Monotonicity properties of certain Laplacian eigenvectors

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Abstract

Nath and Paul (Linear Algebra Appl., 460 (2014), 97-110) have shown that the largest distance Laplacian eigenvalue of a path is simple and the corresponding eigenvector has properties similar to the Fiedler vector. We given an alternative proof, establishing a more general result in the process.

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1 Introduction

Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$. We denote the degree of vertex i by $\delta_i, i = 1, \ldots, n$. Recall that the Laplacian matrix L of T is the $n \times n$ matrix with its (i, j)-element, $i \neq j$, equal to -1, if i and j are adjacent, and zero otherwise. The diagonal elements of L are $\delta_1, \ldots, \delta_n$. It is well-known that L is a positive semidefinite matrix with rank $n - 1$. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of L. The second smallest eigenvalue λ_2 is termed the *algebraic connectivity* of T and the corresponding eigenvector is a Fiedler vector. For basic properties of the Laplacian matrix we refer to [2]. We state the following classical result (see, for example, [2], Chapter 8, [3], Chapter 6).

Theorem 1 *Let T be a tree with $V(T) = \{1, \ldots, n\}$ and let f be a Fiedler vector of T. Let $f(v)$ denote the component of f indexed by the vertex v. Then one of the two cases must occur:*

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Case (i). $f(v) \neq 0$ *for any* v. *Then there is a unique edge* $\{u_1, u_2\}$ *such that* $f(u_1) > 0$ *and* $f(u_2) < 0$. *Moreover, the values of* f *increase along any path starting at* u_1 *and not containing* u_2, *and the values of* f *decrease along any path starting at* u_2 *and not containing* u_1.

Case (ii). $f(v) = 0$ *for some* v. *Then there is a unique vertex* u *such that* $f(u) = 0$ *and* u *is adjacent to a vertex on which* f *takes a nonzero value. Moreover, along any path starting at* u, *the values of* f *either increase, decrease or are identically zero.*

The unique edge in Case (i) is called the *characteristic edge*, while the unique vertex in Case (ii) is called the *characteristic vertex.*

Let T be a tree with $V(T) = \{1, \ldots, n\}$. The distance between vertices $i, j \in V(T)$, denoted d_{ij}, is defined to be the length (the number of edges) of the (unique) path from i to j. We set $d_{ii} = 0, i = 1, \ldots, n$. The distance matrix D is the $n \times n$ matrix with (i, j)-element equal to d_{ij}.

The distance Laplacian of T, denoted D^L, is the $n \times n$ matrix with (i, j)-element $-d_{ij}$ if $i \neq j$, and the (i, i)-element equal to $\sum_{j=1}^{n} d_{ij}, i = 1, \ldots, n$. It has been conjectured by Nath and Paul [4] that if f is an eigenvector corresponding to the largest eigenvalue of D^L, then it has properties similar to that of a Fiedler vector, more specifically, it satisfies either Case (i) or Case (ii) of Theorem 1. The conjecture was confirmed for a path in [4]. In this note we prove a more general statement for a path, employing a technique used by Boman et al[1].

2 Eigenvectors of certain Laplacians

Let A be a symmetric $n \times n$ matrix. The Laplacian associated with A, denoted A^L, is the $n \times n$ matrix with (i, j)-element $-a_{ij}$ if $i \neq j$, and the (i, i)-element equal to $\sum_{j \neq i} a_{ij}, i = 1, \ldots, n$.

In the rest of this paper we show that Conjectures 1 and 2 are true when the tree T is a path. The proof of Conjecture 1 for a path is essentially contained in [1], Theorem 3.2. We give the proof for completeness. We use the same proof technique and prove Conjecture 2 for a path. Let $\mathbf{1}$ denote the vector of appropriate size of all ones. We now state a preliminary result from [1]. The proof is easy and is omitted.

Lemma 2 *Let* S *and* T *be matrices of order* $(n-1) \times n$ *and* $n \times (n-1)$ *respectively, given by*

$$S = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then $ST = I_{n-1}$ *and* $TS = I_n - \mathbf{1}e_1'$, *where* e_1 *is the first column of the identity matrix* I_n.

We now prove a reformulation of a result from [1].

Lemma 3 *Let A be an $n \times n$ symmetric matrix and let $M = SA^LT$, where S and T are as in Lemma 2. Let $0 = \lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A^L. Then*

(i) The eigenvalues of M are $\lambda_2, \ldots, \lambda_n$.

(ii) If y is an eigenvector of A^L such that $y \perp \mathbf{1}$, then Sy is an eigenvector of M corresponding to the same eigenvalue.

(iii) If y is an eigenvector of M, then there exists an eigenvector x of A^L corresponding to the same eigenvalue such that $x \perp \mathbf{1}$ and $y = Sx$.

Proof: (i). Recall that if X and Y are matrices of order $p \times q$ and $q \times p$ respectively, where $p \leq q$, and if $\mu_1, \ldots, \mu_p$ are the eigenvalues of XY, then the eigenvalues of YX are given by $\mu_1, \ldots, \mu_p$, along with 0 repeated $q - p$ times. Note that $M = SA^LT$ and $A^LTS = A^L(I_n - \mathbf{1}e_1{}') = A^L$, since $A^L\mathbf{1} = 0$. It follows that the eigenvalues of M are $\lambda_2, \ldots, \lambda_n$.

(ii). Let $A^Ly = \alpha y$. Note that since $y \neq 0$ and $y \perp \mathbf{1}$, then $Sy \neq 0$. We have $MSy = SA^LTSy = SA^L(I_n - \mathbf{1}e_1{}')y = SA^Ly = \alpha Sy$ and the proof is complete.

(iii). We define the $n \times 1$ vector x as follows. For $i = 1, \ldots, n-1$, we set

$$x_i = \frac{1}{n}\sum_{j=1}^{n-1} jy_j - \sum_{j=i}^{n-1} y_j,$$

and

$$x_n = \frac{1}{n}\sum_{j=1}^{n-1} jy_j.$$

Then it can be verified that $x \perp \mathbf{1}$ and $y = Sx$. Since $My = \alpha y$, then $SA^LTSx = \alpha Sx$, which implies $SA^Lx = \alpha Sx$. Since $x \perp \mathbf{1}$, we conclude $A^Lx = \alpha x$. ■

Lemma 4 *Let A be a symmetric $n \times n$ matrix and let $M = SA^LT$, where S and T are as in Lemma 2.*

(i) Suppose $a_{ij} \geq a_{ik}$ if $j < k < i$ and $a_{ij} \leq a_{ik}$ if $i < j < k$. Then $m_{ij} \geq 0$ for any $i \neq j$.

(ii) Suppose $a_{ij} \leq a_{ik}$ if $j < k < i$ and $a_{ij} \geq a_{ik}$ if $i < j < k$. Then $m_{ij} \leq 0$ for any $i \neq j$.

Proof:

(i). Suppose $a_{ij} \geq a_{ik}$ if $j < k < i$ and $a_{ij} \leq a_{ik}$ if $i < j < k$. For $i < j$,

$$\begin{aligned} m_{ij} &= (SA^LT)_{ij} \\ &= \sum_{k=1}^{n} (SA^L)_{ik} t_{kj} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=j+1}^{n} (a_{ik} - a_{i+1,k}) \\
&\geq 0.
\end{aligned}
$$

For $i > j$, using the fact that A^L has zero row sums,

$$
\begin{aligned}
m_{ij} &= \sum_{k=j+1}^{n} (a_{ik} - a_{i+1,k}) \\
&= \sum_{k=1}^{j} (-a_{ik} + a_{i+1,k}) \\
&\geq 0.
\end{aligned}
$$

The proof of (ii) is similar. ■

An examination of the proof of Lemma 4 reveals that the following result is true, which we state without proof.

Lemma 5 *Let A be a symmetric $n \times n$ matrix and let $M = SA^LT$, where S and T are as in Lemma 2.*

(i) Suppose $a_{ij} > a_{ik}$ if $j < k < i$ and $a_{ij} < a_{ik}$ if $i < j < k$. Then $m_{ij} > 0$ for any $i \neq j$.

(ii) Suppose $a_{ij} < a_{ik}$ if $j < k < i$ and $a_{ij} > a_{ik}$ if $i < j < k$. Then $m_{ij} < 0$ for any $i \neq j$.

Lemma 6 *Let A be a symmetric $n \times n$ matrix such that $a_{ij} \geq 0$ for all $i \neq j$. If λ_n is the largest eigenvalue of A, then A has an eigenvector x for λ_n with $x_i \geq 0, i = 1, \dots, n$. Furthermore, if $a_{ij} > 0$ for all $i \neq j$, then $x_i > 0, i = 1, \dots, n$.*

Proof: The result follows from the Perron-Frobenius Theorem, applied to the matrix $A + \beta I$ for a sufficiently large β. ■

The following is our main result.

Theorem 7 *Let A be a symmetric, nonnegative $n \times n$ matrix with $a_{ij} \geq 0$ for all $i \neq j$ and $a_{ii} = 0, i = 1, \dots, n$. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A^L.*

(i) Suppose $a_{ij} \geq a_{ik}$ if $j < k < i$ and $a_{ij} \leq a_{ik}$ if $i < j < k$. Then A^L has an eigenvector x corresponding to λ_n such that $x \perp \mathbf{1}$ and $x_1 \leq \cdots \leq x_n$.

(ii) Suppose $a_{ij} \leq a_{ik}$ if $j < k < i$ and $a_{ij} \geq a_{ik}$ if $i < j < k$. Then A^L has an eigenvector x corresponding to λ_2 such that $x \perp \mathbf{1}$ and $x_1 \leq \cdots \leq x_n$.

Proof: (i). Let $M = SA^LT$, where S and T are as in Lemma 2. By Lemma 4, $m_{ij} \geq 0$ for all $i \neq j$. By Lemma 3, the largest eigenvalue of M is λ_n, and by Lemma 6, M has an eigenvector y corresponding to λ_n with $y_i \geq 0, i = 1, \dots, n$. By Lemma 3, A^L has an eigenvector $x \perp \mathbf{1}$ corresponding to λ_n such that $y = Sx$. Since $y_i \geq 0, i = 1, \ldots, n$, it follows that $x_1 \leq x_2 < \cdots \leq x_n$.

(ii). Let M be as in (i). By Lemma 4, $m_{ij} \leq 0$ for all $i \neq j$. Let β be sufficiently large so that $\beta I - M$ has all entries nonnegative. By Lemma 3, the eigenvalues of M are $\lambda_2, \ldots, \lambda_n$, and hence $\beta I - M$ has eigenvalues $\beta - \lambda_2, \ldots, \beta - \lambda_n$. The largest eigenvalue of $\beta I - M$ is $\beta - \lambda_2$ and by the Perron-Frobenius Theorem it has an eigenvector y corresponding to this eigenvalue with $y_i \geq 0, i = 1, \ldots, n$. Clearly y must also be an eigenvector of M corresponding to λ_2. By Lemma 3, A^L has an eigenvector $x \perp \mathbf{1}$ corresponding to λ_2 such that $y = Sx$. Since $y_i \geq 0, i = 1, \ldots, n$, it follows that $x_1 \leq x_2 < \cdots \leq x_n$. ■

Corollary 8 *[4] Let T be the path with $V(T) = \{1, \ldots, n\}$ and $E(T) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$. Let D be the distance matrix of T and let λ_n be the largest eigenvalue of D^L with a corresponding eigenvector x. Then either $x_1 \geq \cdots \geq x_n$ or $x_1 \leq \cdots \leq x_n$.*

Proof: Note that D satisfies the condition in Theorem 7, (i). By Lemma 5, if $M = SD^LT$, then $m_{ij} > 0$ for all $i \neq j$ and hence λ_n is a simple eigenvalue of M, and therefore of D^L. It follows that the eigenvector of D^L corresponding to λ_n must be unique up to a scalar multiple. By Theorem 7, for any eigenvector x corresponding to λ_n, either $x_1 \geq \cdots \geq x_n$ or $x_1 \leq \cdots \leq x_n$. ■

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