle circumscribed about the triangle ABC.

Problem 159 [AMM]

Let P be a point in the interior of triangle ABC and let r_1, r_2, r_3 denote the distances from P to the sides of the triangle with lengths a_1, a_2, a_3, respectively. Let R be the circumradius of ABC and let $0<a<1$ be a real number. Let $b=2a/(1-a)$. Prove that

$$r_1^a+r_2^a+r_3^a\le \tfrac{1}{(2R)^a}(a_1^b+a_2^b+a_3^b)^{1-a}$$

Problem 160 [AMM]

Let K be the circumcentre and G the centroid of a triangle with side lengths a, b, c and area Δ.

(a) Show that the distance d from K to G satisfies

$$12\Delta d = a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$

(b) Show that $d(< \frac{abc}{12\Delta}, = \frac{abc}{12\Delta}, > \frac{abc}{12\Delta})$ when the triangle is respectively (acute, right-angled,obtuse).

Problem 161 [AMM]

Let ABC be an acute-triangle and let P be a point in its interior. Denote by a, b, c the lengths of the triangle's sides, by d_a, d_b, d_c the distances from P to the triangle's sides, and by R_a, R_b, R_c the distances from P to the vertices A, B, C respectively. Show that

$$d_a^2+d_b^2+d_c^2 \geq R_a^2 sin^2(A/2)+R_b^2 sin^2(B/2)+R_c^2 sin^2(C/2) \geq (d_a+d_b+d_c)^2/3$$

Problem 162 [Loney]

$A_1A_2\cdots A_n$ is a regular polygon of n sides which is inscribed in a circle, whose radius is a and whose centre is O; prove that the product of the distances of its angular points from a straight line at right angles to OA and at a distance $b(> a)$ from the centre is

$$b^n[\cos^n(\tfrac{1}{2}\sin^{-1}\tfrac{a}{b}) - \sin^n(\tfrac{1}{2}\sin^{-1}\tfrac{a}{b})]^2$$

Problem 163 [Loney]

The radii of an infinite series of concentric circles are $a, \frac{a}{2}, \frac{a}{3}\cdots$. From a point at a distance $c(> a)$ from their common centre a tangent is drawn to each circle. Prove that

$$\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\cdots = \sqrt{\tfrac{c}{\pi a}\sin\tfrac{\pi a}{c}}$$

where $\theta_1, \theta_2, \theta_3, \cdots$ are the angles that the tangents subtend at the common centre.

Problem 164 [Crux]

Construct equilateral triangles $A'BC$, $B'CA$, $C'AB$ exterior to triangle ABC and take points P, Q, R on AA', BB', CC', respectively, such that

$$\tfrac{AP}{AA'} + \tfrac{BQ}{BB'} + \tfrac{CR}{CC'} = 1.$$

Prove that ΔPQR is equilateral.

Problem 165 [Crux]

Given a triangle ABC, we take variable points P on segment AB and Q on segment AC. CP meets BQ in T. Where should P and Q be located so that area of ΔPQT is maximized?

Problem 166 [Crux]

Let ABC be a triangle and A_1, B_1, C_1 the common points of the inscribed circle with the sides BC, CA, AB, respectively. We denote the length of the arc B_1C_1 (not containing A_1) of the incircle by S_a, and similarly define S_b and S_c. Prove that

$$\tfrac{a}{S_a} + \tfrac{b}{S_b} + \tfrac{c}{S_c} \geq \tfrac{9\sqrt{3}}{\pi}$$

Problem 167 [AMM]

A *cevian* of a triangle is a line segment that joins a vertex to the line containing the opposite side. An *equicevian* point of a triangle ABC is a point P (not necessarily inside the triangle) such that the cevians on the lines AP, BP and CP have equal length. Let SBC be an equilateral triangle and let A be chosen in the interior of SBC on the altitude dropped from S.

(a) Show that ABC has two equicevian points.
(b) Show that the common length of the cevians through either of the equicevian points is constant, independent of the choice of A.
(c) Show that the equicevian points divide the cevian through A in a constant ratio, independent of the choice of A.
(d) Find the locus of the equicevian points as A varies.
(e) Let S' be the reflection of S in the line BC. Show that (a), (b) and (c) hold if A moves on any ellipse with S and S' as its foci. Find the locus of the equicevian points as A varies on the ellipse.

Problem 168 [Crux]

Let $ABCD$ be a trapezoid with AD parallel to BC. M, N, P, Q, O are the midpoints of AB, CD, AC, BD, MN, respectively. Circles m, n, p, q all pass through O and are tangent to AB at M, to CD at N, to AC at P, and to BD at Q, respectively. Prove that the centres of m, n, p, q are collinear.

Problem 169 [AMM]

Let ABC be an equilateral triangle inscribed in a circle with radius 1 unit. Suppose P is a point inside the triangle. Prove that $|PA||PB||PC| \leq \frac{32}{27}$. Generalize the result to a regular polygon of n sides. (*Erdos*)

Problem 170

Given two circles. Find the locus of points M such that the ratio of the lengths of the tangents drawn from M to the given circles is a constant k.

Problem 171

In a quadrilateral $ABCD$, P is the intersection point of BC and AD, Q that of CA and BD and R that of AB and CD. Prove that the intersection points of BC and QR, CA and RP, AB and PQ are collinear.

Problem 172

Given two squares whose sides are respectively parallel. Determine the locus of points M such that for any point P of the first square there is a point Q of the second one such that the triangle MPQ is equilateral. Let the side of the first square be a and that of the second square be b. For what relationship between a and b is the desired locus non-empty?

Problem 173 [AMM]

Let $C_1C_2\ldots C_n$ be a regular n-gon and let $C_{n+1} = C_1$. Let O be the inscribed circle. For $1 \leq k \leq n$, let T_k be the point at which O is tangent to C_kC_{k+1}. Let X be a point on the open arc $(T_{n-1}T_n)$ and let Y be a point other than X on O. For $1 \leq i \leq n$, let B_i be the second point at which the line XC_i meets O and let $p_i = |XB_i||XC_i|$. Let M_i be the mid-point of chord T_iT_{i+1} and let N_i be the second point, other than Y, at which YM_i meets O. Let $q_i = |YM_i||YN_i|$. Prove that $\sum\limits_{i=1}^{n} q_i = (\sum\limits_{i=1}^{n} p_i) - p_n$.

Problem 174 [AMM]

Let ABC be an acute triangle, with semi-perimeter p and with inscribed and circumscribed circles of radius r and R, respectively.

(a) Show that ABC has a median of length at most $p/\sqrt{3}$.
(b) Show that ABC has a median of length at most $R + r$.
(c) Show that ABC has an altitude of length at least $R + r$.

Problem 175 [RMO, India]

Let ABC be an acute-angled triangle and CD be the altitude through C. If $AB = 8$ and $CD = 6$ find the distance between the mid-points of AD and BC.

Problem 176 [RMO, India]

Let $ABCD$ be a rectangle with $AB = a$ and $BC = b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 is the radius of the circle passing through B and C and touching AD. Show that

$$r_1 + r_2 \geq \tfrac{5}{8}(a + b)$$

Problem 177 [INMO]

Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meet the circles C_1 and C_2 in A and B respectively. Let Y be the mid-point of AB and QY meet the circles C_1 and C_2 in X and Z respectively. Show that Y is also the mid-point of XZ.

Problem 178 [INMO]

In a triangle ABC angle A is twice angle B. Show that $a^2 = b(b + c)$.

Problem 179 [INMO]

The diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect at P. Let O be the circumcentre of triangle APB and H be the orthocentre of triangle CPD. Show that the points H, P, O are collinear.

Problem 180 [INMO]

Let ABC be a triangle in a plane Σ. Find the set of all points P (distinct from A, B, C) in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP have the same radii.

Problem 181 [INMO]

Let ABC be a triangle right-angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB and AC and the circle S internally. Further let S_2 be the circle touching the lines AB and AC and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2 respectively, show that

$$r_1.r_2 = 4\text{Area}(ABC)$$

Problem 182 [INMO]

Show that there exists a convex hexagon in the plane such that

(a) all its interior angles are equal.
(b) all its sides are 1,2,3,4,5,6 in some order.

Problem 183 [INMO]

Let G be the centroid of a triangle ABC in which the angle C is obtuse and AD and CF be the medians from A and C respectively onto the sides BC and AB. If the four points B, D, G and F are concyclic, show that

$$\tfrac{AC}{BC} > \sqrt{2}$$

If further P is a point on the line BG extended such that $AGCP$ is a parallelogram, show that the triangle ABC and GAP are similar.

Problem 184 [INMO]

A circle passes through a vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units, find the area of the rectangle $ABCD$.

Problem 185 [RMO, India]

In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that

(a) $AD.BC \geq AB.CD$
(b) $AD + BC \geq AB + CD$

Problem 186 [RMO, India]

In the triangle ABC, the incircle touches the sides BC, CA and AB respectively at D, E and F. If the radius of the incircle is 4 units and if BD, CE and AF are consecutive integers , find the sides of the triangle ABC.

Problem 187 [RMO, India]

Let $ABCD$ be a square and M, N points on sides AB, BC, respectively, such that $\angle MDN = 45°$. If R is the midpoint of MN show that $RP = RQ$ where P, Q are the points of intersection of AC with the lines MD and ND.

Problem 188 [RMO, India]

Let AC and BD be two chords of a circle with centre O such that they intersect at right angles inside the circle at the point M. Suppose K and L are the mid-points of the chord AB and CD respectively. Prove that $OKML$ is a parallelogram.

Problem 189 [AMM]

Let P be a convex n-gon inscribed in a circle O and let Δ be a triangulation of P without new vertices. Compute the sum of the squares of distances from the centre O to the incentres of the triangles of Δ and show that this sum is independent of Δ.

Problem 190 [AMM]

Let T_1 and T_2 be triangles such that for $i \in 1, 2$, triangle T_i has circumradius R_i, inradius r_i and side lengths a_i, b_i and c_i. Show that

$$8R_1R_2 + 4r_1r_2 \geq a_1a_2 + b_1b_2 + c_1c_2 \geq 36r_1r_2$$

and determine when equality holds.

Problem 191 [AMM]

Let ABC be an acute triangle. T the mid-point of arc BC of the circle circumscribing ABC. Let G and K be the projections of A and T respectively on BC, let H and L be the projections of B and C on AT and let E be the mid-point of AB. Prove that:

(a) $KH||AC$, $GL||BT$, $GH||TC$, $LK||AB$.
(b)G, H, K and L are concyclic.
(c) The centre of the circle through G, H and K lies on the Euler circle of ABC.

Problem 192 [AMM]

A trapezoid $ABRS$ with $AB||RS$ is inscribed in a non-circular ellipse E with axes of symmetry a and b. The points A and B are reflected through a to points P and Q on E.

(a) Show that P, Q, R and S are concyclic.
(b) Show that if the line PQ intersects the line RS at T, then the angle bisector of $\angle PTR$ is parallel to a.

Problem 193 [RMO, India]

$ABCD$ is a cyclic quadrilateral with $AC \perp BD$; AC meets BD at E. Prove that $EA^2 + EB^2 + EC^2 + ED^2 = 4R^2$.

Problem 194 [RMO, India]

$ABCD$ is a cyclic quadrilateral; x, y, z are the distances of A from the lines BD, BC, CD respectively. Prove that

$$\tfrac{BD}{x} = \tfrac{BC}{y} + \tfrac{CD}{z}$$

Problem 195 [RMO, India]

$ABCD$ is a quadrilateral and P, Q are mid-points of CD, AB respectively. Let AP, DQ meet at X and BP, CQ meet at Y. Prove that

$$\text{area}(ADX)+\text{area}(BCY) = \text{area}(PXQY).$$

Problem 196 [RMO, India]

The cyclic octagon $ABCDEFGH$ has sides a, a, a, a, b, b, b, b respectively. Find the radius of the circle that circumscribes $ABCDEFGH$ in terms of a and b.

Problem 197 [AMM]

Prove that in an acute triangle with angles A, B and C

$$\frac{(1-\cos A)(1-\cos B)(1-\cos C)}{\cos A\cos B\cos C} \geq \frac{8(\tan A+\tan B+\tan C)^3}{27(\tan A+\tan B)(\tan C+\tan A)(\tan B+\tan C)}$$

Problem 198 [Mathscope, Vietnam]

In a triangle ABC, denote by l_a, l_b, l_c the internal angle bisectors, m_a, m_b, m_c the medians and h_a, h_b, h_c the altitudes to the sides a, b, c of the triangle. Prove that

$$\frac{m_a}{l_b+h_b} + \frac{m_b}{h_c+l_c} + \frac{m_c}{l_a+h_a} \geq \frac{3}{2}$$

Problem 199 [Mathscope, Vietnam]

Let AM, BN, CP be the medians of triangle ABC. Prove that if the radius of the incircles of triangles BCN, CAP and ABM are equal in length, then ABC is an equilateral triangle.

Problem 200 [Mathscope, Vietnam]

Given a triangle with incentre I, let l be a variable line passing through I. Let l intersect the ray CB, sides AC, AB at M, N, P respectively. Prove that the value of

$$\frac{AB}{PA.PB} + \frac{AC}{NA.NC} - \frac{BC}{MB.MC}$$

is independent of the choice of l.

Problem 201 [Mathscope, Vietnam]

Let I be the incentre of triangle ABC and let m_a, m_b, m_c be the lengths of the medians from vertices A, B and C, respectively. Prove that

$$\frac{IA^2}{m_a^2} + \frac{IB^2}{m_b^2} + \frac{IC^2}{m_c^2} \leq \frac{3}{4}$$

Problem 202 [Mathscope, Vietnam]

Let R and r be the circumradius and inradius of triangle ABC; the incircle touches the sides of the triangle at three points which form a triangle of perimeter p. Suppose that q is the perimeter of triangle ABC. Prove that

$$\tfrac{r}{R} \leq \tfrac{p}{q} \leq \tfrac{1}{2}$$

Problem 203 [AMM]

Let a, b and c be the lengths of the sides of a triangle and let R and r be the circumradius and inradius of that triangle, respectively. Show that

$$\tfrac{R}{2r} \geq \exp(\tfrac{(a-b)^2}{2c^2} + \tfrac{(b-c)^2}{2a^2} + \tfrac{(c-a)^2}{2b^2})$$

Problem 204 [AMM]

Consider an acute triangle with sides of lengths a, b and c and with an inradius of r and circumradius of R. Show that

$$\tfrac{r}{R} \leq \tfrac{\sqrt{2(2a^2-(b-c)^2)(2b^2-(c-a)^2)(2c^2-(a-b)^2)}}{(a+b)(b+c)(c+a)}.$$

Problem 205 [AMM]

Let a, b and c be the lengths of the sides of a triangle, and let R and r denote the circumradius and inradius of the triangle. Show that

$$\tfrac{R}{2r} \geq (\tfrac{4a^2}{4a^2-(b-c)^2}\tfrac{4b^2}{4b^2-(c-a)^2}\tfrac{4c^2}{4c^2-(a-b)^2})^2$$

Problem 206 [AMM]

Let ABC be a triangle with sides a, b and c all different, and corresponding angles α, β and γ. Show that

(a) $(a+b)\cot(\beta + \frac{\gamma}{2}) + (b+c)\cot(\gamma + \frac{\alpha}{2}) + (c+a)\cot(\alpha + \frac{\beta}{2}) = 0$.

(b) $(a-b)\tan(\beta + \frac{\gamma}{2}) + (b-c)\tan(\gamma + \frac{\alpha}{2}) + (c-a)\tan(\alpha + \frac{\beta}{2}) = 4(R+r)$.

Problem 207 [AMM]

Let r, R and s be the radii of the incircle, circumcircle and semi-perimeter of a triangle. Prove that

$$\sqrt[3]{r^2 s} \leq \sqrt{\tfrac{r^2+4Rr}{3}} \leq \tfrac{s}{3}$$

Problem 208 [AMM]

Let a, b and c be the lengths of the sides of a nondegenerate triangle, let $p = (1/2)(a+b+c)$, and let r and R be the inradius and circumradius of the triangle, respectively. Show that

$$\tfrac{a}{2}(\tfrac{4r-R}{R}) \leq \sqrt{(p-b)(p-c)} \leq \tfrac{a}{2}.$$