A note on singular line graphs

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Abstract: It is shown that if G is a graph with an odd number of spanning trees, then the line graph $\mathcal{L}(G)$ of G has nullity at most one.

1 Introduction

We consider simple graphs, that is, graphs without loops or multiple edges. Given a graph G, its line graph, denoted by $\mathcal{L}(G)$, is a graph H whose vertex set is equal to the edge set of G, with two vertices in H being adjacent if the corresponding edges in G are adjacent (that is, have a common vertex).

For a graph G, let A be its adjacency matrix. We denote by B, the $0-1$ vertex-edge incidence matrix. If we orient each edge of G, then Q will denote the $0, \pm 1$ vertex-edge incidence matrix of the resulting graph. The Laplacian matrix L of G is given by $L = QQ'$, where the superscript denotes transpose. Note that the Laplacian does not depend on the orientation.

By the eigenvalues of a graph we mean the eigenvalues of its adjacency matrix. A graph is said to be singular if its adjacency matrix is singular. Similarly the nullity of a graph means the nullity of its adjacency matrix. Singular line graphs of trees have been studied in the literature. Gutman and Sciriha [2] proved that singular line graphs of trees have 0 as a simple eigenvalue, that is, of multiplicity one. Thus the nullity of the line graph of a tree is at most one. For an alternative proof and related references, see [3]. We give a short proof of a more general version of the result, putting it in proper perspective.

2 Results

We will need the following well-known results, see, for example, [1].

Lemma 1 *Let A and B be matrices of order $m \times n$ and $n \times m$ respectively, where $m \leq n$. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of AB. Then the eigenvalues of BA are $\lambda_1, \ldots, \lambda_m, 0, \ldots, 0$, where the zero is repeated $n - m$ times.*

Lemma 2 (Interlacing of Eigenvalues) *Let A be a symmetric $n \times n$ matrix and let B be a principal $(n-1) \times (n-1)$ submatrix of A. Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of A and B respectively. Then $\lambda_i \geq \mu_i \geq \lambda_{i+1}, i = 1, \ldots, n-1$.*

Theorem 3 (Matrix-Tree Theorem) *Let G be a graph and let L be the Laplacian matrix of G. Then any cofactor of L equals the number of spanning trees of G.*

The following is the main result of this note.

Theorem 4 *Let G be a graph with an odd number of spanning trees. Then the adjacency matrix of the line graph $\mathcal{L}(G)$ of G has nullity at most one.*

Proof: We continue to use the notation introduced earlier. Let B be the $0-1$ incidence matrix of G and let Q be the $0, \pm 1$ incidence matrix after orienting the edges of G. Then $L = QQ'$ is the Laplacian of G. The adjacency matrix of the line graph $\mathcal{L}(G)$ is given by $A(\mathcal{L}(G)) = B'B - 2I$. Thus $B'B = A(\mathcal{L}(G)) + 2I$. We must show that the multiplicity of 2 as an eigenvalue of $B'B$ is at most one. Since $B'B$ and BB' have the same eigenvalues, apart from some zero eigenvalues (see Lemma 1), it is sufficient to show that the multiplicity of 2 as an eigenvalue of BB' is at most one. Let C be the matrix obtained by deleting the first row and column of $BB' - 2I$, and let $L(1,1)$ be the submatrix obtained by deleting the first row and column of L. Since $L = QQ'$, the determinant of C and the determinant of $L(1,1)$ have the same parity. Since $L(1,1)$, which (by Theorem 3) is the number of spanning trees in G, is odd, $L(1,1)$, and hence C, are nonsingular. If 2 is an eigenvalue of BB'

of multiplicity greater than 1, then by Lemma 2, 0 must be an eigenvalues of C, which will contradict that C is nonsingular. Therefore the multiplicity of 2 as an eigenvalue of BB' is at most one and the proof is complete. ■

Example Let k be an odd positive integer and let $\mathcal{C}_k$ denote the cycle on k vertices. Let G_k be the graph obtained by attaching a pendant edge to each vertex in $\mathcal{C}_k$. It can be seen that the line graph of G_k has nullity one. Clearly G_k has k spanning trees. Thus we have an infinite family of graphs, other than trees, with an odd number of spanning trees and with singular line graphs with nullity one.

It may be remarked that the following more general result holds. The proof is similar to that of Theorem 4.

Theorem 5 *Let G be a graph with an odd number of spanning trees and let k be an even integer. Then the multiplicity of k as an eigenvalue of the line graph $\mathcal{L}(G)$ of G is at most one.*

It is shown in [3], Theorem 2.2, that if the line graph of a tree is singular, then the tree has an even number of vertices. We now prove a generalization.

Theorem 6 *Let G be a bipartite graph with an odd number of spanning trees and let $\mathcal{L}(G)$ be singular. Then the number of vertices of G is even.*

Proof: Since $\mathcal{L}(G)$ is singular and $A(\mathcal{L}(G)) = B'B - 2I$, 2 is an eigenvalue of $B'B$, and hence of BB'. Since G is bipartite, BB' and QQ' are similar and hence the Laplacian $L = QQ'$ of G must have 2 as an eigenvalue. Let y be the associated eigenvector of L. Since L is an integral matrix, we may take y to be an integral vector with its components relatively prime. If y, reduced modulo 2, has a zero coordinate, then the corresponding principal submatrix of L, reduced modulo 2, will be singular, contradicting the fact that G has an odd number of spanning trees. Therefore y, reduced modulo 2, has no zero coordinate. Thus each component of y is an odd integer. Since y is orthogonal to $\mathbf{1}$, the vector of all ones, which is the eigenvector of L for the eigenvalue 0, the number of components of y must be even. Hence G has an even number of vertices. ■

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References

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