Exploring mathematical ideas with a deck of cards

R. B. Bapat
Indian Statistical Institute
New Delhi, 110016

The standard deck of cards is a very useful tool for exploring mathematical phenomena arising out of card games, tricks and recreations. Here we describe some such connections between card recreations and topics in combinatorial mathematics. The reader is encouraged to have a deck of cards handy and try out some of the activities mentioned here.

The standard deck consists of 52 cards. The cards are divided into 4 *suits*: Spades, Clubs, Hearts and Diamonds. Each suit has 13 *denominations*: Ace, Two, Three, ..., Ten are the denominations identified by the number of spots. The three *face cards* in each suit are the Jack, the Queen and the King. The cards come in two colours. Spades and Clubs are black while Hearts and Diamonds are red. The interplay between suits, denominations and colour is the basis of most of the card tricks and recreations performed with the deck.

Shuffling

There are various ways to shuffle the deck. The *cut* is the simplest and the most relevant for our purpose. In a cut, the deck is held face down, one or more cards are removed from the top of the deck and the entire set of the removed cards is inserted at the bottom. Equivalently, a set of one or more cards is removed from the bottom and replaced on top, without disturbing the order. If a series of cuts are rapidly performed then it is known as the *overhand shuffle*. A cut leaves the cyclic order of the cards unchanged, no matter how many times it is executed. This property can be used to construct several tricks with the deck.

Here is a simple example. Prearrange the face cards and aces (16 cards in all, which we will refer to as the *picture cards*) in the order: Ace, King, Queen and Jack of one suit, followed by the same cards, in the same order, of another suit, etc. Hold the pile of cards face down. Thus the top card, facing down is an Ace, while the bottom card is a Jack. Hand over the pile of the 16 cards, still face down, to the spectator and ask her to cut the cards any number of times. Then take back the pile from her, execute a few more cuts using overhand shuffle. If possible, make sure that the bottom card is a Jack as you finish shuffling (a quick peek can ensure this). Then deal the cards face up on the table into 4 piles in a sequence. Note that this deal is made as in a typical card game such as the game of bridge. The cards will come out neatly sorted in the four denominations, which is quite magical.

Another shuffle is the *perfect shuffle*. Here the deck is cut in exactly half and the cards are divided into two equal piles. Then the piles are pressed on the table, and mixed into each other perfectly so that the cards from the two piles alternate. If the cards are numbered $1, 2 \ldots, 10$ from top to bottom, then after a perfect shuffle the order will be either $1, 6, 2, 7, 3, 8, 4, 9, 5, 10$, or $6, 1, 7, 2, 8, 3, 9, 4, 10, 5$ depending on whether the bottom pile or the top pile is released first while mixing. Sometimes the terms *in-shuffle* and *out-shuffle* are used for the two types.

Suppose the deck is held face down, with the top card being the Ace of Diamonds. After one in-shuffle the Ace will move to the second position while after two in-shuffles it will move to the fourth position. What will happen if, starting from the beginning, we execute one in-shuffle followed by two out-shuffles and then two in-shuffles? Sounds complicated? Write a 1 for an in-shuffle and a 0 for an out-shuffle. Then the sequence of shuffles mentioned above is coded as 10011, which is 19 in binary notation. The Ace of Diamonds will move to the $19-$th position after the indicated sequence of shuffles! Try to figure out why.

A perfect shuffle is difficult to perform since it is likely that the cards from the two piles do not alternate exactly. If the cards alternate in a random fashion then the shuffle is known as a *riffle shuffle.* The well-known juggler, magician and mathematician Persi Diaconis has investigated properties of the riffle shuffle and, in particular, the number of riffle shuffles required to change a well-ordered deck into a randomly ordered one. He has shown, in joint work with other authors that, using a suitable measure of randomness, seven shuffles are sufficient to get a randomly ordered deck. More importantly, the common practice of using three or four shuffles is shown to be ineffective in mixing the deck well.

Cyclic arrangements

Remove the 13 cards of any one suit, say Hearts, and arrange them in the following order:

$$7, Ace, Queen, 2, 8, 3, Jack, 4, 9, 5, King, 6, 10.$$

Hold the deck of the 13 cards face down, so that the bottom card is 10. Remove one card from the top, insert it at the bottom and turn the next card over, keeping it on the table, face up. It will be the Ace. Now remove the next card from the top, insert it at the bottom and turn the next card over, keeping it on the table, face up. It will be the Two. Continue this process until all the 13 cards are put on the table. They will appear in their natural order: Ace, Two, Three, ..., Jack, Queen and King. Clearly this forms the basis of a card trick. Also there is a more general question one can ask. Suppose instead of removing one card at a time, we remove two cards and place them at the bottom (this is to be done one card at a time), turn the next card over etc. If we want to get the cards in the natural order, what should be the initial arrangement of the deck? More generally, let $a_1, a_2, \ldots, a_{13}$ be a sequence of positive integers. We wish to remove a_1 cards from the top (one at a time as before) place them at the bottom, get the Ace, then remove a_2 cards from the top, place them at the bottom, get the Two etc. What should be the initial arrangement? The reader is invited to find the solution, at least for some specific sequences $a_1, a_2, \ldots, a_{13}$ such as the one where all $a_i's$ are equal.

Hall's Theorem

The assignment problem is a well-known problem in the theory of optimization. Consider n jobs which are to be assigned to n individuals. Each individual is to be assigned exactly one job. Not everyone is suitable for, or capable of, doing all the jobs. However, for each individual i, we do have the list of jobs he/she is capable of doing. The question is: Is it possible to assign the jobs so that everyone is assigned a job he/she can do?

Clearly, if such an assignment exists, then for any subset of k individuals, the union of all the jobs they are capable of, must contain at least k elements. For example, if individual 1 and 2 can both do job 1 but no other job, then an assignment will not be possible. Hall's Theorem asserts that this obvious necessary condition is also sufficient. In other words, if for any subset of k individuals, the union of all the jobs they are capable of, contains at least k elements, then a job assignment is feasible. Hall's Theorem is one of the most important results in combinatorial mathematics and has many applications and extensions.

Now here is a solitaire game based on Hall's Theorem. Arrange the deck of 52 cards in a 13×4 array, all cards face up, in any arbitrary manner. Then it is always possible to choose one card from each row, so that the 13 cards so chosen are from all the 13 denominations. That is, all the 13 denominations are represented. Finding such a choice is a simple solitaire game. Why is such a choice always possible? Consider 13 individuals, corresponding to the 13 rows of the array. Suppose there are 13 jobs, corresponding to the 13 denominations, numbered 1 to 13 for convenience. Let us imagine that the i-th individual can do the j-th job if and only if a card of denomination j appears in row i of the array. A moment's reflection will show that in any k rows of the array, at least k denominations must occur. Thus the condition of Hall's Theorem is satisfied and therefore the choice of 13 cards as indicated above is possible.

An iteration with a fixed point

There is a simple iteration which can be performed with any number of cards, but works particularly well when the number of cards is triangular. Recall that an integer of the form $n(n+1)/2$ is said to be triangular.

To illustrate the iteration, start with 15 cards (the values of the cards are not important) and form any number of piles out of them. The piles can be of any size, not necessarily equal. For example, there may be 4 piles containing $2, 4, 4, 5$ cards to begin with. Pick one card from each pile and put those cards in a new pile besides the existing ones. For example, starting with the piles given above, we will arrive at 5 piles containing $1, 3, 3, 4, 4$ cards respectively. Continue the same procedure, each time forming a new pile, picking one card from each of the existing ones.

You will find that after several iterations you will arrive at 5 piles, containing $1, 2, 3, 4, 5$ cards. This is a fixed point of the iteration, since the next step again produces the same configuration.

The iteration can in general be performed with $1 + 2 + \cdots + n$ cards and will lead to the fixed point of n piles containing $1, 2, \ldots, n$ cards. The proof of this assertion is not easy.

What happens if the number is not triangular? Then we will get into a loop, containing a particular sequence of configurations. But again, given that we start with m cards, it is not easy to say what that sequence will be, unless of course m is triangular. You may want to try with 8 cards and see what happens.

A self-working trick

Many card tricks are based on simple principles which work on their own, without any effort on the part of the person performing them.

Here is an example. Take a deck of 21 cards and place the cards in 3 piles, containing 7 cards each, face up. Ask the spectator to review the cards and mentally make note of one particular card, pointing only to the pile to which the noted card belongs. Pick up the 3 piles and place them on top of each other, making sure that the pile containing the spectator's card is in the centre. Deal the deck again in 3 piles as in the game of bridge. Again request the spectator to only point to the pile containing the noted card. Repeat the procedure one more time. It will so happen that after the second step, the spectator's card will be located exactly in the *middle* of the pile in which it is contained. Then it is easy to identify the noted card.

Observe that when we pick up the 3 piles the first time, the noted card is somewhere in positions 8 to 14 in the deck. The next time it is in positions 10 to 12 and in the final step it reaches the centre of the pile to which it belongs.

The trick also works with 15 or with 27 cards. Will it work with 51 cards?

Josephus type questions

Suppose the numbers 1 to 41 are written along the circumference of a circle in the order $1, 2, \ldots, 39, 40, 41, 1$. Starting at 1, erase every third number. Continue until only one number remains. (When a number is erased, it is no longer taken into account while arriving at every third number.) Which will be that number? You will see that the last two numbers that remain are 16 and 31, while the last number that remains is 31. This problem is related to an historic anecdote involving Josephus, who cleverly used the observation to save himself from the Romans.

Suppose we start with n numbers and follow the same procedure, erasing every k-th number. Let the last remaining number be $f(n, k)$. The function $f(n, k)$ does not have a closed form and its analysis is complicated.

A deck of cards is convenient to model the exercise given in this problem. Each time we transfer $k-1$ cards from the top of the deck to the bottom, one card at a time, and then discard the k-th card.

A simple trick can be designed using the phenomenon mentioned here. Select a deck of 12 cards, ask the spectator to choose one card, note what it is, and then return it to the deck. Make sure that the spectator's card is at the 9-th position from the top when the deck is held face down. Now discard every second card by the procedure mentioned above. That is, transfer one card from the top of the deck to the bottom, discard the next card, and continue until only one card remains. That remaining card will be the card chosen by the spectator! It is easy to see that this trick is based on the observation that $f(12, 2) = 9$.

Latin squares

A Latin square of order n is an arrangement of n symbols in an $n \times n$ array so that every symbol occurs once in each row and column. Theory of Latin squares is an important area of combinatorial mathematics which finds many applications, particularly in design of experiments.

The following example shows a Latin square of order 4 using the symbols $1, 2, 3, 4$:

$$\begin{array}{cccc} 2 & 4 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{array}$$

Two Latin squares of order n are said to be mutually orthogonal if the n^2 pairs formed by taking the symbols in the $i-th$ row and the $j-$th column in the two squares for $i, j = 1, 2, \ldots, n$ are all distinct. For example, the Latin square displayed above and the following Latin square are mutually orthogonal:

$$\begin{array}{cccc} 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \\ 1 & 3 & 2 & 4 \end{array}$$

The term *Graeco-Latin squares* is used to describe the arrangement obtained by superimposing a pair of mutually orthogonal Latin squares.

A deck of cards offers a convenient tool to build Latin squares and Graeco-Latin squares. Arrange the 16 picture cards in a 4×4 array. Then move the cards around so that in each row you have one card of each suit. Continue moving the cards so that in each column there is one card of each suit as well. Now you have an example of a Latin square. Continue rearranging the cards so that in each row there is a card of each denomination (Ace, King, Queen and Jack). Finally, try to get a card of each denomination in every column as well. Now you have in front of you a compact example of a Graeco-Latin square. When the suits are considered as symbols, you have one Latin square. Similarly, when the denominations are treated as symbols, there is another Latin square. Further, the two Latin squares are clearly mutually orthogonal since every suit has precisely one card of each denomination.

Theory of Probability

Choosing cards from a deck at random is a popular theoretical model, along with tossing a coin and casting a die. Concepts in probability theory can be nicely illustrated using the model of drawing cards from a deck.

A card is chosen from the standard deck at random. Let A be the event that it is a Spade and let B be the event that it is an Ace. Are A and B independent? First give an intuitive answer and then check it with numbers.

There are two boxes, one contains two Aces and the other contains an Ace and a King. A box is chosen and a card selected at random. If it happens to be an Ace, what is the probability that the other card in the box is also an

Ace? The immediate reaction is that it should be 1/2, but a careful calculation using conditional probability reveals the answer to be 2/3.

These are simple examples. But one quickly stumbles into more difficult questions such as computing probabilities of different compositions of a hand of poker or very difficult ones such as computing probabilities of winning in a particular solitaire game.

We now describe an intriguing phenomenon that has been observed, but has no simple probabilistic explanation. Hold the deck of 52 cards, well-shuffled, face-up in your hand. Think of a number between 1 to 9 and then count and remove that many cards from the top. Note the value of the new top card and now remove that may cards from the deck. For example, if the top card is a 7, then count seven cards from the top, including the top card, and discard them. Observe the new top card and repeat the process. For the purpose of this game, we are going to assign the value 5 to all the face cards. Continue removing the cards, either until all cards are discarded, or there are not enough cards to discard at some point. Now collect the cards together, *without changing there order,* and restart with some other integer from 1 to 9. Most likely it will be seen that the way the game ends the second time is identical to the way it ended the first time. Try again with some other integer from 1 to 9. It has been observed that with a high probability (about 80 percent) the same sequence of cards starts to appear after a certain initial stage, and as a result the game always ends in the same way.

Gilbreath principle

In his well-known series “Mathematical Games” which ran in the magazine “Scientific American” for several decades, Martin Gardner has described several card recreations. One idea descibed by Gardner is *Gilbreath principle* according to which the cards are prearranged in a particular way and it results in certain magical phenomena. We saw a similar example earlier using the fact that the cards maintain their cyclical order after any number of cuts.

We will give two examples of application of Gilbreath principle. Arrange the 52 cards of the deck so that the suits of the cards appear in the same fixed sequence. For illustration, the sequence may be Diamonds, Clubs, Hearts and Spades. Thus the top card is of Diamonds, followed by a card of Clubs, then Hearts, then Spades, then again a card of Diamonds etc. Hand the prearranged deck over to the spectator, face down, and ask her to deal off about $20 - 25$ cards from the top of the deck, one at a time, so that they form a pile on the table. Now instruct the spectator to hold the two piles, one formed on the table and the one of the remaining cards, and give them a riffle shuffle. The shuffle need not be perfect but it is important that the two piles get mixed into each other, preserving their individual order. Now we are ready to witness the magical effect. In the top four cards of the deck, there will be one card of each suit. The next set of four cards will also contain one card of each suit. Again the next four will have one card of each suit and this will continue to hold for the entire deck!

Let us examine how this effect works. For convenience, suppose the deck has only 16 cards, arranged as

$$DCSHDCSHDCSHDCSH,$$

where we have used D for Diamonds, C for Clubs etc.

Suppose the top 7 cards are dealt, one at a time, face down, to form a pile. Then their order will be reversed and we now have two piles, one containing the cards (from top to bottom)

$$\text{pile 1: } SCDHSCD$$

while the other, containing the cards (again from top to bottom)

$$\text{pile 2: } HDCSHDCSH.$$

When we give a riffle shuffle to the two piles, we essentially mix them, taking some cards from pile 1, then some cards from pile 2 and so on. Note that if we take x cards from pile 1 and y cards from pile 2, where x and y are nonnegative integers adding up to 4, those 4 cards will contain one card of each suit. Now if we remove those 4 cards and continue with the remaining piles, it is as if we started with a deck of 12 cards, and so the entire phenomenon will be repeated.

In the second example, the 13 cards of any one suit, say Diamonds, are removed from the deck and the deck of the remaining 39 cards is used. Arrange the deck as in the previous example, and suppose that the cards are in a particular sequence, say Spades, Hearts and Clubs, when the deck is held face up. Hand over the deck to the spectator, face up, and ask her to divide it in two approximately equal halves. Note that the top card will be a Spades. Instruct her to riffle shuffle the two halves, still holding them face up. While she is doing the exercise, make a note of the top card of the bottom half. Take back the shuffled deck from the spectator and turn it over face down. Now in every group of three cards, starting from the top, the following phenomenon will be observed. If the card noted earlier was of Spades then in any group of three cards there will be at most two Spades. If the noted card was a Hearts then in any group of three cards there will be at most two Clubs. Finally, if the noted card was a Clubs then in any group of three cards there will be at most two

Hearts. It must be remarked that these phenomena are to be combined with other devices to make them appear more magical. For example, the magician could take a bet with the spectator that the cards will appear in a particular way and then manage to win each time.

Tricks based on colour

Here is a well-known puzzle. There are two containers, one containing a litre of water, the other, a litre of milk. A spoon of water is transferred from the first container to the second, mixed, and a spoon of the mixture is transferred back from the second container to the first. The question is, which is greater? The amount of water in the second container or the amount of milk in the first? The answer is of course that they are the same. This requires some thought and some people are never convinced.

A discreet version of the problem is easily modeled by a deck of cards. Separate the cards in two piles, the first containing 26 black cards and the second, 26 red ones. Transfer some cards from the black pile to the red, shuffle, and then transfer an equal number back to the first. Clearly the number of black cards in the second pile must equal the number of red cards in the first pile.

In fact, if the deck is *arbitrarily* divided into two piles of 26 cards each, the number of black cards in one pile must equal the number of red cards in the other.

Now a trick based on these ideas. The magician hands over the deck of 52 cards to the spectator. The deck has some cards facing up and some facing down. The spectator is instructed to cut the deck, or even shuffle it to his satisfaction, and return it to the magician. The magician counts off 26 cards from the top and places the two piles so formed on the table. Then she instructs the spectator to count the number of face up cards in one of the piles. She then instructs him to count the number of face up cards in the second pile and it turns out to be the same!

How does it work? The deck is prearranged by taking 26 face up cards and thoroughly mixing them with the 26 face down cards. As in the red and black phenomenon noted earlier, if the deck is divided into two equal parts, then the number of face up cards in one part equals the number of face down cards in the second. The crucial point is that when the magician counts 26 cards from the top and forms a pile, she turns it over before placing it on the table. That makes the number of face up cards in the two parts equal!

More about the interplay between cards and mathematics can be learnt through the vast literature on the subject, particularly the writings of Martin Gardner, and for more advanced mathematical ideas, the work of Persi

Diaconis.

Suggested reading

1. Persi Diaconis, Mathematical development from the analysis of riffle shuffling, *Groups, Combinatorics and Geometry (Durham, 2001)*, 73-97, World Scientific Publishing, River Edge, NJ, 2003.

2. Dave Bayer and Persi Diaconis, Trailing the dove tail shuffle to its lair, *Annals of Applied Probability,* 2 (1992), no.2, 294-313.

3. Martin Gardner, *Mathematics, Magic and Mystery,* Dover Publications, Inc., New York, 1956.

4. Martin Gardner, *A Gardner's workout,* A K Peters Ltd., Natick, MA, 2001.