Inverses of triangular matrices and bipartite graphs

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Abstract

To a given nonsingular triangular matrix A with entries from a ring, we associate a weighted bipartite graph $G(A)$ and give a combinatorial description of the inverse of A by employing paths in $G(A)$. Under a certain condition, nonsingular triangular matrices A such that A and A^{-1} have the same zero-nonzero pattern are characterized. A combinatorial construction is given to construct outer inverses of the adjacency matrix of a weighted tree.

AMS Classification: 05C50, 15A09, 05C05
Keywords: Triangular matrix, bipartite graph, perfect matching, tree, outer inverse

1 Introduction

Let A be a lower triangular matrix with entries from a ring, which is not necessarily commutative. In the first section of this paper we obtain a combinatorial formula for A^{-1}, when it exists. The formula is in terms of certain paths in the bipartite graph associated with A. We note some consequences of this formula which include expressions for the inverse of a block triangular matrix and a formula for the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching.

In Section 3 we consider lower triangular, invertible, nonnegative matrices A and characterize those such that A and A^{-1} have the same zero-nonzero pattern. This relates to a question posed by Godsil [5] for bipartite graphs. In the final section we provide a combinatorial construction of outer inverses of the adjacency matrix of a weighted tree.

2 Inverses of triangular matrices

Let G be a bipartite graph and let $\mathcal{M}$ be a matching in G. We assume that each edge e of G has a nonzero weight $w(e)$ from a ring (not necessarily commutative). A path in G is said to be *alternating* if the edges are alternately in $\mathcal{M}$ and $\mathcal{M}^c$, with the first and the last edges being in $\mathcal{M}$. A path with only one edge, the edge being in $\mathcal{M}$, is alternating. Let P be the alternating path consisting of the edges $e_1, e_2, \ldots, e_k$ in that order. The *weight* $w(P)$ of P is defined to be $w(e_1)^{-1}w(e_2)w(e_3)^{-1}\cdots w(e_{k-1})w(e_k)^{-1}$, assuming that the inverses exist. Thus, if the weights commute, then $w(P)$ is just the product of the weights of the edges in $P \cap \mathcal{M}^c$ divided by the product of the weights of the edges in $P \cap \mathcal{M}$. The length $\ell(P)$ of P is the number of edges on that. For an alternating path P, we define

$$\epsilon(P) = (-1)^{(\ell(P)-1)/2}.$$

Let A be an $n \times n$ matrix with entries from a ring. We associate a bipartite graph $G(A)$ with A as usual: the vertex set is $\{R_1, \ldots, R_n\} \cup \{C_1, \ldots, C_n\}$ and there is an edge e between R_i to C_j if and only if $a_{ij} \neq 0$, in which case we assign e the weight $w(e) = a_{ij}$. We write vectors as row vectors. The transpose of $\mathbf{x}$ is denoted $\mathbf{x}^\top$.

Theorem 1. *Let A be a lower triangular $n \times n$ matrix with invertible diagonal elements and $\mathcal{M}$ be the unique perfect matching in $G(A)$ consisting of the edges from R_i to C_i, $i = 1, \ldots, n$. Then the entries of $B = A^{-1}$, for $1 \leq j \leq i \leq n$, are given by*

$$b_{ij} = \sum_{P \in \mathcal{P}_{ij}} \epsilon(P) w(P), \tag{1}$$

where $\mathcal{P}_{ij}$ is the set of alternating paths from C_i to R_j in $G(A)$.

Proof. We prove the result by induction on n, the cases $n = 1, 2$ being easy. Assume the result for matrices of order less than n. Partition A and B as

$$A = \begin{pmatrix} A_{11} & \mathbf{0}^\top \\ \mathbf{x} & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \mathbf{0}^\top \\ \mathbf{y} & b_{nn} \end{pmatrix}.$$

Note that $b_{nn} = a_{nn}^{-1}$ and $B_{11} = A_{11}^{-1}$.

By the induction assumption, (1) holds for $1 \leq j \leq i \leq n-1$. Thus we need to verify (1) for the pairs $(n, 1), \ldots, (n, n-1)$.

From $BA = I$ we see that $\mathbf{y}A_{11} + b_{nn}\mathbf{x} = \mathbf{0}$ and hence $\mathbf{y} = -a_{nn}^{-1}\mathbf{x}A_{11}^{-1}$. Therefore

$$y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i b_{ij}, \quad j = 1, \ldots, n-1. \tag{2}$$

Consider any alternating path from C_n to R_j in $G(A)$. Any such path must be composed of the edge from C_n to R_n, followed by an edge from R_n to C_i for some $i \in \{1, \ldots, n-1\}$, and then an alternating path from C_i to R_j.

If P is an alternating path from C_i to R_j, then denote by P' the alternating path from C_n to R_j obtained by concatenating the edge from C_n to R_n, then the edge from R_n to C_i, followed by P. Note that

$$\epsilon(P')w(P') = -\epsilon(P)a_{nn}^{-1}x_i w(P). \tag{3}$$

By the induction assumption, $b_{ij} = \sum \epsilon(P)w(P)$, where the summation is over all alternating paths from C_i to R_j. Hence it follows from (2) and (3) that for $j = 1, \ldots, n-1$,

$$b_{nj} = y_j = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i b_{ij} = -a_{nn}^{-1} \sum_{i=1}^{n-1} x_i \left(\sum_{P \in \mathcal{P}_{ij}} \epsilon(P)w(P) \right) = \sum_{P \in \mathcal{P}_{nj}} \epsilon(P)w(P),$$

completing the proof. □

We note some consequences of Theorem 1. Since the weights are noncommutative, we may take the weights to be square matrices of a fixed order.

This leads to combinatorial formulas for inverses of block triangular matrices. For example, the usual formula

$$\begin{pmatrix} A & O \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}$$

is a consequence of Theorem 1. Another example is the identity

$$\begin{pmatrix} A & O & O & O \\ W & B & O & O \\ X & O & C & O \\ O & Y & Z & D \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A^{-1} & O & O & O \\ -B^{-1}WA^{-1} & B^{-1} & O & O \\ -C^{-1}XA^{-1} & O & C^{-1} & O \\ D^{-1}YB^{-1}WA^{-1} + D^{-1}ZC^{-1}XA^{-1} & -D^{-1}YB^{-1} & -D^{-1}ZC^{-1} & D^{-1} \end{pmatrix}.$$

We note yet another consequence of Theorem 1. Let GF(2) denote the Galois field of order 2. The following result easily follows from Theorem 1.

Corollary 2. *Let A be an $n \times n$ lower triangular matrix over* GF(2) *such that $a_{ii} = 1$, $i = 1, \ldots, n$; and let $B = A^{-1}$. Let $G(A)$ be the graph associated with A. Then $b_{ij} = 1$ if and only if there are an odd number of alternating paths from C_i to R_j in $G(A)$.*

If A is a lower triangular matrix, then

$$\begin{pmatrix} O & A \\ A^{\top} & O \end{pmatrix} \tag{4}$$

is the (weighted) adjacency matrix of a bipartite graph with a unique perfect matching. Conversely the adjacency matrix of a bipartite graph with a unique perfect matching can be put in the form (4) after a relabeling of the vertices. In view of this observation, the unweighted case of Theorem 1 can be seen to be equivalent to Lemma 2.1 of Barik, Neumann and Pati [2]. Our proof technique is different. In the same spirit, Theorem 1 leads to a formula for the inverse of the adjacency matrix of a weighted tree (see Section 4) when the tree has a perfect matching, generalizing a well-known result from [4, 7] (see also [1, Section 3.6]).

Remark 3. Let T be tree with nonsingular weighted adjacency matrix A. Then A^{-1} is the weighted adjacency matrix of a bipartite graph. The graphs that can occur as inverses of nonsingular trees were characterized in [6]. Namely, a graph G is the inverse of some tree if and only if $G \in \mathcal{F}_k$ where $\mathcal{F}_k$ is the family of graphs defined recursively as follows. Set $\mathcal{F}_1 = \{P_2\}$ and for $k \geq 2$ any $G \in \mathcal{F}_k$ is obtained from some $H \in \mathcal{F}_{k-1}$ by taking any vertex u of H and adding two new vertices u' and v where u' is joined to all the neighbors of u and v (a pendant vertex) is joined to u'. The characterization remains valid in the more general setting when the weights of the edges come from a ring (provided the required inverses of the weights exist).

3 Matrices with isomorphic inverses

In this section we consider real matrices. It is an interesting problem to determine the triangular matrices A for which $G(A)$ is isomorphic to $G(A^{-1})$. This problem is in close connection with the one posed by Godsil [5] as described below.

Let G be a bipartite graph on $2n$ vertices which has a unique perfect matching $\mathcal{M}$. Then there is a lower triangular matrix A such that $G = G(A)$. With the additional hypothesis that the graph $G/\mathcal{M}$, obtained from G by contracting the edges in $\mathcal{M}$, is bipartite, Godsil [5] showed that A^{-1} is diagonally similar to a matrix A^+ whose entries are nonnegative and which dominates A, that is $A^+(i,j) \geq A(i,j)$ for all $1 \leq i,j \leq n$. In turn, A^+ can be regarded as the adjacency matrix of a bipartite multigraph G^+ in which G appears as a subgraph. In this framework, Godsil asked for a characterization of the graphs G such that G^+ is isomorphic to G. This was answered in [8], by showing that G and G^+ are isomorphic if and only if G is a corona of bipartite graph. The *corona* of a graph is obtained by creating a new vertex v' for each vertex v such that v' is adjacent to v. The following theorem is a generalization of this result.

Theorem 4. *Let A be a lower triangular matrix with nonnegative entries, $\mathcal{M}$ being the unique matching of $G = G(A)$ and such that $G/\mathcal{M}$ is bipartite. Then A and A^{-1} have the same zero-nonzero pattern if and only if G is a corona of a bipartite graph.*

Proof. If G is a corona, by some rearranging, we may write A as

$$A = \begin{pmatrix} I & O \\ A_0 & I \end{pmatrix},$$

for some A_0. Hence

$$A^{-1} = \begin{pmatrix} I & O \\ -A_0 & I \end{pmatrix},$$

proving the 'if' part of the theorem.

Next, assume that A and A^{-1} have the same zero-nonzero pattern. To show that G is a corona, it suffices to prove that the alternating paths of G are of length at most 3. By contradiction, suppose that G has an alternating path of length larger than 3 and so it has an alternating path of length 5 between R_j and C_i, say. Since $G/\mathcal{M}$ is bipartite, all the alternating paths between R_j and C_i must have the same length mod 4 (note that two alternating paths with different lengths mod 4 between two vertices give rise to an odd cycle in $G/\mathcal{M}$). So, by Theorem 1, the (i, j) entry of A^{-1} is nonzero. Since A and A^{-1} have the same zero-nonzero pattern, the (i, j) entry of A is nonzero and hence R_j and C_i are adjacent. This implies the existence of a triangle in $G/\mathcal{M}$, a contradiction. □

4 Generalized inverses and matchings

Let A be an $m \times n$ matrix with entries from a ring such that $T = G(A)$ is a tree and let $\mathcal{M}$ be a matching in T. When $\mathcal{M}$ is perfect, A is nonsingular and a formula for A^{-1} may be given in terms of alternating paths, as noted at the end of Section 2. When $\mathcal{M}$ is not perfect, we still may define an $n \times m$ matrix $B = (b_{ij})$ using the alternating paths of $\mathcal{M}$ in the same fashion as when $\mathcal{M}$ is a perfect matching. More precisely, if $\{R_1, \ldots, R_m\}$ and $\{C_1, \ldots, C_n\}$ are color classes of T, then for $1 \leq j \leq i \leq n$,

$$b_{ij} = \sum \epsilon(P)w(P),$$

where the summation is over all alternating paths P from C_i to R_j in $G(A)$. We call such a matrix the *path matrix of T with respect to $\mathcal{M}$*. We show that the path matrix turns out be an outer inverse of the adjacency matrix.

Theorem 5. *Let A be an $m \times n$ matrix such that $T = G(A)$ is a tree and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matchings in T with $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Let B_1 and B_2 be $n \times m$ path matrices of T with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then*

$$B_1 A B_2 = B_2 A B_1 = B_2.$$

Proof. Let F_1 and F_2 be the induced forests by T on the vertices saturated by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let A_1 and A_2 be the submatrices of A such that $F_1 = G(A_1)$ and $F_2 = G(A_2)$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are perfect matchings for F_1 and F_2, respectively. Let $|\mathcal{M}_1| = p$ and $|\mathcal{M}_2| = q$. It turns out that, with an appropriate ordering of the vertices,

$$B_1 = \begin{pmatrix} A_1^{-1} & O_{p\times(m-p)} \\ O_{(n-p)\times p} & O_{(n-p)\times(m-p)} \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} A_2^{-1} & O_{q\times(m-q)} \\ O_{(n-q)\times q} & O_{(n-q)\times(m-q)} \end{pmatrix}.$$

Note that A_2^{-1} is also a submatrix of A_1^{-1}, so B_1 is in fact of the form

$$B_1 = \left(\begin{array}{c|c|c} A_2^{-1} & O & O \\ \hline * & * & O \\ \hline O & O & O \end{array}\right).$$

Then

$$AB_1 = \begin{pmatrix} I_{p\times p} & O_{p\times(m-p)} \\ * & O_{(m-p)\times(m-p)} \end{pmatrix}.$$

It follows that

$$B_2 A B_1 = \left(\begin{array}{c|c|c} A_2^{-1} & O & O \\ \hline O & O & O \\ \hline O & O & O \end{array}\right) = B_2.$$

The equality $B_1 A B_2 = B_2$ is proved similarly. □

With the same proof as the theorem above, we can prove even a more general statement as follows.

Theorem 6. *Let A be an $m \times n$ matrix such that $T = G(A)$ is a tree and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matchings in T. If B_1 and B_2 be $n \times m$ path matrices of T with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then*

$$B_1 A B_2 = B_2 A B_1 = C,$$

where C is the path matrix of T with respect to $\mathcal{M}_1 \cap \mathcal{M}_2$.

Recall that the matrix B is called a 2-*inverse* (or an *outer inverse*) of the matrix A if $BAB = B$ (see, for example, [3]). The next result is an immediate consequence of Theorem 5.

Corollary 7. *Let A be a matrix such that $T = G(A)$ is a tree and let $\mathcal{M}$ be a matching in T. If B is the path matrix of T with respect to $\mathcal{M}$, then B is an outer inverse of A.*

Acknowledgments

The first author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India, and also thanks IPM, Tehran, for hospitality during a visit when this research was carried out. The research of the second author was in part supported by a grant from IPM (No. 91050114).

References

[1] R.B. Bapat, *Graphs and Matrices*, Springer, London; Hindustan Book Agency, New Delhi, 2010.

[2] S. Barik, M. Neumann, and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Linear Multilinear Algebra* **54** (2006), 453–465.

[3] A. Ben-Israel and T.N.E. Greville, *Generalized Inverses: Theory and Applications,* Second Edition, Springer, New York, 2003.

[4] F. Buckly, L.L. Doty, and F. Harary, On graphs with signed inverses, *Networks* **18** (1998), 151–157.

[5] C.D. Godsil, Inverses of trees, *Combinatorica* **5** (1985), 33–39.

[6] M. Neumann and S. Pati, On reciprocal eigenvalue property of weighted trees, *Linear Algebra Appl.* (2011) doi:10.1016/j.laa.2011.09.017

[7] S. Pavlikova and J. Krc-Jediny, On the inverse and dual index of a tree, *Linear Multilinear Algebra* **28** (1990), 93–109.

[8] R. Simion and D.-S. Cao, Solution to a problem of C.D. Godsil regarding bipartite graphs with unique perfect matching, *Combinatorica* **9** (1989), 85–89.