# Inverses of $q$-distance matrices of a tree 

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March 3, 2009


#### Abstract

The determinant and the inverse of the distance matrix of a tree has been investigated in the literature, following the classical formulas due to Graham and Pollak for the determinant, and due to Graham and Lovász for the inverse. We consider two $q$-analogs of the distance matrix of a tree and obtain formulas for the inverses of the two distance matrices. Yan and Yeh have previously obtained expressions for the determinants of the two distance matrices. Some related results are proved.


Keywords: tree, distance matrix, q-distance, determinant, inverse

## 1 Introduction

We consider graphs which have no loops or parallel edges. Thus a graph $G=$ $(V(G), E(G))$ consists of a finite set of vertices, $V(G)$, and a set of edges, $E(G)$, each of whose elements is a pair of distinct vertices. A weighted graph is a graph in which each edge is assigned a positive number, called its weight. An unweighted graph, or simply a graph, is thus a weighted graph with each of the edges bearing weight 1 . We will assume familiarity with basic graph-theoretic notions, see, for example, $[1,6]$.

Let $G$ be a connected, weighted graph with vertex set $\{1,2, \ldots, n\}$. The distance between vertices $i$ and $j$, denoted by $d(i, j)$, is defined to be the minimum weight

[^0]of all paths from $i$ to $j$, where the weight of a path is the sum of the weights of the edges in that path. The distance matrix $D$ of $G$ is an $n \times n$ matrix with zeros along the diagonal and with its $(i, j)$-entry equal to $d(i, j)$.

A tree is a simple connected graph without a cycle. The distance matrix of a tree is extensively investigated in the literature. A classical result concerning the determinant of the distance matrix of a tree, due to Graham and Pollak [5], asserts that if $T$ is an unweighted tree on $n$ vertices, then $\operatorname{det}(D)=(-1)^{n-1}(n-1) 2^{n-2}$. Thus, $\operatorname{det}(D)$ is a function dependent on only $n$, the number of vertices of the tree. An extension of the formula to the weighted case has been given in [2]. In the unweighted case, a formula for the inverse of the matrix $D$ was obtained in a subsequent paper by Graham and Lovász [4]. Again, an extension of the formula to the weighted case was provided in [2].

Some $q$-analogs of the distance for a tree were considered in [3,11]. In particular, the following two kinds of $q$-distances have been defined by Yan and Yeh[11]. Consider a weighted tree $T$ with $n$ vertices and with edge weights $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}$. Let $u, v$ be two distinct vertices of $T$ with $d(u, v)=\alpha$. Then let
(i) $d_{q}(u, v)=[\alpha]$, where $[\alpha]=\frac{1-q^{\alpha}}{1-q}$, if $q \neq 1$, and $[\alpha]=\alpha$, if $q=1$. We set $[0]=0$. Note that $[\alpha]=1+q+q^{2}+\cdots+q^{\alpha-1}$ if $\alpha$ is a positive integer.
(ii) $d_{q}{ }^{*}(u, v)=q^{\alpha}$.

We set $d_{q}(u, u)=d_{q}^{*}(u, u)=0$. Let $D_{q}(T)$ and $D_{q}{ }^{*}(T)$ be the $n \times n$ matrices with the $(i, j)$-entry as $d_{q}(i, j)$ and $d_{q}{ }^{*}(i, j)$ respectively, where $1, \ldots, n$ are the vertices of $T$. The two distances have also been considered in [3] in the unweighted case, where the matrix $D_{q}^{*}(T)$ has been termed the exponential distance matrix.

Yan and Yeh[11] obtained formulas for the determinants of the matrices $D_{q}(T)$ and $D_{q}^{*}(T)$. The purpose of this paper is to give formulas for the inverses of these two matrices, when $q \neq \pm 1$. We also obtain a formula for the determinant of $D_{q}(T)$ which is more compact than the one in [11]. Our results generalize formulas obtained in [3] in the unweighted case, to the case of a weighted tree.

## 2 Inverses of $D_{q}{ }^{*}(T)$ and $D_{q}(T)$

We introduce some notation. Let $T$ be a weighted tree with $n$ vertices, $1, \ldots, n$. If $(i j)$ is an edge of the tree $T$ with end points $i$ and $j$, then let $w(i j)$ denote its weight. Let $\mathcal{A}$ be the $n \times n$ matrix defined as follows. The diagonal elements of $\mathcal{A}$
are zero. Let $q \in \mathbb{R}$. The $(i, j)$-element of $\mathcal{A}$ is zero if $i$ and $j$ are not adjacent, and otherwise it is $\frac{q^{w(i j)}}{1-q^{2 w(i j)}}$.

Let $\delta$ be the diagonal matrix with its $(i, i)$-element equal to $\sum_{j \sim i} \frac{q^{2 w(i j)}}{1-q^{2 w(i j)}}$, where $j \sim i$ denotes that $j$ is adjacent to $i, i=1, \ldots, n$. With this notation we have the following.

Theorem 2.1 Let $T$ be a weighted tree with $n$ vertices. Then for $q \neq \pm 1, D_{q}{ }^{*}(T)$ is nonsingular and

$$
\begin{equation*}
D_{q}{ }^{*}(T)^{-1}=I-\mathcal{A}+\delta . \tag{2.1}
\end{equation*}
$$

Proof. We will prove the result by induction on $n$. The result can be easily verified for $n=2$. Let the result be true for a tree with $k$ vertices. Let $\bar{T}$ be a tree on $k+1$ vertices with $k+1$ being a pendant vertex, and the vertex $k$ being adjacent to $k+1$. Let $T=\bar{T} \backslash\{k+1\}$. Suppose $D_{q}{ }^{*}(\bar{T}), \bar{I}, \overline{\mathcal{A}}, \bar{\delta}$ represent the matrices corresponding to $\bar{T}$. Then

$$
D_{q}^{*}(\bar{T})=\left[\begin{array}{cc}
D_{q}^{*}(T) & \underline{\mathbf{q}}  \tag{2.2}\\
\underline{\mathbf{q}}^{t} & 1
\end{array}\right]
$$

where for any $q \in \mathbb{R}$,

$$
\begin{equation*}
\underline{\mathbf{q}}^{t}=\left(q^{d_{q}^{*}(1, k+1)}, q^{d_{q}^{*}(2, k+1)}, \ldots, q^{d_{q}^{*}(k, k+1)}\right) . \tag{2.3}
\end{equation*}
$$

We first show that $D_{q}{ }^{*}(\bar{T})$ is nonsingular if $q \neq \pm 1$. By the Schur formula, applied to (2.2),

$$
\begin{equation*}
\operatorname{det} D_{q}^{*}(\bar{T})=W \operatorname{det} D_{q}^{*}(T), \tag{2.4}
\end{equation*}
$$

where

$$
W=1-\underline{\mathbf{q}}^{t} D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} .
$$

For $1 \leq i \leq k$,

$$
\begin{aligned}
d_{q}^{*}(i, k+1) & =q^{d(i, k+1)} \\
& =q^{d(i, k)+w(k k+1)} \\
& =q^{w(k k+1)} q^{d(i, k)} .
\end{aligned}
$$

Let $\mathbf{e}_{k}$ denote the $k \times 1$ unit vector with 1 at the $k$-th place, and zeros elsewhere. Hence the last column of $D_{q}^{*}(T)$ equals

$$
\begin{equation*}
D_{q}^{*}(T) \mathbf{e}_{k}=\frac{1}{q^{w(k k+1)}} \underline{\mathbf{q}}, \tag{2.5}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
q^{w(k k+1)} \mathbf{e}_{k}=D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} . \tag{2.6}
\end{equation*}
$$

Then

$$
\begin{align*}
W & =1-\underline{\mathbf{q}}^{t} D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} \\
& =1-\underline{\mathbf{q}}^{t} q^{w(k k+1)} \mathbf{e}_{k} \text { by }(2.6) \\
& =1-q^{w(k k+1)} \underline{\mathbf{q}}^{t} \mathbf{e}_{k} \\
& =1-q^{2 w(k k+1)}, \tag{2.7}
\end{align*}
$$

and hence $W \neq 0$ for $q \neq \pm 1$.
It follows from (2.4) that $D_{q}{ }^{*}(\bar{T})$ is nonsingular, $q \neq \pm 1$. Let

$$
D_{q}^{*}(\bar{T})^{-1}=\left[\begin{array}{cc}
D_{q}^{*}(T) & \underline{\mathbf{q}}  \tag{2.8}\\
\underline{\mathbf{q}}^{t} & 1
\end{array}\right]^{-1}=\left[\begin{array}{ll}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}\right]
$$

be a partitioning of $D_{q}{ }^{*}(\bar{T})^{-1}$, conformal with the partitioning of $D_{q}{ }^{*}(\bar{T})$. By the formula for the inverse of a partitioned matrix we have

$$
\begin{gather*}
B_{11}=D_{q}^{*}(T)^{-1}+D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} W^{-1}\left(D_{q}^{*}(T)^{-1} \underline{\mathbf{q}}\right)^{t}  \tag{2.9}\\
B_{12}=-D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} W^{-1} \tag{2.10}
\end{gather*}
$$

and

$$
\begin{equation*}
B_{22}=\frac{1}{W}=\left(1-\underline{\mathbf{q}}^{t} D_{q}^{*}(T)^{-1} \underline{\mathbf{q}}\right)^{-1} \tag{2.11}
\end{equation*}
$$

Let

$$
\bar{I}-\overline{\mathcal{A}}+\bar{\delta}=\left[\begin{array}{ll}
A_{11} & A_{12}  \tag{2.12}\\
A_{21} & A_{22}
\end{array}\right]
$$

be a partitioning of $\bar{I}-\overline{\mathcal{A}}+\bar{\delta}$, conformal with (2.8). We have

$$
\begin{align*}
B_{11} & =D_{q}^{*}(T)^{-1}+D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} W^{-1}\left(D_{q}^{*}(T)^{-1} \underline{\mathbf{q}}\right)^{t} \\
& =D_{q}^{*}(T)^{-1}+q^{w(k k+1)} \mathbf{e}_{k} W^{-1} q^{w(k k+1)} \mathbf{e}_{k}^{t} \text { by }(2.6) \\
& =D_{q}^{*}(T)^{-1}+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \mathbf{e}_{k}^{t} \text { by }(2.7) \tag{2.13}
\end{align*}
$$

and

$$
\begin{equation*}
B_{12}=-D_{q}^{*}(T)^{-1} \underline{\mathbf{q}} W^{-1}=-\frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} . \tag{2.14}
\end{equation*}
$$

Note that

$$
\overline{\mathcal{A}}=\left[\begin{array}{cc}
\mathcal{A} & \frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k}  \tag{2.15}\\
\frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k}^{t} & 0
\end{array}\right]
$$

and

$$
\bar{\delta}=\left[\begin{array}{cc}
\delta+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \mathbf{e}_{k}^{t} & 0  \tag{2.16}\\
0 & \frac{q^{w(2 k k+1)}}{1-q^{2 w(k k+1)}}
\end{array}\right] .
$$

Thus
$\left[\begin{array}{ll}A_{11} & A_{12} \\ A_{21} & A_{22}\end{array}\right]=\left[\begin{array}{cc}I & 0 \\ 0 & 1\end{array}\right]-\left[\begin{array}{cc}\mathcal{A} & \frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \\ \frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k}^{t} & 0\end{array}\right]+\left[\begin{array}{cc}\delta+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \mathbf{e}_{k}^{t} & 0 \\ 0 & \frac{q^{w(2 k k+1)}}{1-q^{2 w(k k+1)}}\end{array}\right]$.
Hence

$$
\begin{equation*}
A_{11}=I-\mathcal{A}+\delta+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \mathbf{e}_{k}^{t} \tag{2.17}
\end{equation*}
$$

By the induction hypothesis,

$$
\begin{equation*}
D_{q}^{*}(T)^{-1}=I-\mathcal{A}+\delta \tag{2.19}
\end{equation*}
$$

It follows from (2.13),(2.18) and (2.19) that

$$
\begin{equation*}
A_{11}=D_{q}^{*}(T)^{-1}+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k} \mathbf{e}_{k}^{t}=B_{11} \tag{2.20}
\end{equation*}
$$

Also, from (2.14),(2.17),

$$
\begin{equation*}
A_{12}=\mathbf{0}-\frac{q^{w(k k+1)}}{1-q^{2 w(k k+1)}} \mathbf{e}_{k}=B_{12} . \tag{2.21}
\end{equation*}
$$

Furthermore, from (2.11) and (2.17),

$$
\begin{equation*}
B_{22}=W^{-1}=\frac{1}{1-q^{2 w(k k+1)}}=1+\frac{q^{2 w(k k+1)}}{1-q^{2 w(k k+1)}}=A_{22} . \tag{2.22}
\end{equation*}
$$

We have shown that $A_{11}=B_{11}, A_{12}=B_{12}$ and $A_{22}=B_{22}$. By symmetry it follows that $A_{21}=B_{21}$. Thus

$$
\left[\begin{array}{ll}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{array}\right]=\left[\begin{array}{ll}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right]
$$

or equivalently,

$$
D_{q}^{*}(\bar{T})^{-1}=\bar{I}-\overline{\mathcal{A}}+\bar{\delta} .
$$

That completes the proof.

Let $T$ be a tree with $n$ vertices, $1, \ldots, n$. Suppose $T$ is unweighted, or equivalently, that the edge weights are all equal to 1 . Let $D$ be the distance matrix, $A$ the adjacency matrix and $\delta$ the diagonal matrix of vertex degrees of $T$. For a real number $q$, the matrix $F$ whose $(i, j)$-entry is $q^{d(i, j)}$ has been termed the exponential matrix in [3]. The following result, proved in [3], follows by setting $w(i j)=1$ in Theorem 2.1.

Corollary 2.2 Let $T$ be an unweighted tree on $n$ vertices and let $F$ be the exponential distance matrix of $T$. If $q \neq \pm 1$ then

$$
F^{-1}=I-\frac{q}{1-q^{2}} A+\frac{q^{2}}{1-q^{2}} \delta .
$$

We now turn to the matrix $D_{q}(T)$ of a weighted tree. For $q \neq-1$, let $\tau$ be the $n \times 1$ vecter with components $\tau_{1}, \ldots, \tau_{n}$ given by

$$
\tau_{i}=1-\sum_{j \sim i} \frac{q^{w(i j)}}{1+q^{w(i j)}} \quad i=1,2, \ldots, n
$$

We first prove the following.
Lemma 2.3 If $q \neq \pm 1$, then $D_{q}(T)$ is nonsingular.
Proof. Observe that

$$
\begin{equation*}
D_{q}(T)=-\frac{D_{q}^{*}(T)-\mathbf{e e}^{t}}{1-q} \tag{2.23}
\end{equation*}
$$

where $\mathbf{e}$ is the vector of all ones.
Evaluating the determinant of the matrix

$$
\left[\begin{array}{cc}
D_{q}^{*}(T) & \mathbf{e} \\
\mathbf{e}^{t} & 1
\end{array}\right]
$$

by Schur formula, applied in two ways, we get

$$
\begin{equation*}
\operatorname{det}\left(D_{q}^{*}(T)\right)\left(1-\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e}\right)=\operatorname{det}\left(D_{q}^{*}(T)-\mathbf{e e}^{t}\right) \tag{2.24}
\end{equation*}
$$

It follows from Theorem 2.1 that

$$
\begin{aligned}
\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e} & =n-\sum_{i=1}^{n} \sum_{j \sim i} \frac{q^{w(i j)}}{1-q^{2 w(i j)}} \sum_{i=1}^{n} \sum_{j \sim i} \frac{2 q^{w(i j)}}{1-q^{2 w(i j)}} \\
& =n-\sum_{i=1}^{n} \sum_{j \sim i} \frac{q^{w(i j)}}{1-q^{w(i j)}}\left(1-q^{w_{i j}}\right) \\
& =n-\sum_{i=1}^{n} \sum_{j \sim i} \frac{q^{w(i j)}}{1+q^{w(i j)}} \\
& =n-\sum_{i=1}^{n}\left(1-\tau_{i}\right) \\
& =\sum_{i=1}^{n} \tau_{i}
\end{aligned}
$$

Hence

$$
\begin{align*}
1-\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e} & =1-\sum_{i=1}^{n} \tau_{i} \\
& =1-\sum_{i=1}^{n}\left(1-\sum_{j \sim i} \frac{q^{w(i j)}}{1+q^{w(i j)}}\right) \\
& =\sum_{i=1}^{n} \sum_{j \sim i} \frac{q^{w(i j)}}{1+q^{w(i j)}}-(n-1) \\
& =\sum_{\{i, j\} \in E(T)}\left(\frac{2 q^{w(i j)}}{1+q^{w(i j)}}-1\right) \\
& =-\sum_{\{i, j\} \in E(T)} \frac{1-q^{w(i j)}}{1+q^{w(i j)}} . \tag{2.25}
\end{align*}
$$

It follows that $1-\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e} \neq 0$ for $q \neq \pm 1$. Since by Theorem 2.1, $D_{q}^{*}(T)$ is nonsingular, we conclude, in view of (2.23) and (2.24), that $D_{q}(T)$ is nonsingular.

In the next result we provide a formula for the inverse of $D_{q}(T), q \neq \pm 1$.
Theorem 2.4 Let $T$ be a weighted tree with $n$ vertices. Then the inverse of distance matrix $D_{q}(T), q \neq \pm 1$, is given by

$$
\begin{equation*}
D_{q}(T)^{-1}=-(1-q)\left(I-\mathcal{A}+\delta+\frac{\tau \tau^{t}}{1-\mathbf{e}^{t} \tau}\right) \tag{2.26}
\end{equation*}
$$

Proof. Let $q \neq \pm 1$. By Lemma 2.3, $D_{q}(T)$ is nonsingular. It follows from (2.23) that

$$
\begin{equation*}
D_{q}(T)^{-1}=-(1-q)\left(D_{q}^{*}(T)-\mathbf{e e}^{t}\right)^{-1} \tag{2.27}
\end{equation*}
$$

Recall the Sherman-Morrison formula which asserts that if $X$ is a nonsingular $n \times n$ matrix, and $\mathcal{U}$ and $\mathcal{V}$ are $n \times 1$ vectors such that $X+\mathcal{U} \mathcal{V}^{t}$ is nonsingular, then

$$
\begin{equation*}
\left(X+\mathcal{U} \mathcal{V}^{t}\right)^{-1}=X^{-1}-\frac{X^{-1} \mathcal{U} \mathcal{V}^{t} X^{-1}}{1+\mathcal{V}^{t} X^{-1} \mathcal{U}} \tag{2.28}
\end{equation*}
$$

Using (2.28) with $X=D_{q}^{*}(T), \mathcal{U}=\mathbf{e}$ and $\mathcal{V}=-\mathbf{e}$ we get

$$
\begin{equation*}
D_{q}(T)^{-1}=-(1-q)\left(D_{q}^{*}(T)^{-1}+\frac{D_{q}^{*}(T)^{-1} \mathbf{e}^{t} D_{q}^{*}(T)^{-1}}{1-\mathbf{e}^{t} D_{q}^{*}(T) \mathbf{e}}\right) \tag{2.29}
\end{equation*}
$$

Since $D_{q}^{*}(T)^{-1} \mathbf{e}=\tau$, it follows from (2.29) that

$$
\begin{equation*}
D_{q}(T)^{-1}=-(1-q)\left(D_{q}^{*}(T)^{-1}+\frac{\tau \tau^{t}}{1-\mathbf{e}^{t} \tau}\right) \tag{2.30}
\end{equation*}
$$

Using Theorem 2.1 and (2.30), (2.26) is proved.
Let $G$ be an unweighted graph with $n$ vertices, labeled $1,2, \ldots, n$. Let $A$ be the adjacency matrix of $G$. Let $\Delta$ be the $n \times n$ diagonal matrix with its $i$-th diagonal entry equal to the degree of vertex $i, i=1,2, \ldots, n$. Then $L=\Delta-A$ is the Laplacian of $G$. For a parameter $q$, the $q$-Laplacian $L_{q}$ of $G$ is defined as

$$
\begin{equation*}
L_{q}=I-q A+q^{2}(\Delta-I)=q L+\left(1-q^{2}\right) I+q(q-1) \Delta . \tag{2.31}
\end{equation*}
$$

The matrix $L_{q}$ has been called the generalized Laplacian of $G$ in $[9,10]$ and it arises in the context of zeta functions of graphs. The matrix was independently introduced in [3] for the case of a tree, where the following result was proved.

Theorem 2.5 Let $T$ be an unweighted tree with $n$ vertices. Let $\mathbf{f}=\mathbf{e}-q(\Delta-I) \mathbf{e}$. Then for $q \neq-1$,

$$
\begin{equation*}
D_{q}(T)^{-1}=\frac{1}{(n-1)(1+q)} \mathbf{f} \mathbf{f}^{\prime}-\frac{1}{1+q} L_{q} . \tag{2.32}
\end{equation*}
$$

It may be verified that for $q \neq \pm 1$, Theorem 2.5 follows from Theorem 2.1. The case $q=1$ needs a similar, separate analysis. We omit the details.

## 3 Determinants of $D_{q}^{*}(T)$ and $D_{q}(T)$

Let $T$ a weighted tree on $n$ vertices with edge weights $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}$. The following formula has been obtained in [3].

## Theorem 3.1

$$
\begin{equation*}
\operatorname{det}\left(D_{q}{ }^{*}(T)\right)=\prod_{i=1}^{n-1}\left(1-q^{2 \alpha_{i}}\right) \tag{3.1}
\end{equation*}
$$

A formula for the determinant of $D_{q}(T)$ has also been given by Yan and Yeh [11] and is stated next.

Theorem 3.2 Let $T$ be a weighted tree with $n \geq 3$ vertices and weights $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}$. Then

$$
\begin{aligned}
\operatorname{det}\left(D_{q}(T)\right) & =(-1)^{n-1}\left(\prod_{i=1}^{n-1}\left[2 \alpha_{i}\right]\right) \\
& \times\left(\frac{\left[\alpha_{1}\right]\left[\alpha_{2}\right]\left[\alpha_{1}+\alpha_{2}\right]}{\left[2 \alpha_{1}\right]\left[2 \alpha_{2}\right]}+\frac{\left[\alpha_{n-1}\right]\left[\alpha_{n-2}\right]\left[\alpha_{n-1}+\alpha_{n-2}\right]}{\left[2 \alpha_{n-1}\right]\left[2 \alpha_{n-2}\right]}+\sum_{i=1}^{n-3} \frac{\left[\alpha_{i}\right]\left[\alpha_{i+2}\right]\left[\alpha_{i}+\alpha_{i+2}\right]}{\left[2 \alpha_{i}\right]\left[2 \alpha_{i+2}\right]}\right)
\end{aligned}
$$

We now derive a formula for $\operatorname{det}\left(D_{q}(T)\right)$, using the formula for $\operatorname{det}\left(D_{q}^{*}(T)\right)$, which is more compact than the one in Theorem 3.2.

Theorem 3.3 Let $T$ be a weighted tree with $n$ vertices and weights $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n-1}$. Then

$$
\begin{equation*}
\operatorname{det}\left(D_{q}(T)\right)=(-1)^{n-1} \prod_{i=1}^{n-1}\left[2 \alpha_{i}\right] \sum_{i=1}^{n-1} \frac{\left[\alpha_{i}\right]}{1+q^{\alpha_{i}}} . \tag{3.2}
\end{equation*}
$$

Proof. Substituting the expression for $\operatorname{det}\left(D_{q}{ }^{*}(T)\right)$ given in (3.1) in (2.24) we get

$$
\begin{equation*}
\operatorname{det}\left(D_{q}(T)\right)=(-1)^{n} \prod_{i=1}^{n-1} \frac{\left(1-q^{2 \alpha_{i}}\right)}{(1-q)^{n}}\left(1-\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e}\right) \tag{3.3}
\end{equation*}
$$

From (2.25) we have,

$$
\begin{equation*}
\left(1-\mathbf{e}^{t}\left(D_{q}^{*}(T)\right)^{-1} \mathbf{e}\right)=-\sum_{i=1}^{n-1} \frac{1-q^{\alpha_{i}}}{1+q^{\alpha_{i}}} . \tag{3.4}
\end{equation*}
$$

It follows from (3.3) and (3.4) that

$$
\begin{aligned}
\operatorname{det}\left(D_{q}(T)\right) & =(-1)^{n-1} \prod_{i=1}^{n-1} \frac{\left(1-q^{2 \alpha_{i}}\right)}{(1-q)^{n-1}} \sum_{i=1}^{n-1} \frac{1-q_{i}^{\alpha}}{(1-q)\left(1+q^{\alpha_{i}}\right)} \\
& =(-1)^{n-1} \prod_{i=1}^{n-1}\left[2 \alpha_{i}\right] \sum_{i=1}^{n-1} \frac{\left[\alpha_{i}\right]}{1+q^{\alpha_{i}}},
\end{aligned}
$$

and the proof is complete.
It can be seen that the formula (3.2) is equivalent to the one in Theorem 3.2, since

$$
\frac{\left[\alpha_{i}\right]\left[\alpha_{j}\right]\left[\alpha_{i}+\alpha_{j}\right]}{\left.\left[2 \alpha_{i}\right] 2 \alpha_{j}\right]}=\frac{1}{2(1-q)}\left[\frac{1-q^{\alpha_{i}}}{1+q^{\alpha_{i}}}+\frac{1-q^{\alpha_{j}}}{1+q^{\alpha_{j}}}\right] .
$$

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    ${ }^{\dagger}$ Department of Mathematics, University of Delhi, Delhi 110007, The support of the Junior Research Fellowship awarded by Council Of Scientific and Industrial Research is gratefully acknowledged.

