# Determinantal divisor rank of an integral matrix 

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Abstract: We define the determinantal divisor rank of an integral matrix to be the number of invariant factors which equal 1. Some properties of the determinantal divisor rank are proved, which are analogous to known properties of the usual rank. These include the Frobenious inequality for the rank of a product and a relation between the rank of a submatrix of a matrix and that of its complementary submatrix in the inverse or a generalized inverse of the matrix.

## 1 Introduction and Preliminaries

We work with matrices over the integers $Z$ but all the results have obvious generalizations to matrices over a principal ideal ring. Let $M_{m, n}(Z)$ be the set of $m \times n$ matrices over $Z$.

We assume familiarity with basic facts concerning integral matrices, see, for example, [6]. For $A \in M_{m, n}(Z)$ we denote the $j$ th determinantal divisor (which by definition is the g.c.d. of the $j \times j$ minors) of $A$ by $d_{j}(A), j=$ $1, \ldots, \min \{m, n\}$, and the $j$ th invariant factor by $s_{j}(A), j=1, \ldots, \operatorname{rank} A$. Recall that $s_{j}(A)=\frac{d_{j}(A)}{d_{j-1}(A)}, j=1, \ldots$, rank $A$, where we set $d_{0}(A)=1$.

If $A \in M_{m, n}(Z)$ then we define the determinantal divisor rank of $A$, denoted by $\delta(A)$, as the maximum integer $k$ such that $d_{k}(A)=1$ if such an integer exists and zero otherwise. Equivalently, $\delta(A)$ is the number of invariant factors of $A$ which equal 1 . This definition is partly motivated by the following result
obtained recently by Zhan [11]. We have modified the notation for consistency.

Theorem 1 Let $r, s, n$ be positive integers with $r, s \leq n$. The matrix $B \in$ $M_{r, s}(Z)$ is a submatrix of some unimodular matrix of order $n$ if and only if $\delta(B) \geq r+s-n$.

Theorem 1 has an uncanny similarity to the following well-known result: An $r \times s$ matrix $B$ over a field is a submatrix of a nonsingular matrix of order $n$ if and only if the rank of $B$ is at least $r+s-n$. We show that this similarity extends to some other results such as the Frobenius inequality for matrix product and a result of Fiedler and Markham [4] and Gustafson [5], and its extension to generalized inverses obtained in [1].

We now introduce some more definitions. If $A$ and $G$ are matrices of order $m \times n$ and $n \times m$ respectively, then $G$ is called a generalized inverse of $A$ if $A G A=A$. We say that $G$ is a reflexive generalized inverse of $A$ if $A G A=A$ and $G A G=G$. For background material on generalized inverses see $[2,3]$.

Generalized inverses of integral matrices have been extensively studied. We say that an integral matrix is regular if it admits an integral generalized inverse. It is well-known (see, for example, [9]) that an integral matrix is regular if and only if all its invariant factors are equal to 1 . Thus an integral matrix is regular if and only if its rank coincides with the determinantal divisor rank.

Let $A \in M_{m, n}(Z)$ and suppose rank $A=r>0$. We say that $A$ admits a rank factorization if there exist matrices $P \in M_{m, r}(Z)$ and $Q \in M_{r, n}(Z)$ such that $P$ has a left-inverse in $M_{r, m}(Z), Q$ has a right-inverse in $M_{n, r}(Z)$ and $A=P Q$.

Then we have the following result. As remarked earlier, the equivalence of (i) and (iii) is known. The other implications are easy to prove.

Lemma 2 Let $A \in M_{m, n}(Z)$. Then the following conditions are equivalent:
(i) $A$ is regular.
(ii) A admits a rank factorizarion.
(iii) Each invariant factor of $A$ is 1 .
(iv) The determinantal divisor rank of $A$ equals the rank of $A$.

We now prove the following:
Theorem 3 Let $A \in M_{m, n}(Z)$ and let $B \in M_{r, s}(Z)$ be a submatrix of $A$. Then (i) $\delta(B) \leq \delta(A)$.
(ii) $r+s-\delta(B) \leq m+n-\delta(A)$.

Proof: First suppose that $m=r+1, n=s$, and without loss of generality assume that $B$ is constituted by the first $r$ rows of $A$. Let $p=\min \{r, s\}$. Let $U B V=\operatorname{diag}\left(s_{1}, \ldots, s_{p}\right)_{r \times s}=D$ be the Smith normal form where $U, V$ are unimodular matrices, $s_{1}, \ldots, s_{k}$ are the invariant factors and $s_{k+1}=\cdots=s_{p}=$ $0, k=\operatorname{rank} B$. Let

$$
\tilde{A}=\operatorname{diag}(U, 1) A V=\left[\begin{array}{c}
D \\
x^{T}
\end{array}\right]
$$

In $\tilde{A}$, subtract $x_{i}$ times the $i$-th row from the last row, $i=1, \ldots, \delta(B)$. Then $\tilde{A}$ is reduced to a matrix where the first $\delta(B)$ elements of the last row equal zero. Clearly, if in the reduced matrix the last row has an entry equal to 1 , then $\delta(\tilde{A})=\delta(A)=\delta(B)+1$. Otherwise, $\delta(A)=\delta(B)$. Therefore (i) is proved in this case.

Let $t=\delta(A)$. If $\tilde{a}_{i_{1} j_{1}}, \ldots, \tilde{a}_{i_{t} j_{t}}$ is a partial diagonal of $\tilde{A}$ of length $t$ then it contains at least $t-1$ entries of $D$. Thus any partial diagonal product of $\tilde{A}$ of length $t$ is either 0 or is divisible by $s_{t-1}(B)$. Thus $s_{t-1}(B) \mid d_{t}(A)=1$. It follows that $s_{1}(B)=\cdots=s_{t-1}(B)=1$ and hence $\delta(B) \geq t-1$. Therefore (ii) is also proved in this case.

When $A$ is obtained by appending a column to $B$, the proof is similar. The general case is proved by proceeding by appending a row or column at a time.

It is clear that the following assertion contained in Theorem 1 is a consequence of Theorem 3, (ii). Let $A \in M_{n, n}(Z)$ be a unimodular matrix and let $B \in M_{r, s}(Z)$ be a submatrix of $A$. Then $\delta(B) \geq r+s-n$.

## 2 Rank of product and sum

If $A$ and $B$ are matrices such that $A B$ is defined then rank $A B \leq \min \{\operatorname{rank}$ $A$, rank $B\}$. The corresponding result for $\delta$ is given next.

Lemma 4 Let $A \in M_{m, n}(Z), B \in M_{n, p}(Z)$. Then $\delta(A B) \leq \min \{\delta(A), \delta(B)\}$.
Proof: By the Cauchy-Binet formula any $t \times t$ minor of $A B$ is either zero or is an integral linear combination of $t \times t$ minors of $A, t=1, \ldots, \min \{m, p\}$. Thus, if $d_{t}(A B)$ is nonzero, then $d_{t}(A) \mid d_{t}(A B)$. Putting $t=\delta(A B)$ we see that $d_{\delta(A B)}(A)=1$ and hence $\delta(A B) \leq \delta(A)$. It can be similarly shown that $\delta(A B) \leq \delta(B)$.

If $A$ and $B$ are matrices of order $m \times n$ and $n \times p$ respectively, then it is well-known that

$$
\operatorname{rank} A B \geq \operatorname{rank} A+\operatorname{rank} B-n .
$$

We now prove the corresponding result for $\delta$.

Theorem 5 Let $A \in M_{m, n}(Z), B \in M_{n, p}(Z)$. Then

$$
\begin{equation*}
\delta(A B) \geq \delta(A)+\delta(B)-n \tag{1}
\end{equation*}
$$

Proof: Let $\delta(A)=r, \delta(B)=s$. Let $w=\min \{m, n\}, z=\min \{n, p\}$. Let

$$
A=U \operatorname{diag}\left(h_{1}, \ldots, h_{r}, h_{r+1}, \ldots, h_{w}\right)_{m \times n} V
$$

and

$$
B=U_{1} \operatorname{diag}\left(k_{1}, \ldots, k_{s}, k_{s+1}, \ldots, k_{z}\right)_{n \times p} V_{1}
$$

be Smith normal forms, where $h_{1}=\cdots=h_{r}=k_{1}=\cdots=k_{s}=1$.

Then

$$
U^{-1} A B V_{1}^{-1}=\operatorname{diag}\left(h_{1}, \ldots, h_{w}\right) V U_{1} \operatorname{diag}\left(k_{1}, \ldots, k_{z}\right) .
$$

It follows from the preceding equation that the submatrix of $V U_{1}$ formed by the first $r$ rows and the first $s$ columns is a submatrix of $U^{-1} A B V_{1}^{-1}$. Since $V U_{1}$ is unimodular, it follows from Theorem 1 that this submatrix must have at least $r+s-n$ invariant factors equal to 1 . Thus by Theorem 3, (i),

$$
\delta(A B)=\delta\left(U^{-1} A B V_{1}^{-1}\right) \geq r+s-n=\delta(A)+\delta(B)-n
$$

and the proof is complete.
If $A, X, B$ are matrices such that $A X B$ is defined, then the well-known Frobenius inequality asserts that

$$
\operatorname{rank} A X B \geq \operatorname{rank} A X+\operatorname{rank} X B-\operatorname{rank} X
$$

The analogous result for $\delta$ is false as seen from the following example.
Let

$$
A=\operatorname{diag}(1,0,1), X=\operatorname{diag}(2,2,3), B=\operatorname{diag}(0,1,1)
$$

Then $\delta(X)=1, \delta(A X)=1, \delta(X B)=1$ and $\delta(A X B)=0$. Thus $\delta(A X B)$ is less than $\delta(A X)+\delta(X B)-\delta(X)$.

We now show that an analogue of the Frobenius inequality is true when $X$ is regular.

Theorem 6 Let $A \in M_{m, n}(Z), X \in M_{n, p}(Z)$ and $B \in M_{p, q}(Z)$ and suppose $X$ is regular. Then

$$
\begin{equation*}
\delta(A X B) \geq \delta(A X)+\delta(X B)-\delta(X) \tag{2}
\end{equation*}
$$

Proof: Let rank $X=r$. Since $X$ is regular, by Lemma 2, each of the $r$ invariant factors of $X$ equals 1 . Let

$$
X=U \operatorname{diag}\left(I_{r}, 0\right)_{n \times p} V
$$

be the Smith normal form of $X$. Then

$$
A X B=A U\left[\begin{array}{c}
I_{r} \\
0
\end{array}\right]_{n \times r}\left[\begin{array}{ll}
I_{r} & 0
\end{array}\right]_{r \times p} V B .
$$

Let $C=A U\left[\begin{array}{c}I_{r} \\ 0\end{array}\right]$ and $D=\left[\begin{array}{ll}I_{r} & 0\end{array}\right] V B$. By Theorem 5,

$$
\begin{equation*}
\delta(C D) \geq \delta(C)+\delta(D)-r \tag{3}
\end{equation*}
$$

Clearly, $\delta(A X)=\delta(C), \delta(X B)=\delta(D)$ and the result follows from (3).

We now turn to the sum of two matrices. If $A \in M_{m, n}(Z), B \in M_{m, n}(Z)$, then it is not true in general that $\delta(A+B) \leq \delta(A)+\delta(B)$. For example, let $A=\operatorname{diag}(2,0), B=\operatorname{diag}(0,3)$. Then $\delta(A)=\delta(B)=0$, whereas $\delta(A+B)=1$.

However, if either $A$ or $B$ is regular, then $\delta(A+B) \leq \delta(A)+\delta(B)$ holds, as we see now. We first prove a preliminary result. We include a proof for completeness, although it is essentially contained in the proof of Theorem 3.

Lemma 7 Let $A_{m, n}(Z)$ and let $C \in M_{m+1, n}(Z)$ be obtained by augmenting $A$ by a row vector. Then $\delta(C) \leq \delta(A)+1$.

Proof: Any $t \times t$ minor of $C$ is an integral linear combination of $(t-1) \times(t-1)$ minors of $A$. Thus $d_{t-1}(A) \mid d_{t}(C)$. Putting $t=\delta(A)$ we see that $\delta(A) \geq \delta(C)-1$ and the proof is complete.

Theorem 8 Let $A \in M_{m, n}(Z), B \in M_{m, n}(Z)$, and suppose either $A$ or $B$ is regular. Then

$$
\delta(A+B) \leq \delta(A)+\delta(B)
$$

Proof: First suppose $B$ is regular. We assume, without loss of generality, that $B$ is in Smith normal form. Since

$$
A+B=\left[I_{m}, I_{m}\right]\left[\begin{array}{l}
A \\
B
\end{array}\right]
$$

we see, in view of Lemma 4, that

$$
\delta(A+B) \leq \delta\left[\begin{array}{l}
A \\
B
\end{array}\right]
$$

The result follows by a repeated application of Lemma 7, keeping in mind that the determinantal divisor rank is unchanged when a matrix is augmented by a zero row. The proof is similar when $A$ is regular.

## 3 Complementary submatrices in a matrix and its inverse

It has been shown by Fiedler and Markham [4] and by Gustafson [5] that if $A$ is a nonsingular $n \times n$ matrix over a field, then the nullity of any submatrix of $A$ equals the nullity of the complementary submatrix in $A^{-1}$. The result was extended to Moore-Penrose inverse by Robinson [10] and then an extension to any generalized inverse was given in [1]. We now prove the analogues of these results for the determinantal divisor rank.

Theorem 9 Let $A \in M_{n, n}(Z)$ be a unimodular matrix, let $B \in M_{n, n}(Z)$ be the inverse of $A$ and suppose $A$ and $B$ are partitioned as

$$
A=\begin{gathered}
s \\
r \\
n-r
\end{gathered}\left(\begin{array}{cc}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right) \text { and } B=\begin{array}{r}
r \\
s \\
n-s
\end{array}\left(\begin{array}{cc}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}\right) .
$$

Then $r-\delta\left(A_{11}\right)=n-s-\delta\left(B_{22}\right)$.
Proof: By Jacobi's identity, any $t \times t$ minor of $A_{11}$ equals the complementary minor in $B, 1 \leq t \leq \min \{r, s\}$. Let the submatrix of $B$ which is complementary to $A_{11}$ be partitioned as

$$
\left.\begin{array}{c}
r-t \\
s-t  \tag{4}\\
n-s
\end{array} \begin{array}{c}
n-r \\
* \\
*
\end{array} B_{22}\right) .
$$

First suppose that $n-r-s+t>0$. It can be seen, by Laplace expansion, that the determinant of the matrix (4) can be expressed as an integral linear combination of minors of $B_{22}$ of order $n-s-r+t$. Thus $d_{n-s-r+t}\left(B_{22}\right) \mid d_{t}\left(A_{11}\right)$.

Setting $t=\delta\left(A_{11}\right)$ we see that $d_{n-s-r+t}\left(B_{22}\right)$ divides 1 and therefore

$$
\begin{equation*}
\delta\left(B_{22}\right) \geq n-s-r+\delta\left(A_{11}\right) . \tag{5}
\end{equation*}
$$

If $n-r-s+\delta\left(A_{11}\right) \leq 0$, then (5) is obvious.
Therefore

$$
r-\delta\left(A_{11}\right) \geq n-s-\delta\left(B_{22}\right) .
$$

Reversing the role of $A$ and $B$ we can prove

$$
r-\delta\left(A_{11}\right) \leq n-s-\delta\left(B_{22}\right)
$$

and hence the result is proved.

If $A$ and $G$ are matrices of order $m \times n$ and $n \times m$ respectively, then it will be convenient to assume that they are partitioned as follows:

$$
A=\begin{gather*}
q_{1}  \tag{6}\\
q_{2} \\
p_{1} \\
p_{2}
\end{gather*}\left(\begin{array}{cc}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right) \text { and } G=\begin{array}{cc}
p_{1} & p_{2} \\
q_{1} \\
q_{2}
\end{array}\left(\begin{array}{cc}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{array}\right)
$$

where $p_{1}+p_{2}=m$ and $q_{1}+q_{2}=n$.
The next result is similar to a result due to Nomakuchi [7] over a field. The proof is based on familiar ideas, see, for example, [8].

Theorem 10 Let $A \in M_{m, n}(Z)$ be a regular matrix of rank $r$ and let $G \in$ $M_{n, m}(Z)$ be a generalized inverse of $A$. Then there exist matrices $B \in M_{m, m-r}(Z), Q \in$ $M_{n-r, n}(Z)$ and $T \in M_{n-r, m-r}(Z)$ such that the matrix

$$
\left[\begin{array}{ll}
A & B \\
Q & T
\end{array}\right]
$$

is unimodular and $G$ is the submatrix formed by the first $n$ rows and the first $m$ columns of its inverse.

Proof: The matrices $A G$ and $G A$ are idempotent. Thus $I_{m}-A G$ and $I_{n}-G A$ are idempotent as well and admit rank factorization by Lemma 2.

Let $I_{m}-A G=B C$ and $I_{n}-G A=P Q$ be rank factorizations. Let $P^{-}$ and $C^{-}$be a left-inverse of $P$ and a right inverse of $C$ respectively and set $T=-P^{-}(G-G A G) C^{-}$.

We have

$$
\left[\begin{array}{cc}
A & B  \tag{7}\\
Q & T
\end{array}\right]\left[\begin{array}{cc}
G & P \\
C & 0
\end{array}\right]=\left[\begin{array}{ll}
A G+B C & A P \\
Q G+T C & Q P
\end{array}\right]
$$

Since $I_{n}-G A=P Q$, then $A P Q=0$ and hence $A P=0$. Similarly we can verify that $A G+B C=I_{m}, Q G+T C=0$ and $Q P=I_{n-r}$. Thus

$$
\left[\begin{array}{ll}
A & B \\
Q & T
\end{array}\right]\left[\begin{array}{ll}
G & P \\
C & 0
\end{array}\right]
$$

equals the identity matrix and the result is proved.
We now prove the main result of this section.

Theorem 11 Let $A \in M_{m, n}(Z)$ be a regular matrix of rank $r$ and let $G \in$ $M_{n, m}(Z)$ be a generalized inverse of $A$. Let $A$ and $G$ be partitioned as in (6). Then

$$
\begin{equation*}
r-p_{1}-q_{1} \leq \delta\left(G_{22}\right)-\delta\left(A_{11}\right) \leq p_{2}+q_{2}-r . \tag{8}
\end{equation*}
$$

Proof: By Theorem 10 there exist matrices $B \in M_{m, m-r}(Z), Q \in M_{n-r, n}(Z)$ and $T \in M_{n-r, m-r}(Z)$ such that the matrix

$$
\left[\begin{array}{ll}
A & B \\
Q & T
\end{array}\right]
$$

is unimodular and $G$ is the submatrix formed by the first $n$ rows and the first $m$ columns of its inverse.

Thus we may write

$$
\left.S=\begin{array}{l}
p_{1} \\
p_{2} \\
n-r
\end{array}\left(\begin{array}{ccc}
q_{1} & q_{2} & m-r \\
A_{11} & A_{12} & B_{1} \\
A_{21} & A_{22} & B_{2} \\
Q_{1} & Q_{2} & T
\end{array}\right), \text { and } S^{-1}=\begin{array}{c}
q_{1} \\
q_{2} \\
m-r
\end{array} \begin{array}{cc}
p_{2} & n-r \\
G_{11} & G_{12} \\
G_{21} & G_{22} \\
C_{1} & C_{2} \\
C_{2}
\end{array}\right)
$$

Since $S$ is unimodular, we have, using Theorem 9, and Theorem 3, (i),

$$
\begin{align*}
p_{1}-\delta\left(A_{11}\right) & =m+q_{2}-r-\delta\left[\begin{array}{cc}
G_{22} & P_{2} \\
C_{2} & 0
\end{array}\right]  \tag{9}\\
& \leq m+q_{2}-r-\delta\left(G_{22}\right) . \tag{10}
\end{align*}
$$

It follows that $\delta\left(G_{22}\right)-\delta\left(A_{11}\right) \leq p_{2}+q_{2}-r$ and the proof of the second inequality in (8) is complete.

We now prove the first inequality. By Theorem 3, (ii) we have

$$
p_{2}+q_{2}-\delta\left(G_{22}\right) \leq\left(q_{2}+m-r\right)+\left(p_{2}+n-r\right)-\delta\left[\begin{array}{cc}
G_{22} & P_{2}  \tag{11}\\
C_{2} & 0
\end{array}\right]
$$

Combining (9) and (11) we see that the first inequality in (8) is true and the proof is complete.

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