Resistance matrix and q-Laplacian of a unicyclic graph

R. B. Bapat
Indian Statistical Institute
New Delhi, 110016, India
e-mail: rbb@isid.ac.in

Abstract: *The resistance distance between two vertices of a graph can be defined as the effective resistance between the two vertices, when the graph is viewed as an electrical network with each edge carrying unit resistance. The concept has several different motivations. The resistance matrix of a graph is a matrix with its (i, j)-entry being the resistance distance between vertices i and j. We obtain an explicit formula for the determinant of the resistance matrix of a unicyclic graph. Some properties of a q-analogue of the Laplacian are also studied, with special attention to the limiting behaviour as q approaches 1. An expression for the inverse of the q-Laplacian of a unicyclic graph is derived.*

1 Introduction and Preliminaries

The classical definition of distance between two vertices in a graph is the length of a shortest path between the two vertices. This definition is well-known and the concept is widely studied. However, it does not capture some features of "distance", such as the degree of communication between the vertices. For example, if there is a multitude of paths between two vertices, intuitively the two vertices should be thought of as having shorter distance.

The concept of *resistance distance*, introduced by Klein and Randić [8], arises naturally from several different considerations and is also mathematically more attractive than the classical distance. For more background information about resistance distance we refer to [2, 5, 8, 14].

It may be remarked that in the case of a tree, the concepts of classical distance and resistance distance coincide. This perhaps explains some very attractive properties of distances in trees which do not carry over to arbitrary graphs. However, if one uses the notion of resistance distance, then some of these attractive properties do have analogs for arbitrary graphs.

We consider only simple graphs, i.e., those with no loops or parallel edges. The results that we obtain can easily be extended to weighted graphs, though we consider only unweighted graphs for simplicity. Now we introduce some notation. Let $G = (V, E)$ be a graph with n vertices, labeled $\{1, 2, \ldots, n\}$. The Laplacian matrix L of G is defined as follows. For $i \neq j$, the (i, j)-entry of L is zero, if vertices i and j are not adjacent, while it is -1, if i and j are adjacent. The (i, i)-entry of L is defined to make the i-th row-sum equal to zero, $i = 1, 2, \ldots, n$. Thus L is a singular matrix. For basic properties of the Laplacian matrix, see [1,10].

Let G be a connected graph with vertex set $\{1, 2, \ldots, n\}$. The resistance distance between two vertices can be defined in a number of different, equivalent ways. We give two definitions here. Let L be the Laplacian of G. Let i, j be vertices of G. The resistance distance r_{ij} between i and j is zero if $i = j$, and if $i \neq j$, then

$$r_{ij} = \frac{detL(i, j; i, j)}{detL(i, i)},$$

where $L(i, i)$ is the submatrix obtained by deleting row i, column i, of L; while $L(i, j; i, j)$ is the submatrix obtained by deleting rows i, j and columns i, j of L. As usual, *det* denotes determinant. Let $\chi(G)$ denote the complexity, that is, the number of spanning trees of G. We remark that by the well-known Matrix-Tree theorem [13], $detL(i, i)$ equals $\chi(G)$ for $i = 1, 2, \ldots, n$.

The second definition is in terms of electrical networks. Think of G as an electrical network in which a unit resistance is placed along each edge. Current is allowed to enter the network only at vertex i and leave the network only at vertex j. Then the resistance distance between i and j is the "effective resistance" between i and j.

If there is a unique (ij)-path in G, then it is clear from the second definition that the resistance distance between i and j equals the length (that is, the number of edges) of the path. This explains why the resistance distance and the classical distance coincide when the graph is a tree.

We now state some known results which will be required. The proofs are omitted and can be found in [2]. Denote the vector of all ones by $\mathbf{1}$, the size of which will be clear from the context. The $n \times n$ matrix of all ones will be denoted J_n. Let G be a connected graph with vertex set $\{1, 2, \ldots, n\}$, and let L be the Laplacian of G. Then L is singular, has rank $n-1$, and any vector in the null space of L is a scalar multiple of $\mathbf{1}$. The matrix $L + \frac{1}{n}J_n$ is nonsingular. We set

$$X = (L + \frac{1}{n}J_n)^{-1}. \tag{1}$$

The resistance matrix R of G is the $n \times n$ matrix with its (i, j)-entry equal to 0 if $i = j$, and r_{ij}, the resistance distance between i and j, otherwise. Let τ be a vector of order $n \times 1$ with its components defined by

$$\tau_i = 2 - \sum_{j:j\sim i} r_{ij}, i = 1, 2, \ldots, n, \tag{2}$$

where $j \sim i$ denotes that j is adjacent to i.

Theorem 1 *Let G be a connected graph with vertex set $\{1, 2, \ldots, n\}$, let L be the Laplacian of G, let R be the resistance matrix of G, and let $X = (x_{ij})$ and τ be defined as in (1),(2). Let $\tilde{X}$ be the $n \times n$ diagonal matrix with $x_{11}, x_{22}, \ldots, x_{nn}$ along the diagonal. Then the following assertions are true:*

(i) $r_{ij} = x_{ii} + x_{jj} - 2x_{ij}, i, j = 1, 2, \ldots, n$

(ii) $R = \tilde{X}J + J\tilde{X} - 2X$

(iii) $\tau = L\tilde{X}\mathbf{1} + \frac{2}{n}\mathbf{1}$

(iv) $\mathbf{1}'\tau = 2$

(v) R is nonsingular and $R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R\tau}\tau\tau'$

(vi) $detR = (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\chi(G)}$

For a graph G with n vertices, D will denote the (classical) distance matrix of G; thus the (i,j)-entry of D is 0, if $i=j$ and it is the length of the shortest path between i and j, if $i \neq j$. When the graph is a tree, one gets the following consequence of Theorem 1. Assertions (iii) and (iv) are well-known results due to Graham and Pollack [7] and Graham and Lovász [6] respectively.

Theorem 2 *Let G be a tree with vertex set $\{1,2,\dots,n\}$, let L be the Laplacian of G, let D be the distance matrix of G, and let τ be defined as in (2). Then the following assertions are true:*

(i) $\tau_i = 2 - \delta_i$, *where* δ_i *is the degree of vertex* $i, i = 1,2,\dots,n$

(ii) $\tau' D \tau = 2(n-1)$

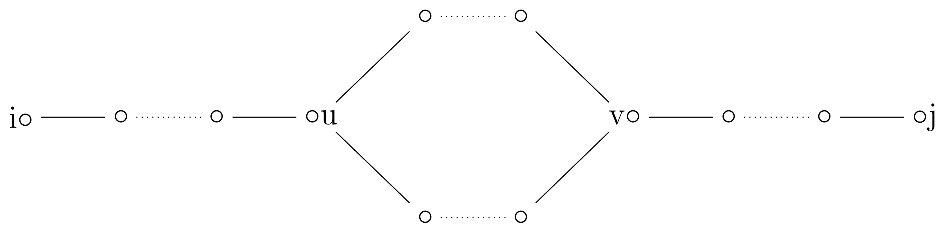
(iii) D *is nonsingular and* $D^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\tau\tau'$

(vi) $detD = (-1)^{n-1}(n-1)2^{n-2}$

2 Determinant of the resistance matrix of a unicyclic graph

We now consider unicyclic graphs. Recall that a graph is unicyclic if it is connected and has a unique cycle. The resistance distance between two vertices in a unicyclic graph G is particularly easy to determine. Clearly, if there is a unique path between vertices i and j of G, then the resistance distance r_{ij} between i and j is the length of the path between i and j.

Suppose a path between vertices i and j meets the cycle in G in at least two vertices. Let u and v be vertices on the cycle such that there is a unique path between i and u, and a unique path between j and v (see Figure). Let the length of the iu-path be a, that of the jv-path be b, and suppose the two paths between u and v have lengths c and d, where $c+d=k$, the length of the cycle. By the interpretation of resistance distance as effective resistance in an electrical network, it follows that r_{ij} is the sum of a, b and the "parallel" sum of c and d. Thus



$$r_{ij} = a + b + \frac{1}{\frac{1}{c} + \frac{1}{d}} = a + b + \frac{cd}{c+d}. \tag{3}$$

Since the resistance matrix R of a unicyclic graph has simple structure, it is natural to seek a precise formula for the determinant of R, which we now proceed to obtain.

We first consider the case of a cycle. Let $G = C_n$, the cycle on the n vertices $\{1, 2, \ldots, n\}, n \geq 3$, (where, of course, i and $i+1$ are adjacent, $i = 1, 2, \ldots, n-1$ and 1 is adjacent to n). If $i, j \in \{1, 2, \ldots, n\}$, then there are two paths between vertices i and j, of lengths $|i - j|$ and $n - |i - j|$. If R denotes the resistance matrix of G, then in view of the preceding discussion, $r_{ii} = 0, i = 1, 2, \ldots, n$; $r_{ij} = r_{ji}$ for all i and j, and

$$r_{ij} = \frac{(j-i)(n-j+i)}{n} \tag{4}$$

if $i < j$.

Let τ be defined as in (2). Then

$$\tau_i = 2 - r_{i-1,i} - r_{i,i+1}, i = 1, 2, \ldots, n;$$

where the subscripts are interpreted modulo n. It follows from (4) that

$$r_{i-1,i} = r_{i,i+1} = \frac{n-1}{n}, i = 1, 2, \ldots, n,$$

and hence $\tau_i = \frac{2}{n}, i = 1, 2, \ldots, n$. This last observation also follows from the fact that $\mathbf{1}'\tau = 2$ (see (iv), Theorem 1) and by symmetry.

Theorem 3 *Let C_n be the cycle on the vertices $\{1, 2, \ldots, n\}, n \geq 3$, and let R be the resistance matrix of C_n. Then* *(i)* $\tau' R\tau = \frac{2(n^2-1)}{3n}$

(ii) $detR = (-1)^{n-1}2^{n-2}\frac{(n^2-1)}{3n^2}$.

Proof: (i). We first consider the case when n is even, say $n = 2m$. It is clear by symmetry that each row-sum of R is the same. Using the expression (4) we see that the first row of R has sum

$$\begin{aligned}
r_{11} + r_{12} + \cdots + r_{1n} &= (r_{12} + r_{13} + \cdots + r_{1m}) \\
&+ r_{1,m+1} + (r_{1,m+2} + r_{1,m+3} + \cdots + r_{1,2m}) \\
&= 2(\frac{1(n-1)}{n} + \frac{2(n-2)}{n} + \cdots + \frac{(m-1)(n-m+1)}{n}) \\
&+ \frac{m(n-m)}{n} \\
&= \frac{2}{n}\cdot n(1 + 2 + \cdots + (m-1)) - \frac{2}{n}(1^2 + 2^2 + \cdots + (m-1)^2) \\
&+ \frac{m(n-m)}{n} \\
&= \frac{2}{n}(\frac{nm(m-1)}{2} - \frac{(m-1)m(2m-1)}{6}) + \frac{m(n-m)}{n} \\
&= \frac{m-1}{2}(2m - \frac{2m-1}{3}) + \frac{m^2}{2m} \\
&= \frac{n^2-1}{6}.
\end{aligned}$$

Thus each row-sum of R is $\frac{n^2-1}{6}$. Since $\tau_i = \frac{2}{n}, i = 1, 2, \ldots, n$, it follows that

$$\tau' R\tau = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}\tau_i\tau_j = \frac{4}{n^2}\cdot n \cdot \frac{n^2-1}{6} = \frac{2(n^2-1)}{3n}.$$

The proof when n is odd is similar and we omit the details.

(ii). By (vi), Theorem 1, $detR = (-1)^{n-1}2^{n-3}\frac{\tau' R\tau}{\chi(C_n)}$. Since $\tau' R\tau = \frac{2(n^2-1)}{3n}$ by the first part and since $\chi(C_n) = n$, the result follows. ■

We give an example. The resistance matrix of C_5 is

$$\frac{1}{5}\begin{bmatrix} 0 & 1\cdot 4 & 2\cdot 3 & 3\cdot 2 & 4\cdot 1 \\ 1\cdot 4 & 0 & 1\cdot 4 & 2\cdot 3 & 3\cdot 2 \\ 2\cdot 3 & 1\cdot 4 & 0 & 1\cdot 4 & 2\cdot 3 \\ 3\cdot 2 & 2\cdot 3 & 1\cdot 4 & 0 & 1\cdot 4 \\ 4\cdot 1 & 3\cdot 2 & 2\cdot 3 & 1\cdot 4 & 0 \end{bmatrix}.$$

According to Theorem 3, (ii), the determinant of the matrix is $\frac{2^3(5^2-1)}{3.5^2} = \frac{64}{25}$. This matrix is a symmetric Toeplitz matrix and a formula for its determinant may be of independent interest.

The following is the main result of this section.

Theorem 4 *Let G be a unicyclic graph with vertices $\{1,2,\ldots,n\}$, suppose the unique cycle C_k of G has length k, and that it is formed by the vertices $\{1,2,\ldots,k\}$; $n \geq k \geq 3$. Let R be the resistance matrix of G. Then*

$$detR = (-1)^{n-1}2^{n-2}\frac{3kn-2k^2-1}{3k^2}. \tag{5}$$

Proof: If $k = n \geq 3$, then G is C_n and the result follows by Theorem 3, (ii). So let $n > k \geq 3$. We prove the result by induction on n. So suppose that the result is true for a graph with $n-1$ vertices. Since $n > k$, then G has at least one pendant vertex. We assume, without loss of generality, that vertex n is pendant and that it is adjacent to vertex $n-1$.

Partition R as

$$\begin{bmatrix} R_{n-1} & z \\ z' & 0 \end{bmatrix},$$

where R_{n-1} is of order $n-1$ and z is $(n-1)\times 1$. If $1 \leq i,j \leq n-1$, then an (ij)-path does not pass through n and therefore R_{n-1} is in fact the resistance matrix of $G \setminus \{n\}$.

Perform the following row and column operations on R. From row n, subtract row $n-1$ and from column n, subtract column $n-1$. For any $1 \leq i \leq n-1$, the resistance distance between i and n is 1 plus the resistance distance between i and $n-1$. Therefore after the row and column operations, the resulting matrix, which has the same determinant as R, is

$$\begin{bmatrix} R_{n-1} & \mathbf{1} \\ \mathbf{1}' & -2 \end{bmatrix}.$$

By Theorem 1, (v), R_{n-1} is nonsingular. Using the well-known Schur formula for the determinant,

$$detR = (detR_{n-1})(-2 - \mathbf{1}'R_{n-1}^{-1}\mathbf{1}). \tag{6}$$

Let $\tilde{L}$ be the Laplacian of $G \setminus \{n\}$ and let $\tilde{\tau}$ be defined for $G \setminus \{n\}$ as in (2). By Theorem 1, (v),

$$R_{n-1}^{-1} = -\frac{1}{2}\tilde{L} + \frac{1}{\tilde{\tau}' R_{n-1}\tilde{\tau}}\tilde{\tau}\tilde{\tau}'. \tag{7}$$

Since $\tilde{L}$ has row-sums zero, (7) and Theorem 1, (iv) give

$$\mathbf{1}' R_{n-1}^{-1}\mathbf{1} = \frac{4}{\tilde{\tau}' R_{n-1}\tilde{\tau}}. \tag{8}$$

It follows from (6) and (8) that

$$detR = (detR_{n-1})(-2 - \frac{4}{\tau' R_{n-1}\tau}) = -2detR_{n-1} - 4\frac{detR_{n-1}}{\tau' R_{n-1}\tau}. \tag{9}$$

Again, (9) and Theorem 1, (vi) lead to

$$detR = -2detR_{n-1} - 4\frac{(-1)^{n-2}2^{n-4}}{k}. \tag{10}$$

By induction assumption,

$$detR_{n-1} = (-1)^{n-2}2^{n-3}\frac{3k(n-1) - 2k^2 - 1}{3k^2}. \tag{11}$$

Substituting (11) in (10) we get

$$\begin{aligned} detR &= (-1)^{n-1}2^{n-2}(\frac{3k(n-1) - 2k^2 - 1}{3k^2} + \frac{1}{k}) \\ &= (-1)^{n-1}2^{n-2}\frac{3kn - 2k^2 - 1}{3k^2} \end{aligned}$$

and (5) is proved. ■

We conclude this section with the remark that a formula for the determinant of the (classical) distance matrix of a unicyclic graph has been obtained in [3].

3 The q-Laplacian

Let G be a graph with n vertices, labeled $\{1, 2, \ldots, n\}$. Let A be the adjacency matrix of G. Thus A is an $n \times n$ matrix with (ij)-entry equal to 1 if i and j are adjacent, and zero if i and j are not adjacent. Also, $a_{ii} = 0, i = 1, 2, \ldots, n$.

Let Δ be the $n \times n$ diagonal matrix with its i-th diagonal entry equal to the degree of vertex $i, i = 1, 2, \ldots, n$. Then $L = \Delta - A$ is the Laplacian of G. For a parameter q, the q-Laplacian L_q of G is defined as

$$L_q = I - qA + q^2(\Delta - I) = qL + (1 - q^2)I + q(q - 1)\Delta. \tag{12}$$

The matrix L_q has been called the generalized Laplacian of G in [9]. The matrix was also introduced in [4] for the case of a tree, in the context of a formula for the inverse of a q-analogue of the distance matrix.

The following result has been obtained by Northshield [11].

Theorem 5 *Let G be a graph with n vertices and m edges, and let L_q be the q-Laplacian of G. If $f(q) = detL_q$, then $f'(q)|_{q=1} = 2(m - n)\chi(G)$.*

Lemma 6 *Let G be a graph with n vertices and m edges, and let L_q be the q-Laplacian of G. Suppose G is connected and not unicyclic. Then*

$$\lim_{q \to 1} (1 - q)L_q^{-1}L = 0. \tag{13}$$

Proof: Let $adjL_q$ be the adjoint of L_q, so that $L_q^{-1} = \frac{adjL_q}{detL_q}$.

Now

$$\lim_{q \to 1} (1 - q)L_q^{-1}L = \lim_{q \to 1} (1 - q)\frac{(adjL_q)L}{detL_q}. \tag{14}$$

Since $\lim_{q \to 1}(1 - q)(adjL_q)L = 0$ and $\lim_{q \to 1} detL_q = 0$, we may apply L'Hospital's rule and then the limit in (14) equals

$$\lim_{q \to 1} \frac{(1 - q)\frac{d}{dq}(adjL_q)L - (adjL_q)L}{\frac{d}{dq}(detL_q)}. \tag{15}$$

Note that

$$\lim_{q \to 1} (1 - q)\frac{d}{dq}(adjL_q)L = 0, \tag{16}$$

and

$$\lim_{q \to 1} (adjL_q)L = (adjL)L = 0. \tag{17}$$

Furthermore, by Theorem 5,

$$\lim_{q\to 1}\frac{d}{dq}(detL_q)=2(m-n)\chi(G), \tag{18}$$

which is nonzero, since G is connected and not unicyclic.

Substituting (16), (17) and (18) in (15) we get (13). ■

We now obtain a limiting property of the inverse of the q-Laplacian, which is motivated by the expression for resistance distance contained in Theorem 1,(i).

Theorem 7 *Let G be a graph with n vertices and m edges, let L_q be the q-Laplacian of G and let $K_q = L_q^{-1}$. Suppose G is connected and not unicyclic. Then for $i, j = 1, 2, \ldots, n$,*

$$\lim_{q\to 1}((K_q)_{ii}+(K_q)_{jj}-2(K_q)_{ij})=r_{ij}, \tag{19}$$

the resistance distance between i and j.

Proof: Since $L_qL_q^{-1}L = L$, therefore, using the definition of L_q,

$$(I-qA+q^2\Delta-q^2I)L_q^{-1}L=L. \tag{20}$$

After a routine simplification, (20) leads to

$$(1-q^2)L_q^{-1}L+(q^2-q)\Delta L_q^{-1}L+qLL_q^{-1}L=L. \tag{21}$$

By Lemma 6 we see that the first two terms on the left hand side in (21) approach 0 as q approaches 1. Therefore

$$\lim_{q\to 1}LL_q^{-1}L=\lim_{q\to 1}LK_qL=L. \tag{22}$$

We now introduce some notation. Fix vertices $i \neq j$ of G. Let e_{ij} be the $n \times 1$ vector with i-th coordinate 1, j-th coordinate -1 and zeros elsewhere. Since $\mathbf{1}'e_{ij} = 0$, e_{ij} is in the column space of L and hence there exists a vector w_{ij} such that $Lw_{ij} = e_{ij}$.

Let X be defined as in (1). Then $L\mathbf{1} = 0$ implies $(L+\frac{1}{n}J_n)\mathbf{1} = \mathbf{1}$ and hence $X\mathbf{1} = \mathbf{1}$. Therefore it easily follows that $LXL = L$. Hence

$$e_{ij}'Xe_{ij}=w_{ij}'LXLw_{ij}=w_{ij}'Lw_{ij}. \tag{23}$$

Now

$$\begin{aligned}
\lim_{q\to 1}((K_q)_{ii} + (K_q)_{jj} - 2(K_q)_{ij}) &= \lim_{q\to 1} e'_{ij} K_q e_{ij} \\
&= \lim_{q\to 1} w'_{ij} L K_q L w_{ij} \\
&= w'_{ij} L w_{ij} \text{ by (22)} \\
&= e'_{ij} X e_{ij} \\
&= x_{ii} + x_{jj} - 2x_{ij} \\
&= r_{ij}, \text{ by Theorem 1, (i),}
\end{aligned}$$

and the proof is complete. ■

4 Inverse of the q-Laplacian of a unicyclic graph

A walk without backtracking is a walk in which any two consecutive edges are distinct. Let G be a graph with the vertices $\{1, 2, \ldots, n\}$. For $m \geq 1$, let A_m be the $n \times n$ matrix whose (i, j)-entry is the number of walks in G of length m with no backtracking from i to j, $i, j = 1, \ldots, n$. We also set $A_0 = I$. Note that $A_1 = A$, the adjacency matrix of G. The following identity has been proved in [12],p.139.

Lemma 8 *Let G be a graph with the vertices $\{1, 2, \ldots, n\}$, and let L_q be the q-Laplacian of G. Then*

$$(\sum_{m=0}^{\infty} A_m q^m) L_q = (1 - q^2) I. \tag{24}$$

For a tree or a unicyclic graph, matrices A_m can be described explicitly and then Lemma 8 may be employed to get an expression for L_q^{-1}.

First consider the case of a tree. As usual, we denote the distance between vertices i and j as r_{ij}. Clearly, $(A_m)_{ii} = 0$ for $m \geq 1$ and $(A_0)_{ii} = 1, i = 1, 2, \ldots, n$. If i, j are distinct vertices of the tree with $d = r_{ij}$, then $(A_d)_{ij} = 1$, whereas $(A_m)_{ij} = 0$ for $m \neq d$. It follows from (24) that

$$(L_q^{-1})_{ii} = \frac{1}{1 - q^2} \text{ and } (L_q^{-1})_{ij} = \frac{q^d}{1 - q^2}, i, j = 1, 2, \ldots, n. \tag{25}$$

The matrix $(q^{r_{ij}})$ has been termed the "exponential distance matrix" of a tree in [4], where its relation with L_q^{-1} is proved in a different way.

We return to unicyclic graphs. For any vertex i of a unicylic graph G, denote by α_i the distance from α_i to the cycle in G. Thus α_i is the least distance from i to a vertex in the cycle.

Theorem 9 *Let G be a unicyclic graph with vertices $\{1, 2, \ldots, n\}$ and suppose the unique cycle C_k of G has length $k, n \geq k \geq 3$. Let α_i denote the distance of i from the cycle, $i = 1, 2, \ldots, n$. Let L_q be the q-Laplacian of G. Let $d(i, j)$ denote the classical distance between i and j. Then for $i, j = 1, 2, \ldots, n$;*

$$(1 - q^2)(L_q^{-1})_{ij} = q^{d(i,j)} + \frac{2q^{\alpha_i+\alpha_j+k}}{1 - q^k}, \tag{26}$$

if an ij-path does not meet the cycle and

$$(1 - q^2)(L_q^{-1})_{ij} = \frac{q^{d(i,j)} + q^{2(\alpha_i+\alpha_j)+k-d(i,j)}}{1 - q^k}, \tag{27}$$

if an ij-path meets the cycle.

Proof: Let i, j be vertices of G and suppose an ij-path does not meet the cycle. There is an ij-path, which is also an ij-walk with no backtracking, of length $d(i, j)$. There are two ij-walks with no backtracking of length $\alpha_i+\alpha_j+k$, two ij-walks with no backtracking of length $\alpha_i + \alpha_j + 2k$, and, in general, two ij-walks with no backtracking of length $\alpha_i + \alpha_j + sk, s \geq 1$. Therefore, by (24), it follows that

$$(1 - q^2)(L_q^{-1})_{ij} = q^{d(i,j)} + 2\sum_{s=1}^{\infty} q^{\alpha_i+\alpha_j+sk} = q^{d(i,j)} + \frac{2q^{\alpha_i+\alpha_j+k}}{1 - q^k}, \tag{28}$$

and (26) is proved. It may be remarked that when $i = j$, (26) is applicable (whether or not i is on the cycle) and we have

$$(1 - q^2)(L_q^{-1})_{ii} = 1 + \frac{2q^{2\alpha_i+k}}{1 - q^k}. \tag{29}$$

The proof of (27) is similar. ■

We are now in a position to complete the case of unicyclic graphs which was left out in Theorem 7. Interestingly, the conclusion is different in case of unicyclic graphs.

Theorem 10 *Let G be a unicyclic graph with vertices $\{1, 2, \ldots, n\}$ and suppose the unique cycle C_k of G has length $k, n \geq k \geq 3$. Let α_i denote the distance of i from the cycle, $i = 1, 2, \ldots, n$. Let L_q be the q-Laplacian of G and let $K_q = L_q^{-1}$. Then for $i, j = 1, 2, \ldots, n$,*

$$\lim_{q \to 1} ((K_q)_{ii} + (K_q)_{jj} - 2(K_q)_{ij}) = r_{ij} + \frac{(\alpha_i - \alpha_j)^2}{k}, \tag{30}$$

where r_{ij} is the resistance distance between i and j.

Proof: Let i, j be vertices of G and suppose an ij-path does not meet the cycle. By Theorem 9,

$$(1-q^2)((K_q)_{ii}+(K_q)_{jj}-2(K_q)_{ij}) = 2+\frac{2q^{2\alpha_i+k}}{1-q^k}+\frac{2q^{2\alpha_j+k}}{1-q^k}-2(q^{d(i,j)}+\frac{2q^{\alpha_i+\alpha_j+k}}{1-q^k}).$$

The result follows by computing the limit of $(K_q)_{ii} + (K_q)_{jj} - 2(K_q)_{ij}$ as $q \to 1$ by L'Hospital's rule. We omit the details. The proof follows similarly when an ij-path meets the cycle. ∎

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