Topics in Diophantine Equations

A thesis by

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"If I could find a proof that you were going to die in five minutes, I would of course be sorry to lose you, but the sorrow would be quite overweighed by the pleasure in the proof."

Godfrey Hardy, talking to Bertrand Russell, who sympathised with him and was not at all offended.

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Contents

Introduction

An old and well known *theorem of Sylvester for consecutive integers* [**56**] states that *a product of k consecutive integers each of which exceeds k is divisible by a prime greater than k.*

In this thesis, we give refinements, extensions, generalisations and applications of this theorem. First we give some notation which will be used throughout the thesis.

Let p_i denote the $i-$th prime number. Thus $p_1 = 2, p_2 = 3, \cdots$. We always write p for a prime number. For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively. Further we put $\omega(1) = 0$ and $P(1) = 1$. For positive real number ν and integers l, d with $d \geq 1$, $\gcd(l, d) = 1$, we denote

$$\pi(\nu) := \sum_{p \leq \nu} 1,$$

$$\pi_d(\nu) := \sum_{\substack{p \leq \nu \\ \gcd(p,d)=1}} 1,$$

$$\pi(\nu, d, l) := \sum_{\substack{p \leq \nu \\ p \equiv l (\bmod\ d)}} 1.$$

We say that a number is *effectively computable* if it can be explicitly determined in terms of given parameters. We write *computable number* for an effectively computable number. Let $d \geq 1, k \geq 2, n \geq 1$ and $y \geq 1$ be integers with $\gcd(n, d) = 1$. We denote by

$$\Delta = \Delta(n, d, k) = n(n + d) \cdots (n + (k-1)d)$$

and we write

$$\Delta(n, k) = \Delta(n, 1, k).$$

Further for $x \geq k$, we write

$$\Delta' = \Delta'(x, k) = \Delta(x - k + 1, k).$$

In the above notation, Sylvester's theorem can be stated as

$$P(\Delta(n, k)) > k \text{ if } n > k. \tag{1}$$

On the other hand, there are infinitely many pairs (n, k) with $n \leq k$ such that $P(\Delta) \leq k$. We observe that (1) is equivalent to

$$\omega(\Delta(n, k)) > \pi(k) \text{ if } n > k. \tag{2}$$

Here we notice that

$$\omega(\Delta(n, k)) \geq \pi(k)$$

since $k!$ divides Δ.

Let $d > 1$. Sylvester [**56**] proved that

$$P(\Delta) > k \text{ if } \ n \geq k + d. \tag{3}$$

Note that (3) includes (1). Langevin [**24**] improved (3) to

$$P(\Delta) > k \text{ if } n > k.$$

Finally Shorey and Tijdeman [**52**] proved that

$$P(\Delta) > k \quad \text{unless} \quad (n, d, k) = (2, 7, 3). \tag{4}$$

We observe that it is necessary to exclude the triple $(2, 7, 3)$ in the above result since $P(2 \cdot 9 \cdot 16) = 3$. The proof of [**52**] for (4) depends on the results on primes in arithmetic progressions. In Chapter 5, we give a proof of (4) which does not depend on these results and the computations required are considerably less.

We give a brief description of the results proved in this thesis. In Chapter 1, we prove Sylvester's Theorem. The proof is due to Erdős [**8**] but we have made simplifications. This proof is elementary and self contained; it does not make use of results from prime number theory. In Chapter 2, we collect together certain estimates on π function and other functions involving primes. In Chapter 3, we give a brief survey on refinements and generalisations of Sylvester's Theorem. These include the statements of our new results. We state here two of our following original results (i) and (ii) appeared in Acta Arith. and Indag. Math., respectively.

(i) Let $n > k$. Then $\omega(\Delta(n, k)) \geq \pi(k) + [\frac{3}{4}\pi(k)] - 1$ except when (n, k) belongs to an explicitly given finite set. (Laishram and Shorey [**18**])
(ii) Let $d > 1$. Then $\omega(\Delta) \geq \pi(2k) - 1$ except when $(n, d, k) = (1, 3, 10)$. (Laishram and Shorey [**19**])

This is best possible for $d = 2$ since $\omega(1 \cdot 3 \cdots (2k - 1)) = \pi(2k) - 1$. The latter result (ii) solves a conjecture of Moree [**29**]. Chapter 4 contains a proof of (i). In Chapter 5, we give a proof of (4). In Chapter 6, we prove (ii).

In 1939, Erdős [**9**] and Rigge [**36**], independently, proved that $\Delta(n, k)$ is divisible by a prime $> k$ to an odd power. As a consequence, we see that product of two or more consecutive positive integers is never a perfect square. In other words, the equation

$$n(n + 1) \cdots (n + k - 1) = y^2$$

does not hold. More generally we consider the equation

$$n(n + d) \cdots (n + (k - 1)d) = by^2. \tag{5}$$

with $P(b) \leq k$. The above equation has been completely solved when $d = 1$ (see Chapter 7). Therefore we suppose that $d > 1$. Erdős conjectured that (5) implies that k is bounded by a computable absolute constant. In Chapter 7, we give a survey of results on Erdős conjecture. Shorey and Tijdeman [**53**] showed that (5) implies that k is bounded by an effectively computable number depending only on $\omega(d)$. Our aim in Chapter 8 is to give an explicit upper bound κ_0 from Laishram [**20**] for k in terms of $\omega(d)$ whenever (5) holds. We show that κ_0 is given by

$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$	$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$
2	500	800	7	2.643×10^5	1.376×10^6
3	700	3400	8	1.172×10^6	6.061×10^6
4	2900	15300	9	5.151×10^6	2.649×10^7
5	13100	69000	10	2.247×10^7	1.149×10^8
6	59000	3.096×10^5	11	9.73×10^7	4.95×10^8

TABLE 1. $\kappa_0(\omega(d))$ for $2 \leq \omega(d) \leq 11$

for $2 \leq \omega(d) \leq 11$ and for $\omega(d) \geq 12$,

$$\kappa_0(\omega(d)) = \begin{cases} 2.25\omega(d)4^{\omega(d)} & \text{if } d \text{ is even} \\ 11\omega(d)4^{\omega(d)} & \text{if } d \text{ is odd.} \end{cases} \tag{6}$$

This original result has been submitted for publication in Publ. Math. Debrecen [**20**].

CHAPTER 1

Sylvester's theorem for consecutive integers

In this chapter, we prove the theorem of Sylvester [**56**] for consecutive integers stated in the Introduction, see (1).

THEOREM 1.0.1. *Let $d = 1$. Then*

$$P(\Delta) > k \text{ if } n > k. \tag{1.0.1}$$

Let us now consider $n \leq k$. For $1 \leq n \leq p_{\pi(k)+1} - k$ where $p_{\pi(k)+1}$ is the smallest prime exceeding k, we see that $P(\Delta) \leq k$ since $n+k-1 < p_{\pi(k)+1}$. Thus it is necessary to assume $n > p_{\pi(k)+1} - k$ for the proof of $P(\Delta) > k$. Then $n = p_{\pi(k)+1} - k + r$ for some $1 \leq r < k$ and hence $p_{\pi(k)+1} = n + k - r$ is a term in Δ, giving $P(\Delta) > k$.

For $x \geq 2k, x = n + k - 1$ and a prime $p > k$, we see that p divides $\binom{x}{k}$ if and only if p divides $\Delta = \Delta(n, k)$. Thus we observe that (1.0.1) is equivalent to the following result.

THEOREM 1.0.2. *If $x \geq 2k$, then $\binom{x}{k}$ contains a prime divisor greater than k.*

Therefore, we shall prove Theorem 1.0.2. The proof is due to Erdős [**8**] but we have made simplifications. This proof is elementary and self contained; it does not make use of results from prime number theory.

1.1. Lemmas for the proof of Theorem 1.0.2

LEMMA 1.1.1. *Let X be a positive real number and k_0 a positive integer. Suppose that $p_{i+1} - p_i < k_0$ for any two consecutive primes $p_i < p_{i+1} \leq p_{\pi(X)+1}$. Then*

$$P(x(x-1)\cdots(x-k+1)) > k$$

for $2k \leq x < X$ and $k \geq k_0$.

PROOF. Let $2k \leq x < X$. We may assume that none of $x, x-1, \cdots, x-k+1$ is a prime, since otherwise the result follows. Thus

$$p_{\pi(x-k+1)} < x - k + 1 < x < p_{\pi(x-k+1)+1} \leq p_{\pi(X)+1}.$$

Hence by our assumption, we have

$$k - 1 = x - (x - k + 1) < p_{\pi(x-k+1)+1} - p_{\pi(x-k+1)} < k_0,$$

which implies $k - 1 < k_0 - 1$, a contradiction. □ □

LEMMA 1.1.2. *Suppose that Theorem 1.0.2 holds for all primes k, then it holds for all k.*

PROOF. Assume that Theorem 1.0.2 holds for all primes k. Let $k_1 \leq k < k_2$ with k_1, k_2 consecutive primes. Let $x \geq 2k$. Then $x \geq 2k_1$ and $x(x-1)\cdots(x-k_1+1)$ has a prime factor $p > k_1$ by our assumption. Further we observe that $p \geq k_2 > k$ since k_1 and k_2 are consecutive primes. Hence p divides $\frac{x\cdots(x-k_1+1)(x-k_1)\cdots(x-k+1)}{k!} = \binom{x}{k}$. □ □

By Lemma 1.1.2, we see that it is enough to prove Theorem 1.0.2 for k prime which we assume from now on. Further we take $x \geq 2k$.

LEMMA 1.1.3. *Let* $p^a \big| \binom{x}{k}$. *Then* $p^a \le x$.

PROOF. We observe that

$$\mathrm{ord}_p \binom{x}{k} = \sum_{\nu=1}^{\infty} \left(\left[\frac{x}{p^\nu}\right] - \left[\frac{x-k}{p^\nu}\right] - \left[\frac{k}{p^\nu}\right] \right).$$

Each of the summand is at most 1 if $p^\nu \le x$ and 0 otherwise. Therefore $\mathrm{ord}_p\binom{x}{k} \le s$ where $p^s \le x < p^{s+1}$. Thus

$$p^a \le p^{\mathrm{ord}_p \binom{x}{k}} \le p^s \le x. \qquad \square \tag{1.1.1}$$

$\square$

LEMMA 1.1.4. *For* $k > 1$, *we have*

$$\binom{2k}{k} > \frac{4^k}{2\sqrt{k}} \tag{1.1.2}$$

and

$$\binom{2k}{k} < \frac{4^k}{\sqrt{2k}}. \tag{1.1.3}$$

PROOF. For $k > 1$, we have

$$\begin{aligned} 1 &> \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{(2k-1)^2}\right) = \frac{2\cdot 4}{3^2}\frac{4\cdot 6}{5^2}\cdots\frac{(2k-2)2k}{(2k-1)^2} \\ &> \frac{1}{4k}\left(\frac{2^k k!}{3\cdot 5 \cdots (2k-1)}\right)^2 = \frac{1}{4k}\left(\frac{4^k (k!)^2}{(2k)!}\right)^2 \end{aligned}$$

implying (1.1.2). Further we have

$$\begin{aligned} 1 &> \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{(2k-2)^2}\right) = \frac{1\cdot 3}{2^2}\frac{3\cdot 5}{4^2}\cdots\frac{(2k-3)(2k-1)}{(2k-2)^2} \\ &> \frac{1}{2k-1}\left(\frac{3\cdot 5 \cdots (2k-1)}{2^k k!}\cdot 2k\right)^2 > \frac{4k^2}{2k}\left(\frac{(2k)!}{4^k (k!)^2}\right)^2 \end{aligned}$$

implying (1.1.3). $\square$ $\square$

LEMMA 1.1.5. *We have*

$$\prod_{p \le x} p \prod_{p \le \sqrt{x}} p \prod_{p \le \sqrt[3]{x}} p \cdots < 4^x. \tag{1.1.4}$$

PROOF. We see that for every prime p and a positive integer a with

$$x < p^a \le 2x,$$

we have

$$\mathrm{ord}_p \left(\binom{2x}{x} \right) = \mathrm{ord}_p \left(\frac{(2x)!}{(x!)^2} \right) = \sum_{i=1}^{a} \left\{ \left[\frac{2x}{p^i}\right] - 2\left[\frac{x}{p^i}\right] \right\} \ge 1 \tag{1.1.5}$$

since

$$\left[\frac{2x}{p^i}\right] - 2\left[\frac{x}{p^i}\right] \ge 0 \text{ and } \left[\frac{2x}{p^a}\right] - \left[\frac{x}{p^a}\right] = 1.$$

Let $\lceil \nu \rceil$ denote the least integer greater than or equal to ν. Let $2^{m-1} \leq x < 2^m$ and we put

$$a_1 = \lceil \frac{x}{2} \rceil, \; a_2 = \lceil \frac{x}{2^2} \rceil, \; \cdots, \; a_h = \lceil \frac{x}{2^h} \rceil, \; \cdots, \; a_m = \lceil \frac{x}{2^m} \rceil = 1.$$

Then

$$a_1 > a_2 > \cdots > a_m$$

and

$$a_h < \frac{x}{2^h} + 1 = \frac{2x}{2^{h+1}} + 1 \leq 2a_{h+1} + 1$$

implying

$$a_h \leq 2a_{h+1}.$$

Also, we have $2a_2 < \frac{x}{2} + 2 \leq a_1 + 2$. Therefore

$$2a_2 \leq a_1 + 1. \tag{1.1.6}$$

Since $2a_1 \geq x$, we see that

$$(1, x] \subseteq \cup_{h=1}^{m} (a_h, 2a_h].$$

Let p and r be given such that $p^r \leq x < p^{r+1}$. Let $1 \leq i \leq r$. Then $p^i \leq x$. It is clear from the above inclusion that there exists k_i such that

$$a_{k_i} < p^i \leq 2a_{k_i}.$$

We observe that $a_{k_i} \neq a_{k_j}$ for $1 \leq j < i \leq r$ since $pa_{k_j} < p^{j+1} \leq p^i \leq 2a_{k_i}$. Thus we see from (1.1.5) that

$$p^r \Big| \binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m}.$$

Hence we have

$$\prod_{p \leq x} p \prod_{p \leq x^{\frac{1}{2}}} p \prod_{p \leq x^{\frac{1}{3}}} p \cdots = \prod_{p^r \leq x < p^{r+1}} p^r \leq \binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m},$$

the middle product being taken over all prime powers p^r with $p^r \leq x < p^{r+1}$. To complete the proof of the lemma, we show that

$$\binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m} < 4^x. \tag{1.1.7}$$

By direct calculation, we check that (1.1.7) holds for $x \leq 10$. For example, when $x = 5$, we have $a_1 = 3, a_2 = 2, a_3 = 1$ so that

$$\binom{2a_1}{a_1} \binom{2a_2}{a_2} \binom{2a_3}{a_2} = 20 \times 6 \times 2 < 4^5.$$

Suppose that $x > 10$ and (1.1.7) holds for any integer less than x. Then

$$\binom{2a_1}{a_1} \binom{2a_2}{a_2} \cdots \binom{2a_m}{a_m} < \binom{2a_1}{a_1} 4^{2a_2 - 1} \tag{1.1.8}$$

which we obtain by applying (1.1.7) with $x = 2a_2 - 1$ and seeing that

$$\lceil \frac{1}{2}(2a_2 - 1) \rceil = a_2, \; \lceil \frac{1}{4}(2a_2 - 1) \rceil = \lceil \frac{a_2}{2} \rceil = a_3, \; \cdots.$$

We obtain from (1.1.3) that

$$\binom{2x}{x} < 4^{x-1}$$

for $x \geq 8$. Hence we see that

$$\binom{2a_1}{a_1}\binom{2a_2}{a_2}\cdots\binom{2a_m}{a_m} < 4^{a_1-1+2a_2-1}. \tag{1.1.9}$$

Now (1.1.7) follows from (1.1.6) and $2a_1 \leq x+1$. $\square$ $\square$

LEMMA 1.1.6. *Assume that*

$$P\left(\binom{x}{k}\right) \leq k \tag{1.1.10}$$

holds. Then we have

(*i*) $x < k^2$ for $k \geq 11$
(*ii*) $x < k^{\frac{3}{2}}$ for $k \geq 37$.

PROOF. We have

$$\binom{x}{k} = \frac{x}{k}\frac{x-1}{k-1}\cdots\frac{x-k+1}{1} > \left(\frac{x}{k}\right)^k.$$

From (1.1.10) and Lemma 1.1.3, we have $\binom{x}{k} \leq x^{\pi(k)}$. Comparing the upper and lower bounds for $\binom{x}{k}$, we derive that

$$x < k^{\frac{k}{k-\pi(k)}}. \tag{1.1.11}$$

For $k \geq 11$, we exclude 1 and 9 to see that there are at most $[\frac{k+1}{2}]-2$ odd primes upto k. Hence $\pi(k) \leq [\frac{k+1}{2}]-1 \leq \frac{k}{2}$ for $k \geq 11$. Further the number of composite integers $\leq k$ and divisible by 2 or 3 or 5 is

$$\begin{aligned}[\frac{k}{2}]+[\frac{k}{3}]+[\frac{k}{5}]-[\frac{k}{6}]-[\frac{k}{10}]-[\frac{k}{15}]+[\frac{k}{30}]-3 &\geq \frac{k}{2}+\frac{k}{3}+\frac{k}{5}+\frac{k}{30}-\frac{k}{6}-\frac{k}{10}-\frac{k}{15}-7\\ &= \frac{11}{15}k-7.\end{aligned}$$

Thus we have $\pi(k) \leq k-1-(\frac{11}{15}k-7) \leq \frac{k}{3}$ for $k \geq 90$. By direct computation, we see that $\pi(k) \leq \frac{k}{3}$ for $37 \leq k < 90$. Hence

$$\frac{k}{k-\pi(k)} \leq \begin{cases} 2 \text{ for } k \geq 11\\ \frac{3}{2} \text{ for } k \geq 37\end{cases}$$

which, together with (1.1.11), proves the assertion of the lemma. $\square$ $\square$

LEMMA 1.1.7. *Let* $x < k^{\frac{3}{2}}$. *Assume that* (1.1.10) *holds. Then*

$$\binom{x}{k} < 4^{k+\sqrt{x}}. \tag{1.1.12}$$

PROOF. We have from Lemma 1.1.3 and (1.1.10) that

$$\binom{x}{k} = \prod_{p^a || \binom{x}{k}} p^a \leq \prod_{p\leq k} p \prod_{p\leq\sqrt{x}} p \prod_{p\leq\sqrt[3]{x}} p \cdots.$$

By (1.1.4), we have

$$\prod_{p\le k} p \prod_{p\le\sqrt{k}} p \prod_{p\le\sqrt[3]{k}} p \cdots < 4^k \tag{1.1.13}$$

and taking $k=\sqrt{x}$,

$$\prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[4]{x}} p \prod_{p\le\sqrt[6]{x}} p \cdots < 4^{\sqrt{x}}. \tag{1.1.14}$$

Since $x<k^{\frac{3}{2}}$, we have $\sqrt[2l-1]{x}\le\sqrt[l]{k}$ for $l\ge 2$. Hence (1.1.13) and (1.1.14) give

$$\prod_{p\le k} p \prod_{p\le\sqrt{x}} p \prod_{p\le\sqrt[3]{x}} p \cdots < 4^{k+\sqrt{x}}$$

implying (1.1.12). $\square$ $\square$

LEMMA 1.1.8. *Let $k\ge 11$ and $x<k^{\frac{3}{2}}$. Assume* (1.1.10). *Then*
(i) $x<4k$
(ii) $k\le 103$ for $\frac{5}{2}k<x<4k$
(iii) $k\le 113$ for $2k\le x\le\frac{5}{2}k$.

PROOF. We have from (1.1.2) that

$$\binom{x}{k}\ge\begin{cases}\binom{4k}{k}=\binom{2k}{k}\frac{4k(4k-1)\cdots(3k+1)}{2k(2k-1)\cdots(k+1)}>\frac{4^k}{2\sqrt{k}}2^k=\frac{8^k}{2\sqrt{k}} \text{ if } x\ge 4k\\ \binom{\lceil\frac{5}{2}k\rceil}{k}=\binom{2k}{k}\frac{\lceil\frac{5}{2}k\rceil(\lceil\frac{5}{2}k\rceil-1)\cdots(\lceil\frac{5}{2}k\rceil-k+1)}{2k(2k-1)\cdots(k+1)}>\frac{4^k}{2\sqrt{k}}\left(\frac{5}{4}\right)^k \text{ if } \frac{5}{2}k<x<4k\\ \binom{2k}{k}>\frac{4^k}{2\sqrt{k}} \text{ if } 2k\le x\le\frac{5}{2}k.\end{cases} \tag{1.1.15}$$

Let $x\ge 4k$. Then $4k<k^{\frac{3}{2}}$ implying $k\ge 17$. Comparing (8.1.2) with the upper bound of $\binom{x}{k}$ given by (1.1.12), we see from $x<k^{\frac{3}{2}}$ that

$$1>\binom{4k}{k}4^{-k-\sqrt{x}}\ge\binom{4k}{k}2^{-2k-2k^{\frac{3}{4}}}>\frac{8^k}{2\sqrt{k}}2^{-2k-2k^{\frac{3}{4}}} \tag{1.1.16}$$

implying

$$2\sqrt{k}>2^{k-2k^{\frac{3}{4}}}.$$

By induction, we see that $2^{\frac{1}{7}k}>2\sqrt{k}$ for $k\ge 23$. Thus $\frac{1}{7}k>k-2k^{\frac{3}{4}}$ giving $k\le 29$. For $k=17,19,23,29$, we see that (1.1.16) is not valid, proving (i).

Let $\frac{5}{2}k<x<4k$. Comparing (8.1.2) with the upper bound of $\binom{x}{k}$ given by (1.1.12), we see that

$$1>\binom{\lceil\frac{5}{2}k\rceil}{k}4^{-k-\sqrt{x}}\ge\binom{\lceil\frac{5}{2}k\rceil}{k}2^{-2k-4k^{\frac{1}{2}}}>\frac{4^k}{2\sqrt{k}}\left(\frac{5}{4}\right)^k 2^{-2k-4k^{\frac{1}{2}}} \tag{1.1.17}$$

implying

$$\left(\frac{5}{4}\right)^k<2\sqrt{k}\;2^{4\sqrt{k}}.$$

Also $2^x>2x$ for $x\ge 3$ so that $2^{\sqrt{k}}>2\sqrt{k}$ for $k\ge 11$. Thus

$$\left(\frac{5}{4}\right)^k<2^{5\sqrt{k}}$$

which gives $k<257$. Further, we check that (1.1.17) does not hold for $107\le k<257$ with k prime. Thus (ii) is valid.

Finally let $2k \leq x \leq \frac{5}{2}k$. In this case, we see that every prime p with $\frac{1}{3}x < p \leq k$ occurs to the second power in the denominator of $\frac{x!}{k!(x-k)!}$ since $2p > \frac{2x}{3} > k$ and $2p > \frac{2x}{3} = x - \frac{x}{3} > x - p \geq x - k$ and it cannot occur to third power in the numerator since $3p > x$. Thus when $p \leq k$ and $p \mid \binom{x}{k}$, then $p \leq \frac{1}{3}x$. Therefore we have

$$\binom{x}{k} = \prod_{p^a || \binom{x}{k}} p^a \leq \prod_{p \leq \frac{x}{3}} p \prod_{p \leq \sqrt{x}} p \prod_{p \leq \sqrt[3]{x}} p \cdots .$$

Since $\frac{1}{3}x > x^{\frac{2}{3}}$ for $x > 27$, we have $\sqrt[l]{\frac{1}{3}x} \geq \sqrt[2l-1]{x}$ for $l \geq 2$. Hence

$$\binom{x}{k} \leq \left(\prod_{p \leq \frac{x}{3}} p \prod_{p \leq \sqrt{\frac{x}{3}}} p \cdots \right) \left(\prod_{p \leq \sqrt{x}} p \prod_{p \leq \sqrt[4]{x}} p \cdots \right).$$

Now we use (1.1.4) with x replaced by $\frac{1}{3}x$ and $\sqrt{x}$ to get

$$\binom{x}{k} < 4^{\frac{1}{3}x + \sqrt{x}} \leq 4^{\frac{5}{6}k + \sqrt{\frac{5}{2}k}}.$$

Comparing this with the lower bound given by (8.1.2), we obtain

$$1 > \binom{2k}{k} 2^{-\frac{5}{3}k - \sqrt{10k}} > \frac{4^k}{2\sqrt{k}} 2^{-\frac{5}{3}k - \sqrt{10k}} \tag{1.1.18}$$

implying

$$2^{\frac{k}{3}} < 2\sqrt{k}\, 2^{\sqrt{10k}} < 2^{\sqrt{k} + \sqrt{10k}}$$

since $2\sqrt{k} < 2^{\sqrt{k}}$. Therefore

$$\frac{k}{3} < \sqrt{k}(1 + \sqrt{10})$$

so that $k \leq 151$. Further we check that (1.1.18) does not hold for $113 \leq k \leq 151$ with k prime, giving (iii). □ □

1.2. Proof of Theorem 1.0.2

Let $x \geq 2k$. Assume that $P(\binom{x}{k}) \leq k$. Then $P(\Delta'(x,k)) = P(k!\binom{x}{k}) \leq k$. We first prove Theorem 1.0.2 for $k \leq 7$. We note that k divides exactly one term of Δ'. Let $p \leq k$. Let $x - i_p$ be the term in which p occurs to the highest power in Δ'. Then we see that

$$\mathrm{ord}_p(x - i) \leq \mathrm{ord}_p(x - i - (x - i_p)) = \mathrm{ord}_p(i - i_p) \tag{1.2.1}$$

for any $0 \leq i < k$, $i \neq i_p$.

Let $k = 2$. Then $x(x-1)$ is divisible by an odd prime, a contradiction. Let $k = 3$. After removing the term divisible by 3 and then the term in which 2 appears to maximal power, we are left with one term divisible only by 2, and by (1.2.1), this term must be ≤ 2. Hence $x - 2 \leq 2$ or $x \leq 4$, which is not possible since $x \geq 6$. Let $k = 5$. After removing the terms divisible by 5, 3 and term in which 2 appears to maximal power, we are left with at least one term divisible by 2 and the term is ≤ 4 by (1.2.1). Therefore $x - 4 \leq 4$, a contradiction since $x \geq 10$. Let $k = 7$. After removing the terms divisible by $7, 5$ and terms in which 3 and 2 appears to maximal power, we are left with at least two terms divisible by 2 or 3 only and we get $x - 6 \leq 4 \cdot 3 = 12$ by (1.2.1). Thus $14 \leq x \leq 18$. Now we check that $P(\Delta') > 7$ in all these cases.

Thus it remains to consider the case $k \geq 11$. From Lemma 1.1.6 and 1.1.8, we have

$$k \leq 113; \ x < k^2 \text{ for } 11 \leq k \leq 31; \ x < 4k \text{ for } 37 \leq k \leq 113. \tag{1.2.2}$$

We check that

$$p_{i+1} - p_i < 15 \quad \text{for} \quad p_{i+1} \leq 457 = p_{\pi(4\times 113)+1}. \tag{1.2.3}$$

Thus Lemma 1.1.1 with $X = 452, k_0 = 15$ implies that $P(\Delta'(x,k)) > k$ for $x < 452$ and $k \geq 15$, a contradiction. Now we consider the case $x < k^2$ with $11 \leq k \leq 31$. We check that

$$p_{i+1} - p_i < \begin{cases} 21 \quad \text{for} \quad p_{i+1} \leq 967 = p_{\pi(31^2)+1} \\ 11 \quad \text{for} \quad p_{i+1} \leq 173 = p_{\pi(13^2)+1} \quad \text{with} \quad (p_i, p_{i+1}) \neq (113, 127). \end{cases} \tag{1.2.4}$$

We apply Lemma 1.1.1 as follows: for $23 \leq k \leq 31$, take $X = 961, k_0 = 21$; for $k = 17, 19$, take $X = 361, k_0 = 15$; for $k = 11, 13$, take $X = 169, k_0 = 11$. Now Theorem 1.0.2 follows from (1.2.3) for $k = 17, 19$ and (1.2.4) except possibly when $113 < x - k + 1 < x < 127$ and $k = 13$. This gives $x = 126$ and $P(\Delta'(x,k)) > k$ holds in this case as well. $\square$

CHAPTER 2

Results from prime number theory

In this chapter, we give the results from Prime Number Theory which we will be using in the subsequent chapters. We begin with the bounds for $\pi(\nu)$ given by Rosser and Schoenfeld, see [**38**, p. 69-71].

LEMMA 2.0.1. *For $\nu > 1$, we have*

(i) $\pi(\nu) < \dfrac{\nu}{\log \nu}\left(1 + \dfrac{3}{2\log \nu}\right)$

(ii) $\pi(\nu) > \dfrac{\nu}{\log \nu - \frac{1}{2}}$ for $\nu \geq 67$

(iii) $\displaystyle\prod_{p^a \leq \nu} p^a < (2.826)^\nu$

(iv) $\displaystyle\prod_{p \leq \nu} p < (2.763)^\nu$

(v) $p_i \geq i \log i$ for $i \geq 2$.

The following sharper estimates are due to Dusart [**4**, p.14]. See also [**5**, p.55], [**6**, p.414].

LEMMA 2.0.2. *For $\nu > 1$, we have*

(i) $\pi(\nu) \leq \dfrac{\nu}{\log \nu}\left(1 + \dfrac{1.2762}{\log \nu}\right) =: a(\nu)$

(ii) $\pi(\nu) \geq \dfrac{\nu}{\log \nu - 1} =: b(\nu)$ for $\nu \geq 5393$.

The following lemma is due to Ramaré and Rumely [**35**, Theorems 1, 2].

LEMMA 2.0.3. *Let $k \in \{3, 4, 5, 7\}$ and*

$$\theta(x, k, l) = \sum_{\substack{p \leq x \\ p \equiv l (\text{mod } k)}} \log p.$$

For $x_0 \leq 10^{10}$, we have

$$\theta(x, k, l) \geq \begin{cases} \frac{x}{\phi(k)}(1 - \epsilon') \text{ for } x \geq 10^{10} \\ \frac{x}{\phi(k)}\left(1 - \frac{\epsilon\phi(k)}{\sqrt{x_0}}\right) \text{ for } 10^{10} > x \geq x_0 \end{cases} \tag{2.0.1}$$

and

$$\theta(x, k, l) \leq \begin{cases} \frac{x}{\phi(k)}(1 + \epsilon') \text{ for } x \geq 10^{10} \\ \frac{x}{\phi(k)}\left(1 + \frac{\epsilon\phi(k)}{\sqrt{x_0}}\right) \text{ for } 10^{10} > x \geq x_0 \end{cases} \tag{2.0.2}$$

where $\epsilon := \epsilon(k)$ and $\epsilon' := \epsilon'(k)$ are given by

k	3	4	5	7
ϵ	1.798158	1.780719	1.412480	1.105822
ϵ'	0.002238	0.002238	0.002785	0.003248

In the next lemma, we derive estimates for $\pi(x,k,l)$ and $\pi(2x,k,l)-\pi(x,k,l)$ from Lemma 2.0.3.

LEMMA 2.0.4. *Let $k \in \{3,4,5,7\}$. Then we have*

$$\pi(x,k,l) \geq \frac{x}{\log x}\left(\mathfrak{c}_1 + \frac{\mathfrak{c}_2}{\log \frac{x}{2}}\right) \quad \textit{for } x \geq x_0 \tag{2.0.3}$$

and

$$\pi(2x,k,l) - \pi(x,k,l) \leq \mathfrak{c}_3 \frac{x}{\log x} \quad \textit{for } x \geq x_0 \tag{2.0.4}$$

where $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ and x_0 are given by

k	3	4	5	7
$\mathfrak{c}_1$	0.488627	0.443688	0.22175	0.138114
$\mathfrak{c}_2$	0.167712	0.145687	0.0727974	0.043768
$\mathfrak{c}_3$	0.013728	0.067974	0.0170502	0.0114886
x_0	25000	1000	2500	1500

PROOF. We have

$$\theta(x,k,l) = \sum_{\substack{p \leq x \\ p \equiv l (\text{mod } k)}} \log p \leq \pi(x,k,l) \log x$$

so that

$$\pi(x,k,l) \geq \frac{\theta(x,k,l)}{\log x}. \tag{2.0.5}$$

Also,

$$\theta(x,k,l) \leq \pi(\frac{x}{2},k,l)\log\frac{x}{2} + \left(\pi(x,k,l) - \pi(\frac{x}{2},k,l)\right)\log x = \pi(x,k,l)\log x - \pi(\frac{x}{2},k,l)\log 2$$

giving

$$\pi(x,k,l)\log x \geq \theta(x,k,l) + \pi(\frac{x}{2},k,l)\log 2.$$

Now we use (2.0.5) for $\frac{x}{2}$ to derive

$$\pi(x,k,l) \geq \frac{x}{\log x}\left(\frac{\theta(x,k,l)}{x} + \frac{\theta(\frac{x}{2},k,l)\log 2}{x}\frac{1}{\log\frac{x}{2}}\right). \tag{2.0.6}$$

Let $k = 3,4,5,7$ and $x_0 := x_0(k)$ be as given in the statement of the lemma. Since $x_0 \leq 50000 \leq (\frac{\epsilon\phi(k)}{\epsilon'})^2$, we have from (2.0.1) that

$$\begin{aligned} \theta(x,k,l) &\geq \frac{x}{\phi(k)}\left(1 - \frac{\epsilon\phi(k)}{\sqrt{x_0}}\right) \quad \text{for } x \geq x_0, \\ \theta(\frac{x}{2},k,l) &\geq \frac{x}{2\phi(k)}\left(1 - \frac{\epsilon\phi(k)}{\sqrt{\frac{x_0}{2}}}\right) \quad \text{for } x \geq x_0. \end{aligned} \tag{2.0.7}$$

This with (2.0.6) implies (2.0.3). Further we also have from (2.0.2) that

$$\theta(2x,k,l) \leq \frac{2x}{\phi(k)}\left(1 + \frac{\epsilon\phi(k)}{\sqrt{2x_0}}\right) \quad \text{for } x \geq x_0.$$

This with (2.0.7), (2.0.6) and

$$\theta(2x,k,l) - \theta(x,k,l) \geq (\pi(2x,k,l) - \pi(x,k,l))\log x$$

implies (2.0.4). □ □

The next lemma gives a lower bound for $\mathrm{ord}_p(k-1)!$.

LEMMA 2.0.5. *For a prime $p < k$, we have*

$$\mathrm{ord}_p(k-1)! \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log\ p}.$$

PROOF. Let $p^r \leq k-1 < p^{r+1}$. Then we have

$$\mathrm{ord}_p(k-1)! = \left[\frac{k-1}{p}\right] + \cdots + \left[\frac{k-1}{p^r}\right].$$

Now, we note that $\left[\frac{k-1}{p^i}\right] \geq \frac{k-1}{p^i} - \frac{p^i-1}{p^i} = \frac{k}{p^i} - 1$ for $i \geq 1$. Hence

$$\mathrm{ord}_p(k-1)! \geq \sum_{i=1}^{r}\left(\frac{k}{p^i} - 1\right) = \frac{k}{p-1}(1 - \frac{1}{p^r}) - r = \frac{k}{p-1} - \frac{1}{p-1}\frac{k}{p^r} - r.$$

Since $p^r \leq k-1 < k \leq p^{r+1}$, we have $r \leq \frac{\log(k-1)}{\log p}$ and $\frac{k}{p^r} \leq p$, which we use in the estimate for $\mathrm{ord}_p((k-1)!)$ above to get the lemma. □ □

We end this chapter with a lemma on Stirling's formula, see Robbins [**37**].

LEMMA 2.0.6. *For a positive integer ν, we have*

$$\sqrt{2\pi\nu}\ e^{-\nu}\nu^{\nu}e^{\frac{1}{12\nu+1}} < \nu! < \sqrt{2\pi\nu}\ e^{-\nu}\nu^{\nu}e^{\frac{1}{12\nu}}.$$

CHAPTER 3

A survey of refinements and extensions of Sylvester's theorem

Let n, d and $k \geq 2$ be positive integers. For a pair (n, k) and a positive integer h, we write $[n, k, h]$ for the set of all pairs $(n, k), \cdots, (n+h-1, k)$ and we set $[n, k] = [n, k, 1] = \{(n, k)\}$.

Let $W(\Delta)$ denote the number of terms in Δ divisible by a prime $> k$. We observe that every prime exceeding k divides at most one term of Δ. On the other hand, a term may be divisible by more than one prime exceeding k. Therefore we have

$$W(\Delta) \leq \omega(\Delta) - \pi_d(k). \tag{3.0.1}$$

If $\max(n, d) \leq k$, we see that $n + (k-1)d \leq k^2$ and therefore no term of Δ is divisible by more than one prime exceeding k. Thus

$$W(\Delta) = \omega(\Delta) - \pi_d(k) \text{ if } \max(n, d) \leq k. \tag{3.0.2}$$

We are interested in finding lower bounds for $P(\Delta)$, $\omega(\Delta)$ and $W(\Delta)$. The first result in this direction is due to Sylvester [**56**] who proved that

$$P(\Delta) > k \text{ if } n \geq d + k. \tag{3.0.3}$$

This immediately gives

$$\omega(\Delta) > \pi_d(k) \text{ if } n \geq d + k. \tag{3.0.4}$$

We give a survey of several results in this direction.

3.1. Improvements of $\omega(\Delta(n, k)) > \pi(k)$

Let $d = 1$. A proof of Sylvester's result is given in Chapter 1. The result of Sylvester was rediscovered by Schur [**48**] and Erdős [**8**]. Let $k = 2$ and $n > 2$. We see that $\omega(n(n+1)) \neq 1$ since $\gcd(n, n+1) = 1$. Thus $\omega(n(n+1)) \geq 2$. Suppose $\omega(n(n+1)) = 2$. Then both n and $n+1$ are prime powers. If either n or $n+1$ is a prime, then $n+1$ or n is a power of 2, respectively. A prime of the form $2^{2^m} + 1$ is called a *Fermat prime* and a prime of the form $2^m - 1$ is called a *Mersenne prime*. Thus we see that either n is a Mersenne prime or $n+1$ is a Fermat prime. Now assume that $n = p^\alpha, n+1 = q^\beta$ where p, q are distinct primes and $\alpha, \beta > 1$. Thus $q^\beta - p^\alpha = 1$, which is Catalan equation. In 1844, Catalan [**2**] conjectured that 8 and 9 are the only perfect powers that differ by 1. Tijdeman [**58**] proved in 1976 that there are only finitely many perfect powers that differ by 1. Catalan's conjecture has been confirmed recently by Mihăilescu [**27**]. Thus $n = 8$ is the only other n for which $\omega(n(n+1)) = 2$. For all other n, we have $\omega(n(n+1)) \geq 3$. Let $k \geq 3$. We observe that

$$\omega(\Delta(n, k)) = \pi(2k) \text{ if } n = k + 1. \tag{3.1.1}$$

If $k+1$ is prime and $2k+1$ is composite, then we observe from (3.1.1) by writing

$$\Delta(k+2, k) = \Delta(k+1, k)\frac{2k+1}{k+1}$$

that

$$\omega(\Delta(k+2, k)) = \pi(2k) - 1. \tag{3.1.2}$$

Let $k+1$ be a prime of the form $3r+2$. Then $2k+1 = 3(2r+1)$ is composite. Since there are infinitely many primes of the form $3r+2$, we see that there are infinitely many k for which $k+1$ is prime and $2k+1$ is composite. Therefore (3.1.2) is valid for infinitely many k. Thus an inequality sharper than $\omega(\Delta(n,k)) \geq \pi(2k) - 1$ for $n > k$ is not valid.

Saradha and Shorey [**41**, Corollary 3] extended the proof of Erdős [**8**] given in Chapter 1 to sharpen (3.0.4) and gave explicit bounds of $\omega(\Delta(n,k))$ as

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{1}{3}\pi(k)\right] + 2 \text{ if } n > k > 2 \tag{3.1.3}$$

unless $(n,k) \in S_1$ where S_1 is the union of sets

$$\begin{cases} [4,3], [6,3,3], [16,3], [6,4], [6,5,4], [12,5], [14,5,3], [23,5,2], \\ [7,6,2], [15,6], [8,7,3], [12,7], [14,7,2], [24,7], [9,8], [14,8], \\ [14,13,3], [18,13], [20,13,2], [24,13], [15,14], [20,14], [20,17]. \end{cases} \tag{3.1.4}$$

Laishram and Shorey [**18**] improved it to $\frac{3}{4}$. Define

$$\delta(k) = \begin{cases} 2 & \text{if } \ 3 \leq k \leq 6 \\ 1 & \text{if } \ 7 \leq k \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\left[\frac{3}{4}\pi(k)\right] - 1 + \delta(k) \geq \left[\frac{1}{3}\pi(k)\right] + 2.$$

We have

THEOREM 3.1.1. *Let $n > k \geq 3$. Then*

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{3}{4}\pi(k)\right] - 1 + \delta(k) \tag{3.1.5}$$

unless

$$(n,k) \in S_1 \cup S_2$$

where S_1 is given by (3.1.4) *and S_2 is the union of sets*

$$\begin{cases} [20,19,3], [24,19], [21,20], [48,47,3], [54,47], [49,48], [74,71,2], [74,72], \\ [74,73,3], [84,73], [75,74], [84,79], [84,83], [90,83], [108,83], [110,83], \\ [90,89], [102,89], [104,89], [108,103], [110,103,2], [114,103,2], [110,104], \\ [114,104], [108,107,12], [109,108,10], [110,109,9], [111,110,7], [112,111,5], \\ [113,112,3], [114,113,7], [138,113], [140,113,2], [115,114,5], [140,114], \\ [116,115,3], [117,116], [174,173], [198,181], [200,181,2], [200,182], \\ [200,193,2], [200,194], [200,197], [200,199,3], [201,200], [282,271,5], \\ [282,272], [284,272,2], [284,273], [278,277,3], [282,277,5], [279,278], \\ [282,278,4], [282,279,3], [282,280], [282,281,7], [283,282,5], \\ [284,283,5], [294,283], [285,284,3], [286,285], [294,293]. \end{cases} \tag{3.1.6}$$

We note here that the right hand sides of (3.1.3) and (3.1.5) are identical for $3 \leq k \leq 18$. Theorem 3.1.1 is an improvement of (3.1.3) for $k \geq 19$. The proof of this theorem uses sharp bounds of π function due to Dusart given by Lemma 2.0.2. We derive the following two results from Theorem 3.1.1. We check that the exceptions in Theorem 3.1.1 satisfy $\omega(\Delta(n,k)) \geq \pi(2k) - 1$. Hence Theorem 3.1.1 gives

Corollary 3.1.2. *Let $n > k$. Then*

$$\omega(\Delta(n,k)) \geq \min\left(\pi(k) + \left[\frac{3}{4}\pi(k)\right] - 1 + \delta(k),\ \pi(2k) - 1\right). \tag{3.1.7}$$

Further all the exceptions in Theorem 3.1.1 except $(n,k) \in \{(114,109),(114,113)\}$ satisfy $\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{2}{3}\pi(k)\right] - 1$. Thus we obtain the following corollary from Theorem 3.1.1.

Corollary 3.1.3. *Let $n > k$. Then*

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{2}{3}\pi(k)\right] - 1 \tag{3.1.8}$$

unless

$$(n,k) \in \{(114,109),(114,113)\}. \tag{3.1.9}$$

The constant $\frac{3}{4}$ in Theorem 3.1.1 can be replaced by a number close to 1 if $n > \frac{17}{12}k$.

Theorem 3.1.4. *Let $k \geq 3$ and $(n,k) \neq (6,4)$. Then we have*

$$\omega(\Delta(n,k)) \geq \pi(2k) \text{ if } n > \frac{17}{12}k. \tag{3.1.10}$$

The inequality (3.1.10) is an improvement of (3.1.3) for $k \geq 10$. We observe that $\frac{17}{12}k$ in Theorem 3.1.4 is optimal since $\omega(\Delta(34,24)) = \pi(48)-1$. Also the assumption $(n,k) \neq (6,4)$ is necessary since $\omega(\Delta(6,4)) = \pi(8) - 1$. We recall that there are infinitely many pairs $(n,k) = (k+2,k)$ satisfying (3.1.2). Thus there are infinitely many pairs (n,k) with $n \leq \frac{17}{12}k$ such that $\omega(\Delta(n,k)) < \pi(2k)$. Let $n = k + r$ with $0 < r \leq k$. We observe that every prime p with $k \leq n-1 < p \leq n+k-1$ is a term of $\Delta(n,k)$. Since $k > \frac{n-1}{2}$, we also see that $2p$ is a term in $\Delta(n,k)$ for every prime p with $k < p \leq \frac{n+k-1}{2}$. Further all primes $\leq k$ divide $\Delta(n,k)$. Thus

$$\omega(\Delta(n,k)) = \pi(2k+r-1) - \pi(k+r-1) + \pi(k + \frac{r-1}{2}) = \pi(2k) + F(k,r)$$

where

$$F(k,r) = \pi(2k+r-1) - \pi(2k) - \left(\pi(k+r-1) - \pi(k + \frac{r-1}{2})\right).$$

We use the above formula for finding some pairs (n,k) as given below when $k < 5000$ and $n \leq 2k$ for which $\omega(\Delta(n,k)) < \pi(2k)$:

$$\begin{aligned}
\omega(\Delta(n,k)) &= \pi(2k) - 1 \text{ if } (n,k) = (6,4),(34,24),(33,25),(80,57)\\
\omega(\Delta(n,k)) &= \pi(2k) - 2 \text{ if } (n,k) = (74,57),(284,252),(3943,3880)\\
\omega(\Delta(n,k)) &= \pi(2k) - 3 \text{ if } (n,k) = (3936,3879),(3924,3880),(3939,3880)\\
\omega(\Delta(n,k)) &= \pi(2k) - 4 \text{ if } (n,k) = (1304,1239),(1308,1241),(3932,3879)\\
\omega(\Delta(n,k)) &= \pi(2k) - 5 \text{ if } (n,k) = (3932,3880),(3932,3881),(3932,3882).
\end{aligned}$$

It is also possible to replace $\frac{3}{4}$ in Theorem 3.1.1 by a number close to 1 if $n > k$ and k is sufficiently large. Let $k < n < \frac{17}{12}k$. Then

$$\omega(\Delta(n,k)) \geq \pi(n+k-1) - \pi(n-1) + \pi(k).$$

Let $\epsilon > 0$ and $k \geq k_0$ where k_0 exceeds a sufficiently large number depending only on ϵ. Using the estimates (i) and (ii) of Lemma 2.0.2, we get

$$\begin{aligned}\pi(n+k-1)-\pi(n-1) &\geq \frac{n+k-1}{\log(n+k-1)-1}-\frac{n}{\log n}-\frac{1.2762n}{\log^2 n}\\ &\geq \frac{n+k-1}{\log n}-\frac{n}{\log n}-\frac{1.2762n}{\log^2 n}\\ &\geq \frac{k-1}{\log n}-\frac{1.2762k}{\log^2 k}\\ &\geq (1-\epsilon)\pi(k).\end{aligned}$$

Thus $\omega(\Delta(n,k)) \geq (2-\epsilon)\pi(k)$ for $k < n < \frac{17}{12}k$ which we combine with Theorem 3.1.4 to conclude the following result.

THEOREM 3.1.5. *Let $\epsilon > 0$ and $n > k$. Then there exists a computable number k_0 depending only on ϵ such that for $k \geq k_0$, we have*

$$\omega(\Delta(n,k)) \geq (2-\epsilon)\pi(k). \tag{3.1.11}$$

Proofs of Theorems 3.1.1 and 3.1.4 are given in *Chapter* 4. We end this section with a conjecture of Grimm [**14**]:

Suppose $n, n+1, \cdots, n+k-1$ are all composite numbers, then there are distinct primes p_{i_j} such that $p_{i_j}|(n+j)$ for $0 \leq j < k$.

This conjecture is open. The conjecture implies that if $n, n+1, \cdots, n+k-1$ are all composite, then $\omega(\Delta(n,k)) \geq k$ which is also open. Let $g(n)$ be the largest integer such that there exist distinct prime numbers $P_0, \cdots P_{g(n)}$ with $P_i|n+i$. A result of Ramachandra, Shorey and Tijdeman [**33**] states that

$$g(n) > c_1\left(\frac{\log n}{\log\log n}\right)^3$$

where $c_1 > 0$ is a computable absolute constant. Further Ramachandra, Shorey and Tijdeman [**34**] showed that

$$\omega(\Delta(n+1,k)) \geq k \ \text{ for } \ 1 \leq k \leq \exp(c_2(\log n)^{\frac{1}{2}})$$

where c_2 is a computable absolute constant.

3.2. Results on refinement of $P(\Delta(n,k)) > k$

Hanson [**16**] improved (1) as $P(\Delta(n,k)) > 1.5k-1$ for $n > k > 1$. The best results in this direction can be found in Langevin [**23**], [**25**]. Sharper estimates have been obtained when k is sufficiently large. See Shorey and Tijdeman [**50**, Chapter 7]. Ramachandra and Shorey [**32**] proved that

$$P(\Delta(n,k)) > c_3 k\log k\left(\frac{\log\log k}{\log\log\log k}\right)^{\frac{1}{2}} \ \text{if } n > k^{\frac{3}{2}}$$

where $c_3 > 0$ is a computable absolute constant. Further it follows from the work of Jutila [**17**] and Shorey [**49**] that

$$P(\Delta(n,k)) > c_4 k\log k\frac{\log\log k}{\log\log\log k} \ \text{if } n > k^{\frac{3}{2}}$$

where c_4 is a computable absolute positive constant. If $n \leq k^{\frac{3}{2}}$, it follows from the results on difference between consecutive primes that $\Delta(n,k)$ has a term which is prime. The proofs are not elementary. The proof of the result of Ramachandra and Shorey depends on Sieve method and the theory of linear forms in logarithms. The proof of the result of Jutila and Shorey depends on estimates from exponential sums and the theory of linear forms in logarithms. Langevin [**21**], [**22**] proved that for any $\epsilon > 0$,

$$P(\Delta(n,k)) > (1-\epsilon)k \log\log k \text{ if } n \geq c_5 = c_5(k,\epsilon)$$

where c_5 is a computable number depending only on k and ϵ. For an account of results in this direction, see Shorey and Tijdeman [**50**, p. 135].

3.3. Sharpenings of (3.0.3) and (3.0.4)

We first state Schinzel's Hypothesis H [**46**]:

Let $f_1(x), \cdots, f_r(x)$ be irreducible non constant polynomials with integer coefficients such that for every prime p, there is an integer a such that the product $f_1(a) \cdots f_r(a)$ is not divisible by p. Then there are infinitely many positive integers m such that $f_1(m), \cdots, f_r(m)$ are all primes.

We assume Schinzel's hypothesis. Then $1+d$ and $1+2d$ are primes for infinitely many d. Therefore

$$\omega(\Delta) = \pi(k), \quad k = 3 \tag{3.3.1}$$

for infinitely many pairs $(n,d) = (1,d)$. Let $f_r(x) = 1 + rx$ for $r = 1,2,3,4$. For a given p, we see that $p \nmid f_1(p)f_2(p) \cdots f_4(p)$. Hence there are infinitely many d such that $1+d, 1+2d, 1+3d, 1+4d$ are all primes. Thus

$$\omega(\Delta) = \pi(k) + 1, \;\; k = 4,5 \tag{3.3.2}$$

for infinitely many pairs $(n,d) = (1,d)$.

Langevin [**24**] sharpened (3.0.3) to

$$P(\Delta) > k \text{ if } n > k.$$

Shorey and Tijdeman [**52**] improved the above result as

$$P(\Delta) > k \;\; \text{unless} \;\; (n,d,k) = (2,7,3). \tag{3.3.3}$$

Further Shorey and Tijdeman [**51**] proved that

$$\omega(\Delta) \geq \pi(k). \tag{3.3.4}$$

Thus (3.3.4) is likely to be best possible when $k = 3$ by (3.3.1). A proof of (3.3.3) is given in *Chapter* 5. Moree [**29**] sharpened (3.3.4) to

$$\omega(\Delta) > \pi(k) \text{ if } k \geq 4 \text{ and } (n,d,k) \neq (1,2,5). \tag{3.3.5}$$

We observe that (3.3.5) implies (3.3.3) for $k \geq 4$. If $k = 4,5$, then (3.3.5) is likely to be best possible by (3.3.2).

Saradha and Shorey [**42**] showed that for $k \geq 4$, Δ is divisible by at least 2 distinct primes exceeding k except when $(n,d,k) \in \{(1,5,4),(2,7,4),(3,5,4), (1,2,5),(2,7,5),(4,7,5), (4,23,5)\}$. Further Saradha, Shorey and Tijdeman [**45**, Theorem 1] improved (3.3.5) to

$$\omega(\Delta) > \frac{6}{5}\pi(k) + 1 \text{ for } k \geq 6 \tag{3.3.6}$$

unless $(n,d,k) \in V_0$ where V_0 is

$$\{(1,2,6),(1,3,6),(1,2,7),(1,3,7),(1,4,7),(2,3,7),(2,5,7),(3,2,7),$$
$$(1,2,8),(1,2,11),(1,3,11),(1,2,13),(3,2,13),(1,2,14)\}.$$

In fact they derived (3.3.6) from

$$W(\Delta) > \frac{6}{5}\pi(k) - \pi_d(k) + 1 \text{ for } k \geq 6 \tag{3.3.7}$$

unless $(n, d, k) \in V_0$. It is easy to see that the preceding result is equivalent to [**45**, Theorem 2]. We have no improvement for (3.3.7) when $k = 6, 7$ and 8. For $k \geq 9$, Laishram and Shorey [**19**] sharpened (3.3.7) as

THEOREM 3.3.1. *Let $k \geq 9$ and $(n, d, k) \notin V$ where V is given by*

$$\begin{cases} n = 1, \ d = 3, \ k = 9, 10, 11, 12, 19, 22, 24, 31; \\ n = 2, \ d = 3, \ k = 12; \ n = 4, \ d = 3, \ k = 9, 10; \\ n = 2, \ d = 5, \ k = 9, 10; \ n = 1, \ d = 7, \ k = 10. \end{cases} \tag{3.3.8}$$

Then

$$W(\Delta) \geq \pi(2k) - \pi_d(k) - \rho \tag{3.3.9}$$

where

$$\rho = \rho(d) = \begin{cases} 1 \text{ if } d = 2, n \leq k \\ 0 \text{ otherwise.} \end{cases} .$$

When $d = 2$ and $n = 1$, we see that

$$\omega(\Delta) = \pi(2k) - 1$$

and

$$W(\Delta) = \pi(2k) - \pi_d(k) - 1$$

by (3.0.2), for every $k \geq 2$. This is also true for $n = 3, d = 2$ and $2k + 1$ not a prime. Therefore (3.3.9) is best possible when $d = 2$. We see from Theorem 3.3.1 and (3.0.1) that

$$\omega(\Delta) \geq \pi(2k) - \rho \ \text{ if } \ (n, d, k) \notin V. \tag{3.3.10}$$

For $(n, d, k) \in V$, we see that $\omega(\Delta) = \pi(2k) - 1$ except at $(n, d, k) = (1, 3, 10)$. This is also the case for $(n, d, k) \in V_0$ with $k = 6, 7, 8$. Now, we apply Theorem 3.3.1, (3.3.6) for $k = 6, 7, 8$ and (3.3.5) for $k = 4, 5$ to derive

COROLLARY 3.3.2. *Let $k \geq 4$. Then*

$$\omega(\Delta) \geq \pi(2k) - 1 \tag{3.3.11}$$

except at $(n, d, k) = (1, 3, 10)$.

This solves a conjecture of Moree [**29**]. Proof of Theorem 3.3.1 is given in *Chapter* 6.

CHAPTER 4

Refinement of Sylvester's theorem for consecutive integers: Proof of Theorems 3.1.1 and 3.1.4

In this chapter we prove Theorems 3.1.1 and 3.1.4. We give a sketch of the proof. We first show that it is enough to prove Theorem 3.1.1 for k which are primes and Theorem 3.1.4 for k such that $2k-1$ is a prime. The sharp estimates of π function due to Dusart given in Lemma 2.0.2 have been applied to count the number of terms in $\Delta'(x,k)$ which are primes and the number of terms of the form ap with $2 \le a \le 6$ and $p > k$. The latter contribution is crucial for keeping the estimates well under computational range. It has been applied in the interval $2k \le x < 7k$. In fact this interval has been partitioned into several subintervals and it has been applied to each of those subintervals. This leads to sharper estimates. See Lemmas 4.2.6, 4.2.7, 4.2.9. For covering the range $x \ge 7k$, the ideas of Erdős [**8**] have been applied, see Lemmas 4.2.3, 4.2.5, 4.2.8.

4.1. An Alternative Formulation

As remarked in Chapter 3, we prove Theorem 3.1.1 for $k \ge 19$ and Theorem 3.1.4 for $k \ge 10$. Further we derive these two theorems from the following more general result.

Theorem 4.1.1. **(a)** *Let $k \ge 19$, $x \ge 2k$ and $(x,k) \notin S_3$ where S_3 is the union of all sets $[x,k,h]$ such that $[x-k+1,k,h]$ belongs to S_2 given by* (3.1.6)*. Let $f_1 < f_2 < \cdots < f_\mu$ be all the integers in $[0,k)$ satisfying*

$$P((x-f_1)\cdots(x-f_\mu)) \le k. \tag{4.1.1}$$

Then

$$\mu \le k - \left[\frac{3}{4}\pi(k)\right] + 1. \tag{4.1.2}$$

(b) *Let $k \ge 10$, $x > \frac{29}{12}k - 1$. Assume* (4.1.1)*. Then we have*

$$\mu \le k - M(k) \tag{4.1.3}$$

where

$$M(k) = \max(\pi(2k) - \pi(k), \left[\frac{3}{4}\pi(k)\right] - 1). \tag{4.1.4}$$

Thus, under the assumptions of the theorem, we see that the number of terms in $\Delta' = x(x-1)\cdots(x-k+1)$ divisible by a prime $> k$ is at least $k-\mu$. Since each prime $> k$ can divide at most one term, there are at least $k-\mu$ primes $> k$ dividing Δ'. Thus

$$\omega(\Delta') \ge \pi(k) + k - \mu.$$

Putting $x = n+k-1$, we see that $\Delta' = \Delta$ and hence

$$\omega(\Delta) \ge \pi(k) + k - \mu$$

and the Theorems 3.1.1 for $k \ge 19$ and Theorem 3.1.4 for $k \ge 10$ follow from (4.1.2) and (4.1.3).

4.2. Lemmas

LEMMA 4.2.1. *We have*

$$M(k) = \begin{cases} \left[\frac{3}{4}\pi(k)\right] - 1 & \text{if} \quad k \in \mathfrak{K}_1 \\ \pi(2k) - \pi(k) & \text{otherwise} \end{cases} \tag{4.2.1}$$

where $\mathfrak{K}_1$ *is given by*

$$\begin{aligned} \mathfrak{K}_1 = \{&19, 20, 47, 48, 73, 74, 83, 89, 107, 108, 109, 110, 111, 112, 113, 114, \\ &115, 116, 173, 199, 200, 277, 278, 281, 282, 283, 284, 285, 293\}. \end{aligned} \tag{4.2.2}$$

PROOF. By Lemma 2.0.2 (i) and (ii), we have

$$\pi(2k) - \pi(k) - \left[\frac{3}{4}\pi(k)\right] + 1 \geq \frac{2k}{\log(2k) - 1} - \frac{7}{4}\frac{k}{\log k}\left(1 + \frac{1.2762}{\log k}\right) + 1$$

for $k \geq 2697$. The right hand side of the above inequality is an increasing function of k and it is non-negative at $k = 2697$. Hence $\pi(2k) - \pi(k) \geq \left[\frac{3}{4}\pi(k)\right] - 1$ for $k \geq 2697$ thereby giving $M(k) = \pi(2k) - \pi(k)$ for $k \geq 2697$. For $k < 2697$, we check that (4.2.1) is valid. □

LEMMA 4.2.2. *(i) Let $k' < k''$ be consecutive primes. Suppose Theorem 4.1.1* **(a)** *holds at k'. Then it holds for all k with $k' \leq k < k''$.*
(ii) Let $k_1 < k_2$ be such that $2k_1 - 1$ and $2k_2 - 1$ are consecutive primes. Suppose Theorem 4.1.1 **(b)** *holds at k_1. Then Theorem 4.1.1* **(b)** *holds for all k with $k_1 \leq k < k_2$, $k \notin \mathfrak{K}_1$.*

PROOF. Firstly, we see that (4.1.2) and (4.1.3) are equivalent to

$$W(\Delta') \geq \left[\frac{3}{4}\pi(k)\right] - 1 \tag{4.2.3}$$

and

$$W(\Delta') \geq M(k), \tag{4.2.4}$$

respectively.

Suppose that Theorem 4.1.1 **(a)** holds at k' for k' prime. Let k as in the statement of the Lemma and $x \geq 2k$. Then $x \geq 2k_1$ and $\Delta' = x(x-1)\cdots(x-k'+1)(x-k')\cdots(x-k+1)$. Thus

$$W(\Delta') \geq W(x(x-1)\cdots(x-k'+1)) \geq \left[\frac{3}{4}\pi(k')\right] - 1 = \left[\frac{3}{4}\pi(k)\right] - 1$$

implying (i). We now prove (ii). Assume that Theorem 4.1.1 **(b)** holds at k_1. Let k be as in the statement of the lemma. Further let $x \geq \frac{29}{12}k - 1 \geq \frac{29}{12}k_1 - 1$. Since $k \notin \mathfrak{K}_1$, we have $M(k) = \pi(2k) - \pi(k)$ by Lemma 4.2.1. Also $\pi(2k_1) = \pi(2k_1 - 1) = \pi(2k-1) = \pi(2k)$. Therefore

$$W(\Delta') \geq W(x(x-1)\cdots(x-k_1+1)) \geq M(k_1) = \pi(2k_1) - \pi(k_1) \geq \pi(2k) - \pi(k)$$

implying (4.2.4). □

The next lemma is a generalisation of Lemma 1.1.7. We need some notations. Let $P_0 > 0$ and $\nu \geq 0$ with $g_1, g_2, \cdots g_\nu$ be all the integers in $[0, k)$ such that each of $x - g_i$ with $1 \leq i \leq \nu$ is divisible by a prime exceeding P_0. Further we write

$$(x - g_1)\cdots(x - g_\nu) = GH \tag{4.2.5}$$

with $\gcd(G, H) = 1$, $\gcd(H, \prod_{p \leq P_0} p) = 1$. Then we have

LEMMA 4.2.3. *If* $x < P_0^{\frac{3}{2}}$, *then*

$$\binom{x}{k} \leq (2.83)^{P_0+\sqrt{x}} x^{\nu} \left(G \prod_{p>P_0} p^{\mathrm{ord}_p(k!)} \right)^{-1}. \tag{4.2.6}$$

PROOF. Let $p^a || \binom{x}{k}$. From (1.1.1), we have $p^{ord_p\binom{x}{k}} = p^a \leq x$. Therefore

$$\prod_{p \leq P_0} p^{\mathrm{ord}_p\binom{x}{k}} \leq \prod_{\substack{p \leq P_0 \\ p^a \leq x}} p^a \leq \prod_{p \leq P_0} p \prod_{p \leq x^{\frac{1}{2}}} p \prod_{p \leq x^{\frac{1}{3}}} p \cdots. \tag{4.2.7}$$

From Lemma 2.0.1 (iii) with $\nu = \sqrt{x}$ and $\nu = P_0$, we get

$$\prod_{p \leq x^{\frac{1}{2}}} p \prod_{p \leq x^{\frac{1}{4}}} p \prod_{p \leq x^{\frac{1}{6}}} p \cdots < (2.83)^{\sqrt{x}} \tag{4.2.8}$$

and

$$\prod_{p \leq P_0} p \prod_{p \leq P_0^{\frac{1}{2}}} p \prod_{p \leq P_0^{\frac{1}{3}}} p \cdots < (2.83)^{P_0},$$

respectively. Since $x < P_0^{\frac{3}{2}}$, we have $P_0^{\frac{1}{l}} > x^{\frac{1}{2l-1}}$ for $l \geq 2$ so that the latter inequality implies

$$\prod_{p \leq P_0} \prod_{p \leq x^{\frac{1}{3}}} p \prod_{p \leq x^{\frac{1}{5}}} p \cdots < (2.83)^{P_0}. \tag{4.2.9}$$

Combining (4.2.7), (4.2.8) and (4.2.9), we get

$$\prod_{p \leq P_0} p^{\mathrm{ord}_p\binom{x}{k}} \leq (2.83)^{P_0+\sqrt{x}}. \tag{4.2.10}$$

By (4.2.5), we have

$$\prod_{p > P_0} p^{\mathrm{ord}_p\binom{x}{k}} = \frac{(x-g_1)\cdots(x-g_\nu)}{G \prod_{p>P_0} p^{\mathrm{ord}_p(k!)}}. \tag{4.2.11}$$

Further we observe that

$$(x-g_1)\cdots(x-g_\nu) < x^{\nu}. \tag{4.2.12}$$

Finally, we combine (4.2.10), (4.2.11) and (4.2.12) to conclude (4.2.6). □

Lemma 4.2.3 with $G \geq 1, P_0 = k$ and $\nu = k - \mu$ implies the following Corollary, see Saradha and Shorey [**41**, Lemma 3].

COROLLARY 4.2.4. *Let* $x < k^{\frac{3}{2}}$. *Assume that* (4.1.1) *holds. Then*

$$\binom{x}{k} \leq (2.83)^{k+\sqrt{x}} x^{k-\mu}.$$

LEMMA 4.2.5. *Assume* (4.1.1) *and*

$$\mu \geq k - M(k) + 1 \tag{4.2.13}$$

where $M(k)$ *is given by* (4.1.4). *Then we have*

(*i*) $x < k^{\frac{3}{2}}$ *for* $k \geq 71$
(*ii*) $x < k^{\frac{7}{4}}$ *for* $k \geq 25$
(*iii*) $x < k^2$ *for* $k \geq 13$

(iv) $x < k^{\frac{9}{4}}$ *for* $k \geq 10$.

PROOF. Since $(x-f_1)\cdots(x-f_\mu)$ divides $\binom{x}{k}k!$, we observe from (4.1.1) and (1.1.1) that

$$(x-f_1)\cdots(x-f_\mu) \leq \left(\prod_{p\leq k} p^{\mathrm{ord}_p\binom{x}{k}}\right) k! \leq \left(\prod_{p\leq k} x\right) k! = x^{\pi(k)}k!. \tag{4.2.14}$$

Also

$$(x-f_1)\cdots(x-f_\mu) \geq (x-f_\mu)^\mu \geq (x-k+1)^\mu > x^\mu\left(1-\frac{k}{x}\right)^\mu.$$

Comparing this with (4.2.14), we get

$$k! > x^{\mu-\pi(k)}\left(1-\frac{k}{x}\right)^\mu. \tag{4.2.15}$$

Let $k \geq 71$. We assume that $x \geq k^{\frac{3}{2}}$ and we shall arrive at a contradiction. From (4.2.15), we have

$$k! > k^{\frac{3}{2}(\mu-\pi(k))}\left(1-\frac{1}{\sqrt{k}}\right)^\mu \tag{4.2.16}$$

and since $\mu \leq k$,

$$k! > k^{\frac{3}{2}(\mu-\pi(k))}\left(1-\frac{1}{\sqrt{k}}\right)^k. \tag{4.2.17}$$

We use (4.2.17), (4.2.13), (4.2.1) and Lemmas 2.0.2 (i) and 2.0.6 to derive for $k \geq 294$ that

$$1 > 2.718k^{\frac{1}{2}-\frac{3}{\log 2k}(1+\frac{1.2762}{\log 2k})}(1-\frac{1}{\sqrt{k}})$$

since $\exp\left(\frac{\log 0.3989k}{k}-\frac{1}{12k^2}\right) \geq 1$. The right hand side of above inequality is an increasing function of k and it is not valid at $k = 294$. Thus $k \leq 293$. Further we check that (4.2.17) is not valid for $71 \leq k \leq 293$ except at $k = 71, 73$ by using (4.2.13) with $\mu = k - M(k) + 1$ and the exact values of $k!$ and $M(k)$. Let $k = 71, 73$. We check that (4.2.16) is not satisfied if (4.2.13) holds with equality sign. Thus we may suppose that (4.2.13) holds with strict inequality. Then we find that (4.2.17) does not hold. This proves (i). For the proofs of (ii), (iii) and (iv), we may assume that $x \geq k^{\frac{7}{4}}$ for $25 \leq k \leq 70$, $x \geq k^2$ for $13 \leq k \leq 24$ and $x \geq k^{\frac{9}{4}}$ for $k = 10, 11, 12$, respectively, and arrive at a contradiction. □

The next four lemmas show that under the hypothesis of Theorem 4.1.1, k is bounded. Further we show that Theorem 4.1.1 **(a)** is valid for primes k if $x \leq \frac{29}{12}k-1$ and Theorem 4.1.1 **(b)** is valid for all $k \in \mathfrak{K}$ where

$$\mathfrak{K} = \mathfrak{K}_1 \cup \{k \big| k \geq 10 \text{ and } 2k-1 \text{ is a prime}\}. \tag{4.2.18}$$

LEMMA 4.2.6. **(a)** *Let* $k \geq 19$ *be a prime,* $2k \leq x \leq \frac{29}{12}k-1$ *and* $(x,k) \notin S_3$*. Then Theorem 4.1.1*(**a**) *is valid.*
(b) *Let* $k \geq 10$, $\frac{29}{12}k-1 < x < 3k$*. Then Theorem 4.1.1*(**b**) *holds for all* $k \in \mathfrak{K}$.

PROOF. Let $2k \leq x < 3k$. We observe that every prime p with $k \leq x-k < p \leq x$ is a term of Δ'. Since $k > \frac{x-k}{2}$, we also see that $2p$ is a term in Δ' for every prime p with $k < p \leq \frac{x}{2}$. Thus

$$W(\Delta') \geq \pi(x) - \pi(x-k) + \pi\left(\frac{x}{2}\right) - \pi(k). \tag{4.2.19}$$

The contribution of $\pi(\frac{x}{2}) - \pi(k)$ in the above expression is necessary to get an upper bound for k which is not very large.

(a) Let $2k \le x \le \frac{29}{12}k - 1$ with $(x,k) \notin S_3$. We will show that (4.2.3) holds. Let $(2+t_1)k \le x < (2+t_2)k$ with $0 \le t_1 < t_2 \le 1$ and $t_2 - t_1 \le \frac{1}{4}$. Then we have from (4.2.19) that

$$W(\Delta') \ge \pi(2k + t_1 k) - \pi(k + t_2 k) + \pi(k + \frac{t_1 k}{2}) - \pi(k).$$

Hence it is enough to prove

$$\pi((2+t_1)k) - \pi((1+t_2)k) + \pi((1+\frac{t_1}{2})k) - \pi(k) - \left[\frac{3}{4}\,\pi(k)\right] + 1 \ge 0. \tag{4.2.20}$$

Using Lemma 2.0.2 (i), (ii) and

$$\frac{\log Y}{\log Z} = 1 + \frac{\log(\frac{Y}{Z})}{\log Z} \text{ and } \frac{\log Y}{\log Z - 1} = 1 + \frac{1 + \log(\frac{Y}{Z})}{\log Z - 1},$$

we see that the left hand side of (4.2.20) is at least

$$\begin{aligned} &\sum_{i=1}^{2} b\left(\frac{2+t_1}{i}k\right) - a((1+t_2)k) - \frac{7}{4}\,a(k) + 1 \\ =&\frac{k}{(\log(2+t_1)k)^2}\left\{f(k,t_1,t_2) - g(k,t_1,t_2) - \frac{7}{4}g(k,t_1,0)\right\} + 1 \end{aligned} \tag{4.2.21}$$

for $k \ge 5393$, where

$$f(k,t_1,t_2) = (1.5t_1 - t_2 + \frac{1}{4})(\log(2+t_1)k) + \sum_{i=1}^{2} \frac{(2+t_1)(1+\log i)}{i}\left(1 + \frac{1+\log i}{\log((2+t_1)k/i) - 1}\right)$$

and

$$g(k,t_1,t_2) = (1+t_2)\left(1 + \frac{\log(\frac{2+t_1}{1+t_2})}{\log((1+t_2)k)}\right)\left(1.2762 + \log\left(\frac{2+t_1}{1+t_2}\right) + \frac{1.2762\log(\frac{2+t_1}{1+t_2})}{\log((1+t_2)k)}\right).$$

Then we have

$$kf'(k,t_1,t_2) = (1.5t_1 - t_2 + \frac{1}{4}) - \sum_{i=1}^{2}\left(\frac{2+t_1}{i}\right)\left(\frac{1+\log i}{\log((2+t_1)k/i) - 1}\right)^2.$$

We write

$$1.5t_1 - t_2 + \frac{1}{4} = 0.5t_1 - (t_2 - t_1) + \frac{1}{4}$$

to observe that the left hand side is positive unless $(t_1, t_2) = (0, \frac{1}{4})$ and we shall always assume that $(t_1, t_2) \ne (0, \frac{1}{4})$.

Let $k_0 = k_0(t_1, t_2)$ be such that $kf'(k, t_1, t_2)$ is positive at k_0. Since $kf'(k, t_1, t_2)$ is an increasing function of k, we see that $f(k, t_1, t_2)$ is also an increasing function of k for $k \ge k_0$. Also $g(k, t_1, t_2)$ is a decreasing function of k. Hence (4.2.21) is an increasing function of k for $k \ge k_0$. Let $k_1 = k_1(t_1, t_2) \ge k_0$ be such that (4.2.21) is non-negative at k_1. Then (4.2.20) is valid for $k \ge k_1$. For $k < k_1$, we check inequality (4.2.20) by using the exact values of $\pi(\nu)$. Again for k not satisfying (4.2.20), we take $x = 2k + r$ with $t_1 k \le r < t_2 k$ and check that the right hand side of (4.2.19) is at least the right hand side of (4.2.3).

Let $2k \le x < \frac{49}{24}k$. Then $t_1 = 0, t_2 = \frac{1}{24}$ and we find $k_1 = 5393$ by (4.2.21). For $k < 5393$ and k prime, we check that (4.2.20) holds except at the following values of k:

$$\begin{cases} 19, 47, 71, 73, 83, 89, 103, 107, 109, 113, 151, 167, 173, 191, 193, 197, \\ 199, 269, 271, 277, 281, 283, 293, 449, 463, 467, 491, 503, 683, 709. \end{cases}$$

Thus (4.2.3) is valid for all primes k except at above values of k. For these values of k, we take $x = 2k + r$ with $0 \le r < \frac{k}{24}$ and show that the right hand side of (4.2.19) is at least the right hand side of (4.2.3) except at $(x, k) \notin S_3$.

We divide the interval $[\frac{49}{24}k, \frac{29}{12}k)$ into following subintervals

$$\left[\frac{49}{24}k, \frac{25}{12}k\right), \left[\frac{25}{12}k, \frac{13}{6}k\right), \left[\frac{13}{6}k, \frac{9}{4}k\right), \left[\frac{9}{4}k, \frac{19}{8}k\right) \text{ and } \left[\frac{19}{8}k, \frac{29}{12}k\right).$$

We find $k_1 = 5393$ for each of these intervals. For $k < 5393$ and k prime, we check that (4.2.20) holds except at following values of k for the intervals:

$$\left[\frac{49}{24}k, \frac{25}{12}k\right) : \begin{cases} 19, 47, 67, 71, 73, 79, 83, 103, 107, 109, 113, 131, 151, 167, 181, 199, \\ 211, 263, 271, 277, 293, 467, 683 \end{cases}$$

$$\left[\frac{25}{12}k, \frac{17}{8}k\right) : \{19, 71, 83, 101, 103, 107, 113, 179, 181, 199, 257, 281, 283, 467, 683$$

$$\left[\frac{17}{8}k, \frac{13}{6}k\right) : \{19, 37, 47, 61, 73, 89, 113, 197$$

$$\left[\frac{13}{6}k, \frac{9}{4}k\right) : \{19, 43, 61, 67, 83, 89, 113, 139, 193, 197, 199, 257, 281, 283$$

$$\left[\frac{9}{4}k, \frac{19}{8}k\right) : \begin{cases} 19, 23, 31, 43, 47, 61, 79, 83, 109, 113, 139, 151, 167, 193, 197, 199, \\ 239, 283, 359 \end{cases}$$

and there are no exceptions for the subinterval $\left[\frac{19}{8}k, \frac{29}{12}k\right)$. Now we apply similar arguments as in the case $2k \le x < \frac{49}{24}k$ to each of the above subintervals to complete the proof.

For the proof of **(b)**, we divide $\frac{29}{12}k - 1 < x < 3k$ into subintervals $\left(\frac{29}{12}k - 1, \frac{5}{2}k\right)$, $\left[\frac{5}{2}k, \frac{21}{8}k\right)$, $\left[\frac{21}{8}k, \frac{11}{4}k\right)$ and $\left[\frac{11}{4}k, 3k\right)$. We apply the arguments of **(a)** to each of these subintervals to conclude that the right hand side of (4.2.19) is at least the right hand side of (4.2.4). Infact we have the inequality

$$\pi((2 + t_1)k) - \pi((1 + t_2)k) + \pi((1 + \frac{t_1}{2})k) - \pi(k) - M(k) \ge 0 \tag{4.2.22}$$

analogous to that of (4.2.20). As in **(a)**, using (4.2.1), we derive that $k_1 = 5393$ in each of these intervals. For $k < 5393$ and $k \in \mathfrak{K}$, we check that (4.2.22) hold except at the following values of k for the intervals:

$\left(\frac{29}{12}k - 1, \frac{5}{2}k\right)$: $\{54, 55, 57, 73, 79, 142\}$,
$\left[\frac{5}{2}k, \frac{21}{8}k\right)$: $\{12, 52, 55, 70\}$,
$\left[\frac{21}{8}k, \frac{11}{4}k\right)$: $\{22, 27\}$
$\left[\frac{11}{4}k, 3k\right)$: $\{10, 12, 19, 21, 22, 24, 37, 54, 55, 57, 59, 70, 91, 100, 121, 142, 159\}$.
Now we proceed as in **(a)** to get the required result. □

Lemma 4.2.7. *Let $3k \le x < 7k$. Then Theorem 4.1.1* **(b)** *holds for $k \in \mathfrak{K}$.*

We prove a stronger result that Theorem 4.1.1 **(b)** holds for all $k \ge 29000$ and for $k \in \mathfrak{K}$.

PROOF. Let $3k \le x < 7k$. We show that (4.2.4) holds. Let $(s+t_1)k \le x < (s+t_2)k$ with integers $3 \le s \le 6$ and $t_1, t_2 \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ such that $t_2 - t_1 = \frac{1}{4}$. Then Δ' contains a term equal to ip with $\frac{x-k}{i} < p \le \frac{x}{i}$ for each i with $1 \le i < s$ and a term equal to sp for $k < p \le \frac{x}{s}$. Therefore

$$W(\Delta') \ge \sum_{i=1}^{s-1} \left(\pi\left(\frac{x}{i}\right) - \pi\left(\frac{x-k}{i}\right)\right) + \pi\left(\frac{x}{s}\right) - \pi(k). \tag{4.2.23}$$

Since $x \ge (s+t_1)k$ and $x - k < (s-1+t_2)k$, we observe from (4.2.23) that

$$W(\Delta') \ge \sum_{i=1}^{s-1} \left(\pi\left(\frac{s+t_1}{i}k\right) - \pi\left(\frac{s-1+t_2}{i}k\right)\right) + \pi\left(\frac{s+t_1}{s}k\right) - \pi(k).$$

Hence it is enough to show

$$\sum_{i=1}^{s-1} \left(\pi\left(\frac{s+t_1}{i}k\right) - \pi\left(\frac{s-1+t_2}{i}k\right)\right) + \pi\left(\frac{s+t_1}{s}k\right) - \pi(k) - M(k) \ge 0. \tag{4.2.24}$$

Using (4.2.1) and Lemma 2.0.2 (i), (ii), we see that the left hand side of (4.2.24) is at least

$$\begin{aligned} &\sum_{i=1}^{s-1} \left(b\left(\frac{s+t_1}{i}k\right) - a\left(\frac{s-1+t_2}{i}k\right)\right) + b\left(\frac{s+t_1}{s}k\right) - a(2k) \\ =&\frac{k}{(\log(s+t_1)k)^2}\left\{ F(k,s,t_1,t_2) - \sum_{i=1}^{s-1} G(k,s,t_1,t_2,i) - G(k,s,t_1,1,\frac{s}{2})\right\} \end{aligned} \tag{4.2.25}$$

for $k \ge 5393$, where

$$\begin{aligned} F(k,s,t_1,t_2) = &\left(\sum_{i=1}^{s-1} \left(\frac{1+t_1-t_2}{i}\right) + \frac{t_1}{s} - 1\right)(\log(s+t_1)k) + \\ &+ \sum_{i=1}^{s} \frac{(s+t_1)(1+\log i)}{i}\left(1 + \frac{1+\log i}{\log((s+t_1)k/i) - 1}\right) \end{aligned}$$

and

$$\begin{aligned} G(k,s,t_1,t_2,i) = &\left(\frac{s-1+t_2}{i}\right)\left(1 + \frac{\log\left(\frac{(s+t_1)i}{s-1+t_2}\right)}{\log\left(\frac{s-1+t_2}{i}k\right)}\right) \times \\ &\left(1.2762 + \log\left(\frac{(s+t_1)i}{s-1+t_2}\right) + \frac{1.2762\log\left(\frac{(s+t_1)i}{s-1+t_2}\right)}{\log\left(\frac{s-1+t_2}{i}k\right)}\right). \end{aligned}$$

Then

$$kF'(k,s,t_1,t_2) = \left(\sum_{i=1}^{s-1}\left(\frac{1+t_1-t_2}{i}\right) + \frac{t_1}{s} - 1\right) - \sum_{i=1}^{s} \frac{(s+t_1)}{i}\left(\frac{1+\log i}{\log((s+t_1)k/i) - 1}\right)^2.$$

If $s = 2$, we note that F and G are functions similar to f and g appearing in Lemma 4.2.6. As in Lemma 4.2.6, we find $K_1 := K_1(s, t_1, t_2)$ such that (4.2.25) is non negative at $k = K_1$ and it is increasing for $k \ge K_1$. Hence (4.2.24) is valid for $k \ge K_1$. For $k < K_1$, we check inequality (4.2.24) by using the exact values of π function in (4.2.24) for k with $2k-1$ prime or primes k given by (4.2.2). Again for k not satisfying (4.2.24), we take $x = sk + r$ with $t_1 k \le r < t_2 k$ and check that the right hand side of (4.2.23) is at least the right hand side of (4.2.4).

Let $3k \le x < \frac{13}{4}k$. Here $t_1 = 0,\ t_2 = \frac{1}{4}$ and and we find $K_1 = 29000$. We check that (4.2.24) holds for $3 \le k < 29000$ except at $k = 10, 12, 19, 22, 40, 42, 52, 55, 57, 100, 101,\ 126, 127, 142$. For these values of k, putting $x = 3k + r$ with $0 \le r < \frac{1}{4}k$, we show that the right hand side of (4.2.23) is at least the right hand side of (4.2.4). Hence the assertion follows in $3k \le x < \frac{13}{4}k$. For $x \ge \frac{13}{4}k$, we apply similar arguments to intervals $(s + t_1)k \le x < (s + t_2)k$ with integers $3 \le s \le 6$ and $t_1, t_2 \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ such that $t_2 - t_1 = \frac{1}{4}$. We find $K_1 = 5393$ for each of these intervals except for $6k \le x < \frac{25}{4}k$ where $K_1 = 5500$. □

In view of Lemmas 4.2.6 and 4.2.7, it remains to prove Theorem 4.1.1 for $x \ge 7k$ which we assume. Further we may also suppose (4.2.13). Otherwise (4.1.3) follows. Now we derive from Lemma 4.2.5 that $x < k^{\frac{9}{4}}$. On the other hand, we prove $x \ge k^{\frac{9}{4}}$. This is a contradiction. We split the proof of $x \ge k^{\frac{9}{4}}$ in the following two lemmas.

Lemma 4.2.8. *Assume* (4.1.1), (4.2.13) *with* $x \ge 7k$. *Then* $x \ge k^{\frac{3}{2}}$ *for* $k \in \mathfrak{K}$.

Proof. We prove it by contradiction. We assume (4.1.1), (4.2.13) and $7k \le x < k^{\frac{3}{2}}$. Then $k \ge 50$. Further by Corollary 4.2.4 and $\binom{x}{k} \ge \binom{7k}{k}$, we have

$$\binom{7k}{k} < (2.83)^{k+k^{\frac{3}{4}}} k^{\frac{3}{2}(M(k)-1)} \tag{4.2.26}$$

since $x < k^{\frac{3}{2}}$. We observe from Lemma 2.0.6 that

$$\begin{aligned}\binom{7k}{k} = \frac{(7k)!}{k!(6k)!} &> \frac{\sqrt{14\pi k}\exp^{-7k}(7k)^{7k}\exp^{\frac{1}{84k+1}}}{\sqrt{2\pi k}\exp^{-k}k^k\exp^{\frac{1}{12k}}\sqrt{12\pi k}\exp^{-6k}(6k)^{6k}\exp^{\frac{1}{72k}}} \\ &> \frac{0.4309}{\sqrt{k}}\exp^{\frac{1}{84k+1}-\frac{7}{72k}}(17.65)^k.\end{aligned}$$

Combining this with (4.2.26), we get

$$1 > \exp\left(\log(0.4309k) + \frac{1}{84k+1} - \frac{7}{72k}\right)(17.65)^k(2.83)^{-k-k^{\frac{3}{4}}}k^{-\frac{3}{2}M(k)}. \tag{4.2.27}$$

Using (4.2.1), Lemma 1(i), (ii) and $\exp\left(\frac{\log(0.4309k)}{k} + \frac{1}{84k^2+k} - \frac{7}{72k^2}\right) \ge 1$, we derive for $k \ge 5393$ that

$$\begin{aligned}1 &> 6.2367(2.83)^{-k^{-\frac{1}{4}}}k^{-\frac{3}{\log 2k}(1+\frac{1.2762}{\log 2k})+\frac{3}{2(\log k-1)}} \\ &> 6.2367\ \exp\left(\frac{3}{2} + \frac{3}{2\log k - 2}\right)(2.83)^{-k^{-\frac{1}{4}}}k^{-\frac{3}{\log 2k}(1+\frac{1.2762}{\log 2k})} \\ &> 27.95(2.83)^{-k^{-\frac{1}{4}}}k^{-\frac{3}{\log 2k}(1+\frac{1.2762}{\log 2k})} := h(k)\end{aligned}$$

since $\exp\left(\frac{3}{2\log k - 2}\right) > 1$ for $k \ge 3$. We see that $h(k)$ is an increasing function of k and $h(k) > 1$ at $k = 5393$. Therefore $k < 5393$. By using the exact values of $M(k)$, we now check that (4.2.27) does not hold for $50 \le k < 5393$ and $k \in \mathfrak{K}$. □

Lemma 4.2.9. *If* (4.1.1) *and* (4.2.13) *holds and* $x \ge k^{\frac{3}{2}}$, *then* $x \ge k^{\frac{9}{4}}$ *for* $k \in \mathfrak{K}$.

Proof. We prove by contradiction. Assume (4.1.1), (4.2.13) and $k^{\frac{3}{2}} \le x < k^{\frac{9}{4}}$. We derive from Lemma 4.2.5 that $k \le 70$. Let $k = 10, 11, 12, 13$. By Lemmas 4.2.5, 4.2.7 and 4.2.8, we can take

$\max(7k, k^{\frac{3}{2}}) \le x < k^{\frac{9}{4}}$ for $k = 10, 11, 12$ and $\max(7k, k^{\frac{3}{2}}) \le x < k^2$ for $k = 13$. For these values of x and k, we find that

$$W(\Delta') \ge \sum_{i=1}^{6} \left(\pi\left(\frac{x}{i}\right) - \pi\left(\frac{x-k}{i}\right) \right) \ge M(k)$$

contradicting (4.2.13).

Therefore we assume that $k \ge 14$. Let $k^{\frac{3}{2}} \le x < k^{\frac{25}{16}}$. By Lemma 4.2.7 and 4.2.8, we can take $x \ge \max(7k, k^{\frac{3}{2}})$ so that we can assume $k \ge 32$. Then

$$\binom{x}{k} \ge \binom{\max(7k, \lceil k^{\frac{3}{2}} \rceil)}{k}$$

where $\lceil \nu \rceil$ denotes the least integer $\ge \nu$. From (1.1.1), we have $\mathrm{ord}_p(\binom{x}{k}) \le \left[\frac{\log x}{\log p}\right] \le \left[\frac{25}{16}\frac{\log k}{\log p}\right]$ and hence

$$\binom{x}{k} \le \left(\prod_{i=1}^{\pi(k)} p_i^{\left[\frac{25}{16}\frac{\log k}{\log p_i}\right]} \right) x^{k-\mu} < \left(\prod_{i=1}^{\pi(k)} p_i^{\left[\frac{25}{16}\frac{\log k}{\log p_i}\right]} \right) k^{\frac{25}{16}(M(k)-1)}$$

by (4.2.13). Combining the above estimates for $\binom{x}{k}$, we get

$$\binom{\max(7k, \lceil k^{\frac{3}{2}} \rceil)}{k} < \left(\prod_{i=1}^{\pi(k)} p_i^{\left[\frac{25}{16}\frac{\log k}{\log p_i}\right]} \right) k^{\frac{25}{16}(M(k)-1)}$$

which is not possible for $32 \le k \le 70$. By similar arguments, we arrive at a contradiction for $\max(7k, k^{\frac{25}{16}}) \le x < k^{\frac{26}{16}}$ in $23 \le k \le 70$, $\max(7k, k^{\frac{26}{16}}) \le x < k^{\frac{27}{16}}$ in $17 \le k \le 70$ and $\max(7k, k^{\frac{27}{16}}) \le x < k^{\frac{7}{4}}$ in $14 \le k \le 70$ except at $k = 16$. Let $k = 16$ and $\max(7k, k^{\frac{27}{16}}) \le x < k^{\frac{7}{4}}$. Then we observe that

$$W(\Delta') \ge \sum_{i=1}^{6} \left(\pi\left(\frac{x}{i}\right) - \pi\left(\frac{x-16}{i}\right) \right) \ge 5 = M(16)$$

contradicting (4.2.13).

Now we consider $x \ge k^{\frac{7}{4}}$. We observe that $k^{\frac{7}{4}} \ge 7k$ since $k \ge 14$. Further we derive from Lemma 4.2.5 that $k \le 24$. We apply similar arguments for $14 \le k \le 24$ as above to arrive at a contradiction in the intervals $k^{\frac{7}{4}} \le x < k^{\frac{15}{8}}$ except at $k = 16$, $k^{\frac{15}{8}} \le x < k^{\frac{31}{16}}$ and $k^{\frac{31}{16}} \le x < k^2$. The case $k = 16$ and $k^{\frac{7}{4}} \le x < k^{\frac{15}{8}}$ is excluded as earlier. $\square$

4.3. Proof of Theorem 4.1.1

Suppose that the hypothesis of Theorem 4.1.1 **(b)** is valid and $k \ge 10$. By Lemmas 4.2.6 **(b)**, 4.2.7, 4.2.8 and 4.2.9, we see that Theorem 4.1.1 **(b)** is valid for all $k \in \mathfrak{K}$. Thus (4.2.4) holds for all $k \in \mathfrak{K}$ and $x > \frac{29}{12}k - 1$. Let $k \notin \mathfrak{K}$ and $k_1 < k$ be the largest integer with $2k_1 - 1$ prime. Then $k_1 \ge 10$. For $x > \frac{29}{12}k - 1 > \frac{29}{12}k_1 - 1$, we see that (4.2.4) is valid at (x, k_1). By Lemma 4.2.2 (ii), (4.2.4) is valid at (x, k) too. Hence Theorem 4.1.1 **(b)** is valid for all k.

Suppose that the hypothesis of Theorem 4.1.1 are satisfied and $k \ge 19$. We have from Lemma 4.2.6 **(a)** that (4.2.3) holds for (x, k) with k prime, $x \le \frac{29}{12}k - 1$ and $(x, k) \notin S_3$. By Theorem 4.1.1(**b**), (4.2.4) and hence (4.2.3) is valid for all k and $x > \frac{29}{12}k - 1$. Thus (4.2.3) holds for (x, k) with k prime and $(x, k) \notin S_3$. Let k be a composite number and $k' < k$ be the greatest prime. Then $k' \ge 19$. Suppose $(x, k') \notin S_3$. Then (4.2.3) is valid at (x, k') and hence valid at (x, k) by Lemma 4.2.2 (i). Suppose now that $(x, k') \in S_3$. Then we check the validity of (4.2.3) at (x, k). We see that (4.2.3) does not hold only if $(x, k) \in S_3$. We explain this with two examples. Let $k = 20$. Then $k' = 19$. Since $(42, 19) \in S_3$, we check the validity of (4.2.3) at $(42, 20)$ which is true. Hence $(42, 20) \notin S_3$. Again let $k = 72$. Then $k' = 71$. Since $(145, 71) \in S_3$, we check the validity of (4.2.3) at $(145, 72)$ and see that (4.2.3) does not hold at $(145, 72)$ which is an element of S_3. This completes the proof. $\square$

4.4. Corollary 3.1.3 revisited

We remark here that Corollary 3.1.3 can also be obtained by imitating the proof of Theorem 3.1.1 and using the weaker bounds for prime function given by Lemma 2.0.1 instead of that given by Lemma 2.0.2. We present here few details. By (3.1.3), it is enough to prove Corollary 3.1.3 for $k \geq 19$. Assume (4.1.1). Now as in Lemma 4.2.5, if we have $\mu \geq k - [\frac{2}{3}\pi(k)]$, then

$$x < k^{\frac{3}{2}} \text{ for } k \geq 62; \; x < k^{\frac{7}{4}} \text{ for } k \geq 25; \; x < k^2 \text{ for } k \geq 19. \tag{4.4.1}$$

As in Lemmas 4.2.8 and 4.2.9, we see that if $x \geq 7k$, then $x \geq k^2$. This contradicts (4.4.1). Thus from now on, we consider only $x < 7k$. Analogous to Lemma 4.2.6 **(a)**, we have

$$W(\Delta') \geq \left[\frac{2}{3}\pi(k)\right] - 1 \tag{4.4.2}$$

for all k prime, $2k \leq x < 3k$ except when $(x, k) = (222, 109), (226, 113)$. Infact we split $[2k, 3k)$ into 13 subintervals

$$\left[2k, \frac{49}{24}k\right), \left[\frac{49}{24}k, \frac{37}{18}k\right), \left[\frac{37}{18}k, \frac{25}{12}k\right), \left[\frac{25}{12}k, \frac{19}{9}k\right), \left[\frac{19}{9}k, \frac{15}{7}k\right), \left[\frac{15}{7}k, \frac{11}{5}k\right),$$
$$\left[\frac{11}{5}k, \frac{9}{4}k\right), \left[\frac{9}{4}k, \frac{7}{3}k\right), \left[\frac{7}{3}k, \frac{29}{12}k\right) \left[\frac{29}{12}k, \frac{5}{2}k\right), \left[\frac{5}{2}k, \frac{21}{8}k\right), \left[\frac{21}{8}k, \frac{11}{4}k\right), \left[\frac{11}{4}k, 3k\right)$$

and bound $k \leq 150000$ and we check that (4.4.2) holds at all primes k. We note here that the equation analogous to (4.2.20) is taken as

$$\pi(2k) - \pi\left(\frac{25}{24}k\right) - \left[\frac{2}{3}\,\pi(k)\right] + 1 \geq 0$$

while using the bounds of π function given by Lemma 2.0.1. This is necessary to reduce the bound for k. Next we take k prime and $3k \leq x < 7k$. Here we split this interval into subintervals of length $\frac{k}{8}$ and arguing as in Lemma 4.2.7, we bound $k \leq 60000$. We also note here that when $t_1 = 0$, we take the equation

$$\sum_{i=1}^{s-1}\left(\pi\left(\frac{s}{i}k\right) - \pi\left(\frac{s-1+\frac{1}{8}}{i}k\right)\right) - \left[\frac{2}{3}\,\pi(k)\right] + 1 \geq 0$$

analogous to (4.2.24) while applying the bounds of π function given by Lemma 2.0.1. We observe here that this bound can be reduced further by taking subintervals of smaller lengths than $\frac{k}{8}$. We check that (4.4.2) holds for all primes $k \leq 60000$. Now as in Lemma 4.2.2 (i), we see that (4.4.2) is valid for all k. $\square$

CHAPTER 5

An analogue of Sylvester's theorem for arithmetic progressions: Proofs of (3.3.5) and (3.3.3)

In this chapter, we prove (3.3.5) viz,

$$\omega(\Delta) > \pi(k) \text{ if } k \geq 4 \text{ and } (n, d, k) \neq (1, 2, 5)$$

and derive (3.3.3) i.e.,

$$P(\Delta) > k \ \text{ if } k \geq 3 \text{ and } (n, d, k) \neq (2, 7, 3).$$

The proof depends on the combinatorial arguments of Sylvester and Erdős. In particular it depends on their fundamental inequality which we shall explain below. We sharpen this inequality. Further we use the estimates of π function due to Dusart given in Lemma 2.0.2. The proof of (3.3.3) for $k = 3$ depends on solving some special cases of Catalan equation.

5.1. Fundamental inequality of Sylvester and Erdős

For $0 \leq i < k$, let

$$n + id = B_i B_i' \tag{5.1.1}$$

where B_i and B_i' are positive integers such that $P(B_i) \leq k$ and $\gcd(B_i', \prod_{p \leq k} p) = 1$. Let $\mathcal{S} \subset \{B_0, \cdots, B_{k-1}\}$. Let $p \leq k$ be such that $p \nmid d$ and p divides at least one element of $\mathcal{S}$. Choose $B_{i_p} \in \mathcal{S}$ such that p does not appear to a higher power in the factorisation of any other element of $\mathcal{S}$. Let $\mathcal{S}_1$ be the subset of $\mathcal{S}$ obtained by deleting from $\mathcal{S}$ all such B_{i_p}. Let $\mathfrak{P}$ be the product of all the elements of $\mathcal{S}_1$. For any $i \neq i_p$, we have $\text{ord}_p(B_i) = \text{ord}_p(n + id) \leq \text{ord}_p((n + id) - (n + i_p d)) = \text{ord}_p(i - i_p)$. Therefore

$$\text{ord}_p(\mathfrak{P}) \leq \text{ord}_p\left(\prod_{i \in \mathcal{S}_1} (i - i_p)\right) \leq \text{ord}_p(i_p!(k - 1 - i_p)!) \leq \text{ord}_p((k - 1)!). \tag{5.1.2}$$

Hence

$$\mathfrak{P} \leq \prod_{p \nmid d} p^{\text{ord}_p((k-1)!)} \tag{5.1.3}$$

which is the fundamental inequality of Sylvester and Erdős. □

5.2. Refinement of fundamental inequality of Sylvester and Erdős

The following lemma is a refinement of a fundamental inequality (5.1.3) of Sylvester and Erdős.

Lemma 5.2.1. *Let $\mathcal{S}, \mathcal{S}_1, \mathfrak{P}$ be as in Section 5.1 and let a' be the number of terms in $\mathcal{S}_1$ divisible by 2. Also we denote*

$$n_0 = \gcd(n, k - 1)$$

and

$$\theta = \begin{cases} 1 \text{ if } 2|n_0 \\ 0 \text{ otherwise.} \end{cases} \tag{5.2.1}$$

Then

$$\mathfrak{P} \leq n_0 \prod_{p \nmid d} p^{\mathrm{ord}_p((k-2)!)}. \tag{5.2.2}$$

Further for d odd, we have

$$\mathfrak{P} \leq 2^{-\theta} n_0 2^{a' + \mathrm{ord}_2([\frac{k-2}{2}]!)} \prod_{p \nmid 2d} p^{\mathrm{ord}_p((k-2)!)}. \tag{5.2.3}$$

We shall use only (5.2.2) in the proof of (3.3.5).

Proof. Let $p < k$, $p \nmid d$ be such that p divides at least one element of $\mathcal{S}$. Let $r_p \geq 0$ be the smallest integer such that $p \mid n + r_p d$. Write $n + r_p d = pn_1$. Then

$$n + r_p d, n + r_p d + pd, \cdots, n + r_p d + p[\frac{k - 1 - r_p}{p}]d$$

are all the terms in Δ divisible by p. Let $B_{r_p + pi_p}$ be such that p does not divide any other term of $\mathcal{S}$ to a higher power. Let a_p be the number of terms in $\mathcal{S}_1$ divisible by p. We note here that $a_p \leq [\frac{k-1-r_p}{p}]$. For any $B_{r_p+pi} \in \mathcal{S}_1$, we have $\mathrm{ord}_p(B_{r_p+pi}) = \mathrm{ord}_p(n + r_p d + pid) \leq \mathrm{ord}_p((n + r_p d + pid)) - (n + r_p d + pi_p d)) = 1 + \mathrm{ord}_p(i - i_p)$. Therefore

$$\mathrm{ord}_p(\mathfrak{P}) \leq a_p + \mathrm{ord}_p \left(\prod_{\substack{i=0 \\ i \neq i_p}}^{[\frac{k-1-r_p}{p}]} (i - i_p) \right) \leq a_p + \mathrm{ord}_p \left(i_p! [\frac{k - 1 - r_p}{p} - i_p]! \right) \tag{5.2.4}$$

Thus

$$\mathrm{ord}_p(\mathfrak{P}) \leq a_p + \mathrm{ord}_p([\frac{k - 1 - r_p}{p}]!). \tag{5.2.5}$$

Let $p \nmid n$. Then $r_p \geq 1$ and hence $a_p \leq [\frac{k-2}{p}]$. From (5.2.5), we have

$$\mathrm{ord}_p(\mathfrak{P}) \leq [\frac{k-2}{p}] + \mathrm{ord}_p([\frac{k-2}{p}]!) = \mathrm{ord}_p((k-2)!). \tag{5.2.6}$$

Let $p = 2$. Then $a_2 = a'$ so that

$$\mathrm{ord}_2(\mathfrak{P}) \leq a' + \mathrm{ord}_2([\frac{k-2}{2}]!). \tag{5.2.7}$$

Let $p|n$. Then $r_p = 0$. Assume that $p \nmid (k-1)$. Then from (5.2.5), we have

$$\mathrm{ord}_p(\mathfrak{P}) \leq a_p + \mathrm{ord}_p([\frac{k-2}{p}]!). \tag{5.2.8}$$

Assume $p|(k-1)$ and let $i_0 \in \{0, \frac{k-1}{p}\}$ with $i_0 \neq i_p$ be such that $\mathrm{ord}_p(n + pi_0 d) = \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1))$. If $\mathrm{ord}_p(n) = \mathrm{ord}_p(k-1)$, we take $i_0 = 0$ if $i_p \neq 0$ and $i_0 = \frac{k-1}{p}$ otherwise. From (5.2.4), we have

$$\mathrm{ord}_p(\mathfrak{P}) \leq \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + a_p - 1 + \mathrm{ord}_p \left(\prod_{\substack{i=0 \\ i \neq i_0, i_p}}^{\frac{k-1}{p}} (i - i_p) \right).$$

Thus

$$\mathrm{ord}_p(\mathfrak{P}) \leq \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + a_p - 1 + \mathrm{ord}_p((\frac{k-1-p}{p})!). \tag{5.2.9}$$

From (5.2.8) and (5.2.9), we conclude

$$\mathrm{ord}_p(\mathfrak{P}) \leq \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + [\frac{k-2}{p}] + \mathrm{ord}_p([\frac{k-2}{p}]!)$$

since $a_p \leq [\frac{k-1}{p}]$. Thus

$$\mathrm{ord}_p(\mathfrak{P}) \leq \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + \mathrm{ord}_p((k-2)!). \tag{5.2.10}$$

Now (5.2.2) follows from (5.2.6) and (5.2.10). Let $p = 2$. By (5.2.8) and (5.2.9), we have in case of even n that

$$\mathrm{ord}_2(\mathfrak{P}) \leq \min(\mathrm{ord}_2(n), \mathrm{ord}_2(k-1)) - \theta + a' + \mathrm{ord}_2([\frac{k-2}{2}]!)$$

which, together with (5.2.6), (5.2.7) and (5.2.10), implies (5.2.7). □

5.3. Lemmas

The following Lemma is a consequence of Lemma 5.2.1.

LEMMA 5.3.1. *Let $\alpha \geq 0$ and $m \geq 0$. Suppose $W(\Delta) \leq m$. Then there exists a set $\mathfrak{T} = \{n + i_h d | 0 \leq h \leq t,\ i_0 < i_1 < \cdots < i_t\}$ such that $1 + t := |\mathfrak{T}| \geq k - m - \pi_d(k)$ satisfying*

$$d^t \leq \frac{n_0}{n} \frac{\prod_{p \nmid d} p^{\mathrm{ord}_p((k-2)!)}}{(\alpha + i_1) \cdots (\alpha + i_t)} \quad \text{if} \quad n = \alpha d \tag{5.3.1}$$

and

$$\frac{(n + i_0 d) \cdots (n + i_t d)}{2^a} \leq 2^{-\theta} n_0 2^{\mathrm{ord}_2([\frac{k-2}{2}]!)} \prod_{p \nmid 2d} p^{\mathrm{ord}_p((k-2)!)} \text{ if } d \text{ is odd} \tag{5.3.2}$$

where a is the number of even elements in $\mathfrak{T}$.

We shall use only (5.3.1) in the proof of (3.3.5).

PROOF. Let $\alpha > 0$ be given by $n = \alpha d$. Let $\mathfrak{S}$ be the set of all terms of Δ composed of primes not exceeding k. Then $|\mathfrak{S}| \geq k - m$. For every p dividing an element of $\mathfrak{S}$, we delete an $f(p) \in \mathfrak{S}$ such that

$$\mathrm{ord}_p(f(p)) = \max_{s \in \mathfrak{S}} \mathrm{ord}_p(s).$$

Then we are left with a set $\mathfrak{T}$ with $1 + t := |\mathfrak{T}| \geq k - m - \pi_d(k)$ elements of $\mathfrak{S}$. Let

$$\mathcal{P} := \prod_{\nu=0}^{t} (n + id) \geq (n + i_0 d)(\alpha + i_1) \cdots (\alpha + i_t) d^t.$$

We now apply Lemma 5.2.1 with $\mathcal{S} = \mathfrak{S}$ and $\mathcal{S}_1 = \mathfrak{T}$ so that $\mathfrak{P} = \mathcal{P}$. Thus the estimates (5.2.2) and (5.2.3) are valid for $\mathcal{P}$. Comparing the upper and lower bounds of $\mathcal{P}$, we have (5.3.1) and further (5.3.2) for d odd. □

The next lemma is an analogue of Lemma 1.1.2 for $d > 1$.

LEMMA 5.3.2. *Let $k_1 < k_2$ be such that k_1 and k_2 are consecutive primes. Suppose* (3.3.5) *holds at k_1. Then it holds for all k with $k_1 \leq k < k_2$.*

PROOF. Assume that (3.3.5) holds at k_1. Let k be as in the statement of the lemma. From $\Delta(n,d,k) = n(n+d)\cdots(n+(k_1-1)d)(n+k_1d)\cdots(n+(k-1)d)$, we have

$$\omega(\Delta(n,d,k)) \geq \omega(\Delta(n,d,k_1)) > \pi(k_1) = \pi(k)$$

since $k_1 < k_2$ are consecutive primes. □

5.4. Proof of (3.3.5) for $k = 4$ and primes $k \geq 5$

Suppose $\omega(\Delta) \leq \pi(k)$. Then $W(\Delta) \leq \pi(k) - \pi_d(k)$. Thus $m = \pi(k) - \pi_d(k)$ so that $t \geq k - \pi(k) - 1$ in Lemma 5.3.1. Let $\alpha > 0$ be given by $n = \alpha d$. From (5.3.1) and since $n_0 \leq n$, we have

$$d^{k-\pi(k)-1} \leq \frac{(k-2)!\prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}}{(\alpha+1)\cdots(\alpha+k-\pi(k)-1)}. \tag{5.4.1}$$

Since $\alpha > 0$, this gives

$$d^{k-\pi(k)-1} \leq \frac{(k-2)!\prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}}{(k-\pi(k)-1)!} < (k-2)^{\pi(k)-1}\prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}. \tag{5.4.2}$$

Hence

$$d < (k-2)^{\frac{\pi(k)-1}{k-\pi(k)-1}}. \tag{5.4.3}$$

Using Lemma 2.0.2 (i), we derive that

$$d < \exp\left[\frac{\frac{\log(k-2)}{\log k}(1+\frac{1.2762}{\log k}) - \frac{\log(k-2)}{k}}{1 - \frac{1}{\log k}(1+\frac{1.2762}{\log 2k}) - \frac{1}{k}}\right]. \tag{5.4.4}$$

We see that the right hand side of the above inequality is a non increasing function of k and < 2 at $k = 43$. Thus $d < 2$ for $k \geq 43$. Hence we need to consider only $k < 43$. By using exact values of $\pi(k)$, we get from (5.4.3) that $d = 2$, $k = 5, 7$. Taking $d = 2, k = 5, 7$ in the first inequality of (5.4.2), we get $d = 2$, $k = 5$. Let $d = 2, k = 5$ and $n > 4$. Then $\alpha > 2$ and we get from (5.4.1) that $2 \leq 1$, a contradiction. For $d = 2, k = 5$ and $n = 1, 3$, we check that (3.3.5) holds except at $(1, 2, 5)$. □

5.5. Proof of (3.3.5)

By the preceding Section and Lemma 5.3.2, we see that (3.3.5) is valid for all $k \geq 4$, $(n,d,k) \neq (1,2,5)$ except possibly at $(1,2,6)$. We check that (3.3.5) is valid at $(1,2,6)$. □

5.6. Proof of (3.3.3)

Let $(n,d,k) = (1,2,5)$. Then we see that (3.3.3) holds. For $(n,d,k) \neq (1,2,5)$, we have by (3.3.5) and Lemma 5.3.2 that there is a prime $> k$ dividing Δ for $k \geq 4$. Thus (3.3.3) is valid for all $k \geq 4$. Let $k = 3$ and assume that $P(n(n+d)(n+2d)) \leq 3$. If d is even, then $n, n+d, n+2d$ are all odd and 3 does not divide all of them. Hence there is a prime $p > 3$ dividing $n(n+d)(n+2d)$. Assume d is odd. Then we have the following possibilities.

$$\begin{aligned} &n = 1, n+d = 2^a \text{ and } n+2d = 3^b \text{ implying } 2^{a+1} - 3^b = 1 \\ &n = 2^a, n+d = 3^b \text{ and } n+2d = 2^c \text{ implying } 3^b = 2^{a-1}(2^{c-a}+1) \end{aligned} \tag{5.6.1}$$

where a, b, c are positive integers. In the first case, we see that $a > 1, b > 1$ since $d > 1$. Thus we have $3^b \equiv -1 (\mathrm{mod}\ 8)$. This is not possible since $3^b \equiv 1, 3 (\mathrm{mod}\ 8)$. In the second case, we get $a = 1$

giving $3^b = 2^{c-1} + 1$. Since $d > 1$, we have $b > 1, c > 3$ so that b is even since $3^b \equiv 1(\text{mod } 4)$. Hence $2^{c-1} = (3^{\frac{b}{2}} - 1)(3^{\frac{b}{2}} + 1)$ which implies $3^{\frac{b}{2}} - 1 = 2, 3^{\frac{b}{2}} + 1 = 2^{c-2}$ giving $b = 2, c = 4$. Hence we see that $P(n(n + d)(n + 2d)) > 3$ except at $(n, d, k) = (2, 7, 3)$. $\square$

CHAPTER 6

Refinement of an analogue of Sylvester's theorem for arithmetic progressions: Proof of Theorem 3.3.1

In this chapter, we prove Theorem 3.3.1. We give a sketch of the proof. The proof of Theorem 3.3.1 depends on the sharpening of the upper bound for $\mathfrak{P}$ in the fundamental inequality (5.1.3) of Sylvester and Erdős which we described in Lemma 5.2.1. Further we also give a better lower bound for $\mathfrak{P}$, see (6.2.12). Comparing the upper and lower bounds for $\mathfrak{P}$, we bound n, d and k. When $d \leq 7$, we also need to use estimates on primes in arithmetic progression due to Ramaré and Rumely given in Lemma 2.0.4. We apply these estimates to count the number of terms of Δ which are of the form ap where $1 \leq a < d$, $\gcd(a, d) = 1$ and $p > k$, see Lemma 6.1.3. For the finitely many values of n, d, k thus obtained, we check the validity of (3.3.9) on a computer.

6.1. Lemmas for the proof of Theorem 3.3.1

The following lemma is analogue of Lemma 4.2.2 (ii) for $d > 1$.

Lemma 6.1.1. *Let $k_1 < k_2$ be such that $2k_1 - 1$ and $2k_2 - 1$ are consecutive primes. Suppose (3.3.9) holds at k_1. Then it holds for all k with $k_1 \leq k < k_2$.*

Proof. Assume that (3.3.9) holds at k_1. Let k be as in the statement of the lemma. Then $\pi(2k_1) = \pi(2k)$. From $\Delta(n, d, k) = n(n + d)\cdots(n + (k_1 - 1)d)(n + k_1 d)\cdots(n + (k - 1)d)$, we have

$$W(\Delta(n, d, k)) \geq W(\Delta(n, d, k_1)) \geq \pi(2k_1) - \pi_d(k_1) - \rho \geq \pi(2k) - \pi_d(k) - \rho$$

since $\pi_d(k) \geq \pi_d(k_1)$. □

Lemma 6.1.2. *Let $\max(n, d) \leq k$. Let $1 \leq r < k$ with $\gcd(r, d) = 1$ be such that*

$$W(\Delta(r, d, k)) \geq \pi(2k) - \rho.$$

Then for each n with $r < n \leq k$ and $n \equiv r(\mathrm{mod}\ d)$, we have

$$W(\Delta(n, d, k)) \geq \pi(2k) - \rho.$$

Proof. For $r < n \leq k$, we write

$$\begin{aligned}\Delta(n, d, k) &= \frac{r(r + d)\cdots(r + (k - 1)d)(r + kd)\cdots(n + (k - 1)d)}{r(r + d)\cdots(n - d)}\\ &= \Delta(r, d, k)\frac{(r + kd)\cdots(n + (k - 1)d)}{r(r + d)\cdots(n - d)}.\end{aligned}$$

We observe that $p \mid \Delta(n, d, k)$ for every prime $p > k$ dividing $\Delta(r, d, k)$. □

Lemma 6.1.3. *Let $d \leq k$. For each $1 \leq r < d$ with $\gcd(r, d) = 1$, let r' be such that $rr' \equiv 1(\mathrm{mod}\ d)$. Then*
(a) For a given n with $1 \leq n \leq k$, Theorem 3.3.1 holds if

$$\sum_{\substack{1 \leq r < d \\ \gcd(r,d)=1}} \pi\left(\frac{n + (k - 1)d}{r}, d, nr'\right) - \pi(2k) + \rho \geq 0 \tag{6.1.1}$$

is valid.

(b) For a given n with $k < n < 1.5k$, Theorem 3.3.1 holds if

$$\sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1)-d+1}{r}, d, nr'\right) - \pi(2k) + \pi(k,d,n) - \pi(1.5k,d,n) \ge 0 \tag{6.1.2}$$

is valid.

(c) For a given n with $k < n \le 2k$, Theorem 3.3.1 holds if

$$\sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1)-d+1}{r}, d, nr'\right) - \pi(2k) + \pi(k,d,n) - \pi(2k,d,n) \ge 0 \tag{6.1.3}$$

is valid.

PROOF. Let $1 \le r < d \le k$, $\gcd(r,d) = 1$. Then for each prime $p \equiv nr' (\mathrm{mod}\ d)$ with $\max(k, \frac{n-1}{r}) < p \le \frac{n+(k-1)d}{r}$, there is a term $rp = n + id$ in $\Delta(n,d,k)$. Therefore

$$W(\Delta(n,d,k)) \ge \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \left(\pi\left(\frac{n+(k-1)d}{r}, d, nr'\right) - \pi(\max(k, \frac{n-1}{r}), d, nr')\right). \tag{6.1.4}$$

Since

$$\sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi(k,d,nr') = \pi_d(k), \tag{6.1.5}$$

it is enough to prove (6.1.1) for deriving (3.3.9) for $1 \le n \le k$. This gives (a).

Let $k < n < k'$ where $k' = 1.5k$ or $2k+1$. Then from (6.1.4) and (6.1.5), we have

$$\begin{aligned} W(\Delta(n,d,k)) &\ge \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \left(\pi\left(\frac{k+1+(k-1)d}{r}, d, nr'\right) - \pi(\max(k, \frac{k'-1}{r}), d, nr')\right) \\ &\ge \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1)-d+1}{r}, d, nr'\right) - \pi(k'-1,d,n) - \pi_d(k) + \pi(k,d,n) \end{aligned}$$

since $r' = 1$ for $r = 1$. Hence it suffices to show (6.1.2) for proving (3.3.9) for $k < n < 1.5k$ or (6.1.3) for proving (3.3.9) for $k < n \le 2k$. Hence (b) and (c) are valid. □

6.2. Proof of Theorem 3.3.1 for k with $2k-1$ prime

Let

$$\chi = \chi(n) = \begin{cases} \min\left(1, \frac{k-1}{n}\prod\limits_{p\mid 2d} p^{-\mathrm{ord}_p(k-1)}\right) & \text{if } 2 \nmid n \\ \min\left(2^{\theta-1}, \frac{k-1}{n}\prod\limits_{p\mid d} p^{-\mathrm{ord}_p(k-1)}\right) & \text{if } 2 \mid n \end{cases} \tag{6.2.1}$$

and

$$\chi_1 = \chi_1(n) = \min\left\{1, \frac{k-1}{n}\prod_{p\mid d} p^{-\mathrm{ord}_p(k-1)}\right\}. \tag{6.2.2}$$

We observe that χ is non increasing function of n even and n odd separately. Further χ_1 is a non increasing function of n. We also check that

$$\frac{n_0}{n} \leq \chi \leq \chi_1 \tag{6.2.3}$$

and $\chi(1) = 1$, $\chi(2) = 2^{\theta-1}$.

We take $(n, d, k) \notin V$, $n > k$ when $d = 2$ so that $\rho = 0$. We assume that (3.3.9) is not valid and we shall arrive at a contradiction. We take $m = \pi(2k) - \pi_d(k) - 1$ in Lemma 5.3.1. Then $t \geq k - \pi(2k)$ in Lemma 5.3.1 and we have from (5.3.1) and (6.2.3) that

$$d^{k-\pi(2k)} \leq \chi_1(n) \frac{(k-2)! \prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}}{(\alpha+1)\cdots(\alpha+k-\pi(2k))} \tag{6.2.4}$$

where $n = \alpha d$ which is also the same as

$$\prod_{i=1}^{k-\pi(2k)} (n+id) \leq \chi_1(n)(k-2)! \prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}. \tag{6.2.5}$$

From (6.2.4), we have

$$d^{k-\pi(2k)} \leq \begin{cases} \chi_1(\alpha d)[\alpha]!(k-2)\cdots([\alpha]+k-\pi(2k)+1) \prod_{p|d} p^{-\mathrm{ord}_p(k-2)!} \text{ if } [\alpha] \leq \pi(2k)-3, \\ \chi_1(\alpha d)[\alpha]! \prod_{p|d} p^{-\mathrm{ord}_p(k-2)!} \text{ if } [\alpha] = \pi(2k)-2, \\ \chi_1(\alpha d) \frac{[\alpha]!}{(k-1)k(k+1)\cdots([\alpha]+k-\pi(2k))} \prod_{p|d} p^{-\mathrm{ord}_p(k-2)!} \text{ if } [\alpha] \geq \pi(2k)-1. \end{cases} \tag{6.2.6}$$

We observe that the right hand sides of (6.2.4), (6.2.5) and (6.2.6) are non-increasing functions of $n = \alpha d$ when d and k are fixed. Thus (6.2.6) and hence (6.2.4) and (6.2.5) are not valid for $n \geq n_0$ whenever it is not valid at $n_0 = \alpha_0 d$ for given d and k. This will be used without reference throughout this chapter. We obtain from (6.2.4) and $\chi_1 \leq 1$ that

$$d^{k-\pi(2k)} \leq (k-2)\cdots(k-\pi(2k)+1) \prod_{p|d} p^{-\mathrm{ord}_p(k-2)!} \tag{6.2.7}$$

which implies that

$$d^{k-\pi(2k)} \leq \begin{cases} (k-2)\cdots(k-\pi(2k)+1)2^{-\mathrm{ord}_2(k-2)!} \text{ if } d \text{ is even}, \\ (k-2)\cdots(k-\pi(2k)+1) \text{ if } d \text{ is odd} \end{cases} \tag{6.2.8}$$

and

$$d \leq (k-2)^{\frac{\pi(2k)-2}{k-\pi(2k)}} \prod_{p|d} p^{\frac{-\mathrm{ord}_p(k-2)!}{k-\pi(2k)}}. \tag{6.2.9}$$

Using Lemmas 2.0.2 (i) and 2.0.5, we derive from (6.2.9) that

(6.2.10)

$$d \leq \exp\left[\frac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - \frac{2\log(k-2)}{k}}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] \prod_{p|d} p^{-\max\{0,\left(\frac{k-1-p}{p-1}-\frac{\log(k-2)}{\log p}\right)/\left(k-\frac{2k}{\log 2k}(1+\frac{1.2762}{\log 2k})\right)\}}$$

which implies

$$d \le \begin{cases} \exp\left[\dfrac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k})-\frac{2\log(k-2)}{k}-\left((1-\frac{3}{k})\log 2-\frac{\log(k-2)}{k}\right)}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] & \text{for } d \text{ even,} \\ \exp\left[\dfrac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k})-\frac{2\log(k-2)}{k}}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] & \text{for } d \text{ odd.} \end{cases} \tag{6.2.11}$$

We use the inequalities (6.2.5)-(6.2.11) at several places.

Let d be odd. Then for n even, $2 \mid n+id$ if and only if i is even and for n odd, $2 \mid n+id$ if and only if i is odd. Let $b = k-\pi(2k)+1-a$ and $a_0 = \min(k-\pi(2k)+1, [\frac{k-2+\theta}{2}])$. We note here that $a \le [\frac{k-2+\theta}{2}]$ where θ is given by (5.2.1). Let n_e, d_e, n_o and d_o be positive integers with n_e even and n_o odd. Let $n \ge n_e$ and $d \le d_e$ for n even, and $n \ge n_o$ and $d \le d_o$ for n odd. Assume (5.3.2). The left hand side of (5.3.2) is greater than

$$\begin{cases} \frac{n}{2} d^{k-\pi(2k)} \displaystyle\prod_{i=1}^{a-1}\left(\frac{n_e}{2d_e}+i\right)\prod_{j=1}^{b}\left(\frac{n_e}{d_e}+2j-1\right) := \frac{n}{2}d^{k-\pi(2k)}F(a) & \text{if } n \text{ is even} \\ n d^{k-\pi(2k)} \displaystyle\prod_{i=1}^{a}\left(\frac{n_o}{2d_o}+i-\frac{1}{2}\right)\prod_{j=1}^{b-1}\left(\frac{n_o}{d_o}+2j\right) := nd^{k-\pi(2k)}G(a) & \text{if } n \text{ is odd.} \end{cases} \tag{6.2.12}$$

Let $A_e := \min\left(a_0, \lceil\frac{2}{3}(k-\pi(2k))+\frac{n_e}{6d_e}+\frac{1}{3}\rceil\right)$ and $A_o := \min\left(a_0, \lceil\frac{2}{3}(k-\pi(2k))+\frac{n_o}{6d_o}-\frac{1}{6}\rceil\right)$. By considering the ratios $\frac{F(a+1)}{F(a)}$ and $\frac{G(a+1)}{G(a)}$, we see that the functions $F(a)$ and $G(a)$ take minimal values at A_e and A_o, respectively. Thus (5.3.2) with (6.2.3) implies that

$$d^{k-\pi(2k)}F(A_e) \le 2^{-\theta+1}\chi(n_e)2^{\mathrm{ord}_2([\frac{k-2}{2}]!)}\prod_{p\nmid 2d}p^{\mathrm{ord}_p(k-2)!} \text{ for } n \text{ even} \tag{6.2.13}$$

since $\chi(n) \le \chi(n_e)$ and

$$d^{k-\pi(2k)}G(A_o) \le \chi(n_o)2^{\mathrm{ord}_2([\frac{k-2}{2}]!)}\prod_{p\nmid 2d}p^{\mathrm{ord}_p((k-2)!)} \text{ for } n \text{ odd} \tag{6.2.14}$$

since $\chi(n) \le \chi(n_o)$. In the following two lemmas, we bound d if (3.3.9) does not hold.

LEMMA 6.2.1. *Let d be even. Assume that* (3.3.9) *does not hold. Then $d \le 4$.*

PROOF. Let d be even. By (6.2.11), $d \le 6$ for $k \ge 860$. For $k < 860$, we use (6.2.8) to derive that

$$\begin{aligned} &d \le 12 \text{ for } k \ge 9;\ d \le 10 \text{ for } k = 100;\ d \le 8 \text{ for } k > 57; \\ &d \le 6 \text{ for } k > 255,\ k \ne 262, 310, 331, 332, 342. \end{aligned} \tag{6.2.15}$$

Let d be a multiple of 6. Then we see from (6.2.10) that $k \le 100$. Again for $k \le 100$, (6.2.7) does not hold. Let d be a multiple of 10. Then we see from (6.2.15) that $k = 100$ and $k \le 57$. Again, (6.2.7) does not hold at these values of k.

Let $d = 8$. By (6.2.15), we may assume that $k \le 255$ and $k = 262, 310, 331, 332, 342$. Let $n \le k$. From Lemma 6.1.2, we need to consider only $n = 1, 3, 5, 7$ and (3.3.9) is valid for these values of n. Let $n = k+1$. Then, we see that (6.2.5) does not hold. Thus (6.2.5) is not valid for all $n > k$. Hence $d \le 4$. □

LEMMA 6.2.2. *Let d be odd. Assume that* (3.3.9) *does not hold. Then $d \le 53$ and d is prime.*

PROOF. Let d be odd. We may assume that $d > 53$ whenever d is prime. Firstly we use (6.2.11) and then (6.2.8) to derive that $d \le 15$ for $k \ge 2164$, $d \le 59$ for $k \ge 9$ except at $k = 10, 12$, and $d \le 141$ for $k = 10, 12$.

We further bring down the values of d and k by using (6.2.13) and (6.2.14). We shall be using (6.2.13) with $n_e = 2, \chi(n_e) = 2^{\theta-1}$ and (6.2.14) with $n_o = 1, \chi(n_o) = 1$ unless otherwise specified. Let $k < 2164$. We take $d_e = d_o = 59$ when $k \neq 10, 12$ and $d_e = d_o = 141$ for $k = 10, 12$. Let n be even. From (6.2.13), we derive that

$$
\begin{aligned}
&d \le 27 \text{ for } k \ge 9, k \neq 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57;\\
&d \le 57 \text{ for } k = 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57;\\
&d \le 21 \text{ for } k > 100, k \neq 106, 117, 121, 136, 139, 141, 142, 147, 159;\\
&d \le 17 \text{ for } k > 387, k \neq 415, 420, 432, 442, 444;\\
&d \le 15 \text{ for } k > 957, k \neq 1072, 1077, 1081.
\end{aligned}
\tag{6.2.16}
$$

Further we check that (6.2.16) holds for n odd using (6.2.14). Let $d > 3$ with $3 \mid d$. Then $k \le 1600$ by (6.2.10) and $k \le 850$ by (6.2.7). Further we apply (6.2.13) and (6.2.14) with $d_e = d_o = 57$ to conclude that $d = 9$, $k \le 147$, $k = 157, 159, 232, 234$ and $d = 15$, $k = 10$. The latter case is excluded by applying (6.2.13) and (6.2.14) with $d_e = d_o = 15$. Let $d = 9$. Suppose $n \le k$. We check that (3.3.9) is valid for $1 \le n < 9$ and $\gcd(n, 3) = 1$. Now we apply Lemma 6.1.2 to find that (3.3.9) is valid for all $n \le k$. Let $n > k$. Taking $n_e = 2\lceil \frac{k+1}{2} \rceil, n_o = 2\lceil \frac{k}{2} \rceil + 1, d_e = d_o = 9$, we see that (6.2.13) and (6.2.14) are not valid for $n > k$.

Let $d > 15$ with $5 \mid d$ and $3 \nmid d$. Then $k \le 159$ by (6.2.16). Now, by taking $d_e = d_o = 55$, we see that (6.2.13) and (6.2.14) do not hold unless $k = 10, d = 25$ and n odd. We observe that (6.2.14) with $n_o = 3$ and $d_o = 25$ is not valid at $k = 10$. Thus $(n, d, k) = (1, 25, 10)$ and we check that (3.3.9) holds. Let $d > 7$ and $3 \nmid d, 5 \nmid d$. Then we see from (6.2.16) that $d = 49$ and $k = 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57$. Taking $d_e = d_o = 49$, we see that both (6.2.13) and (6.2.14) do not hold. Thus $d < 57$ and the least prime divisor of d when $d \notin \{3, 5, 7\}$ is at least 11. Hence d is prime and $d \le 53$. □

In view of Lemmas 6.2.1 and 6.2.2, it suffices to consider $d = 2, 4$ and primes $d \le 53$. We now consider some small values of d.

LEMMA 6.2.3. *Let $d = 2, 3, 4, 5$ and 7. Assume that $n \le k$ and $(n, d, k) \notin V$. Then* (3.3.9) *holds.*

PROOF. First, we consider the case $1 \le n \le k$ and $(n, d, k) \notin V$. By Lemma 6.1.2, we may assume that $1 \le n < d$ and $\gcd(n, d) = 1$. Let $d = 2$. Then

$$\pi(n + 2(k-1), 2, 1) - \pi(2k) + 1 = \pi(n + 2k - 2) - 1 - \pi(2k-1) + 1 \ge 0.$$

Now the assertion follows from Lemma 6.1.3. Let $d = 3, 4, 5$ or 7. We may assume that k is different from those given by $(n, d, k) \in V$, otherwise the assertion follows by direct computations. By using the bounds for $\pi(x, d, l)$ and $\pi(x)$ from Lemmas 2.0.4 and 2.0.2, we see that the left hand side of (6.1.1) is at least

$$k\left\{\sum_{i=1}^{d-1} \frac{\left(\frac{d}{i} - \frac{d-1}{ik}\right)}{\log \frac{1+dk-d}{i}} \left(\mathfrak{c}_1 + \frac{\mathfrak{c}_2}{\log \frac{1+dk-d}{2i}}\right) - \frac{2}{\log 2k}\left(1 + \frac{1.2762}{\log 2k}\right)\right\} \tag{6.2.17}$$

for $k \ge \frac{d-1}{d}(1 + x_0)$ at $d = 3, 5, 7$ and

$$k\left\{\sum_{i=1,3} \frac{\left(\frac{4}{i} - \frac{3}{ik}\right)}{\log \frac{4k-3}{i}} \left(\mathfrak{c}_1 + \frac{\mathfrak{c}_2}{\log \frac{4k-3}{2i}}\right) - \frac{2}{\log 2k}\left(1 + \frac{1.2762}{\log 2k}\right)\right\} \tag{6.2.18}$$

for $k \ge \frac{3}{4}(1 + x_0)$ at $d = 4$. Here x_0 is as given in Lemma 2.0.4. We see that (6.2.17) and (6.2.18) are increasing functions of k and (6.2.17) is non negative at $k = 20000, 2200, 1500$ for $d = 3, 5$ and 7, respectively, and (6.2.18) is non negative at $k = 751$. Therefore, by Lemma 6.1.3, we conclude that k is less than $20000, 751, 2200$ and 1500 according as $d = 3, 4, 5$ and 7, respectively. Further we recall that $n < d$. For these values of n and k, we check directly that (3.3.9) is valid. □

Therefore, by Lemma 6.2.3, we conclude that $n > k$ when $d = 2, 3, 4, 5$ and 7.

LEMMA 6.2.4. *Let $d = 2, 3, 4, 5$ and 7. Assume that $k < n \leq 2k$ if $d \neq 2$ and $k < n < 1.5k$ if $d = 2$. Then (3.3.9) holds.*

PROOF. Let $d = 2$ and $k < n < 1.5k$. By Lemma 6.1.3, it suffices to prove (6.1.2). By using the bounds for $\pi(k)$ from Lemma 2.0.2, we see that the left hand side of (6.1.2) is at least

$$k\left\{\frac{3}{\log 3k - 1} + \frac{1}{\log k - 1} - \frac{2}{\log 2k}\left(1 + \frac{1.2762}{\log 2k}\right) - \frac{1.5}{\log 1.5k}\left(1 + \frac{1.2762}{\log 1.5k}\right)\right\} - 1$$

for $k \geq 5393$ since $\pi(3k - 1, 2, 1) = \pi(3k) - 1$. We see that the above expression is an increasing function of k and it is non negative at $k = 5393$. Thus (6.1.2) is valid for $k \geq 5393$. For $k < 5393$, we check using exact values of π function that (6.1.2) is valid except at $k = 9, 10, 12$. For these values of k, we check directly that (3.3.9) is valid since $k < n < 1.5k$.

Let $d = 3, 4, 5, 7$ and $k < n \leq 2k$. By Lemma 6.1.3, it suffices to prove (6.1.3). By using the bounds for $\pi(x, d, l), \pi(2x, d, l) - \pi(x, d, l)$ and $\pi(k)$ from Lemmas 2.0.4 and 2.0.2, respectively, we see that (6.1.3) is valid for $k \geq 20000, 4000, 2500, 1500$ at $d = 3, 4, 5$ and 7, respectively. Thus we need to consider only $k < 20000, 4000, 2500, 1500$ for $d = 3, 4, 5$ and 7, respectively. Taking $n_e = 2\lceil\frac{k+1}{2}\rceil, n_o = 2\lceil\frac{k}{2}\rceil + 1, d_e = d_o = d$ for $d = 3, 5, 7$ in (6.2.13) and (6.2.14), and $n = k + 1$ for $d = 4$ in (6.2.5), we see that

$$\begin{aligned}
&k \leq 3226 \text{ or } k = 3501, 3510, 3522 \text{ when } d = 3\\
&k \leq 12 \text{ or } k = 16, 22, 24, 31, 37, 40, 42, 52, 54, 55, 57, 100, 142 \text{ when } d = 4\\
&k \leq 901 \text{ or } k = 940 \text{ when } d = 5\\
&k \leq 342 \text{ when } d = 7.
\end{aligned}$$

For these values of k, we check that (3.3.9) holds whenever $k < n < 1.5k$. Hence we may assume that $n \geq 1.5k$. Taking $n_e = 2\lceil\frac{1.5k}{2}\rceil, n_o = 2\lceil\frac{1.5k-1}{2}\rceil + 1, d_e = d_o = d$ for $d = 3, 5, 7$ in (6.2.13) and (6.2.14), and $n = \lceil 1.5k\rceil$ for $d = 4$ in (6.2.5), we see that

$$\begin{aligned}
&k \in \{54, 55, 57\} \text{ when } d = 3\\
&k \in \{10, 22, 24, 40, 42, 54, 55, 57, 70, 99, 100, 142\} \text{ when } d = 5\\
&k \in \{10, 12, 24, 37, 40, 42, 54, 55, 57, 100\} \text{ when } d = 7.
\end{aligned}$$

For these values of k, we check directly that (3.3.9) holds for $1.5k \leq n \leq 2k$. □

LEMMA 6.2.5. *Let $d = 2, 3, 4, 5$ and 7. Assume $n > 2k$ if $d \neq 2$ and $n \geq 1.5k$ if $d = 2$. Then (3.3.9) holds.*

PROOF. Let $d = 2$ and $n \geq 1.5k$. Then we take $\alpha = \frac{1.5k}{2}$ so that $n \geq \alpha d$. Further we observe that $\alpha \geq \pi(2k) - 1$. Then we see from (6.2.6) and (6.2.2) that

$$2^{k-\pi(2k)} \leq \frac{\lceil .75k\rceil!}{1.5k^2(k+1)\cdots(\lceil .75k\rceil + k - \pi(2k))}\, 2^{-\mathrm{ord}_2(k-1)!}. \tag{6.2.19}$$

Now we apply Lemmas 2.0.6, 2.0.5 and 2.0.2 (i) in (6.2.19) to derive that

$$2 \le \left(\frac{\frac{8}{3}\sqrt{2\pi}\,\exp(-.75k)(.75(k+1))^{.75(k+1)+\frac{1}{2}}\,\exp(\frac{1}{9k})2^{\pi(2k)}}{k^2(k+1)^{.75k-\pi(2k)}}\right)^{\frac{1}{2k-\frac{\log(k-1)}{\log 2}}}$$

$$\le \exp\left(\frac{\frac{2\log 2(k+1)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - .75 + .75\log .75 + \frac{1}{9k^2} + \frac{1.25\log(k+1)-2\log k+1.54017}{k}}{2-\frac{\log(k-1)}{k\log 2}}\right)$$

for $k \ge 9$. This does not hold for $k \ge 700$. Thus $k < 700$. Further using (6.2.5) with $n = \lceil 1.5k \rceil$, we get $k \in \{16, 24, 54, 55, 57, 100, 142\}$. For these values of k, taking $n = 2k+1$, we see that (6.2.5) is not valid. Thus $n \le 2k$. Now we check that (3.3.9) holds for these values of k and $1.5 \le n \le 2k$.

Let $d = 3, 4, 5$ and 7 and $n > 2k$. Then we take $\alpha = \frac{2k+1}{d}$ so that $n \ge \alpha d$. We proceed as in the case $d = 2$ to derive from (6.2.5) that $k < 70, 69, 162$ and 1515 for $d = 3, 4, 5$ and 7, respectively. Let $d = 3, 5$ and 7. We use (6.2.13) and (6.2.14) with $n_e = 2k+2, n_o = 2k+1$ and $d_e = d_o = d$ if $d = 3, 5, 7$, respectively to get $d = 5, k = 10$ and n even. Let $k = 10, d = 5$ and n even. We take $n_e = 2k+6, d_e = 5$ to see that (6.2.13) holds. Hence $n \le 2k+4$. Now we check directly that (3.3.9) is valid for $n = 2k+2, 2k+4$. Finally we consider $d = 4$ and $k < 69$. Taking $n = 2k+1$, we see that (6.2.5) is not valid. Thus (3.3.9) holds for all $n > 2k$. $\square$

By Lemmas 6.2.1, 6.2.2, 6.2.3, 6.2.4, and 6.2.5, it remains to consider

$$11 \le d \le 53, d \text{ prime}.$$

We prove Theorem 3.3.1 for these cases in the next section.

6.2.1. The Case d$\ge$ 11 with d prime. Our strategy is as follows. Let $U_0, U_1, \cdots$ be sets of positive integers. For any two sets U and V, we denote $U - V = \{u \in U | u \notin V\}$. Let d be given. We take $d_e = d_o = d$ always unless otherwise specified. We apply steps $1-5$ as given below.

1. Let $d = 11, 13$. We first use (6.2.10) to bound k. We reduce this bound considerably using (6.2.7). For $d > 13$, we use (6.2.16) to bound k. Then we apply (6.2.13) and (6.2.14) with $n_e = n_e^{(0)} = 2, n_o = n_o^{(0)} = 1$ to bring down the values of k still further. Let U_0 be these finite set of values of k.
2. For each $k \in U_0$, we check that (3.3.9) is valid for $1 \le n < d$. Hence by Lemma 6.1.2, we get $n > k$.
3. For $k \in U_0$, we apply (6.2.5) with $n = k+1$ to find a subset $U_0' \subsetneq U_0$.
4. For $k \in U_0'$, we apply (6.2.13) and (6.2.14) with $n_e = n_e^{(1)} = 2\lceil \frac{k+1}{2} \rceil, n_o = n_o^{(1)} = 2\lceil \frac{k}{2} \rceil + 1$ to get a subset $U_1 \subsetneq U_0'$.
5. Let $i \ge 2$. For $k \in U_{i-1}$, we apply (6.2.13) and (6.2.14) with suitable values of $n_e = n_e^{(i)}$ and $n_o = n_o^{(i)}$ to get a subset $U_i \subsetneq U_{i-1}$. Thus for $k \in U_{i-1} - U_i$, we have $k < n < \max(n_e^{(i)}, n_o^{(i)})$ and we check that (3.3.9) is valid for these values of n and k. We stop as soon as $U_i = \phi$.

We explain the above strategy for $d = 11$. From (6.2.10), we get $k \le 11500$ which is reduced to $k \le 5589$ by (6.2.7). By taking $n_e^{(0)} = 2, n_o^{(0)} = 1$, we get

$$U_0 = \{k | k \le 2977, k = 3181, 3184, 3187, 3190, 3195, 3199\}.$$

We now check that (3.3.9) is valid for $1 \le n < 11$ for each $k \in U_0$ so that we conclude $n > k$. By Step 3, we get $U_0' = \{k | k \le 252\}$. Further by step 4, we find

$$U_1 = \{9, 10, 12, 16, 21, 22, 24, 27, 31, 37, 40, 42, 45, 52, 54, 55, 57, 70, 91, 99, 100, 121, 142\}.$$

Now we take

$$n_e^{(2)} = 2\lceil \frac{1.5k}{2} \rceil, \ n_o^{(2)} = 2\lceil \frac{1.5k-1}{2} \rceil + 1$$

to get $U_2 = \{10, 22, 37, 42, 54, 55, 57\}$. Then we have

$$k < n < 1.5k \text{ for } k \in U_1 - U_2. \tag{6.2.20}$$

Next we take $n_e^{(3)} = 2k+2, n_o^{(3)} = 2k+1$ to get $U_3 = \{10, 22, 55\}$ and we have

$$k < n < 2k \text{ for } k \in U_2 - U_3. \tag{6.2.21}$$

Finally we take $n_e^{(4)} = 4k, n_o^{(4)} = 4k+1$ to get $U_4 = \phi$ and hence

$$k < n < 4k \text{ for } k \in U_3 \tag{6.2.22}$$

and our procedure stops here since $U_4 = \phi$. Now we check that (3.3.9) holds for k and n as given by (6.2.20), (6.2.21) and (6.2.22).

We follow steps $1-5$ with the same parameters as for $d = 11$ in the cases $d = 13, 17, 19$ and 23. Let $23 < d \leq 53$, d prime. We modify our steps $1-5$ slightly to cover all these values of d simultaneously. For each of $k \in U_0$, we check that (3.3.9) is valid for $1 \leq n \leq \min(d, k)$ and coprime to d. Thus $n > k$. Now we apply step 4 with $d_e = d_o = 53$ to get $U_1 = \{10, 12, 16, 24, 37, 55, 57\}$. In step 5, we take $n_e^{(2)} = 2\lceil \frac{3k+1}{2} \rceil, n_o^{(2)} = 2\lceil \frac{3k}{2} \rceil + 1, d_e = d_o = 53$ to see that that $U_2 = \phi$. Thus

$$k < n < 3k \text{ for } k \in U_1. \tag{6.2.23}$$

Now we check that (3.3.9) holds for k and n as given by (6.2.23) for every d with $23 < d \leq 53$ and d prime. □

6.3. Proof of Theorem 3.3.1

By the preceding section, Theorem 3.3.1 is valid for all k such that $2k-1$ is prime. Let k be any integer and $k_1 < k < k_2$ be such that $2k_1 - 1, 2k_2 - 1$ are consecutive primes. By Lemma 6.1.3, we see that (3.3.9) is valid except possibly for those triples (n, d, k) with $(n, d, k_1) \in V$. We check the validity of (3.3.9) at those (n, d, k). For instance, let $k = 11$. Then $k_1 = 10$. We see that $(1, 3, 10), (4, 3, 10), (2, 5, 10), (1, 7, 10) \in V$. We check that (3.3.9) does not hold at $(1, 3, 11)$ and (3.3.9) holds at $(4, 3, 11), (2, 5, 11)$ and $(1, 7, 11)$. Thus $(1, 3, 11) \in V$. We find that all the exceptions to Theorem 3.3.1 are given by V. □

CHAPTER 7

Squares in arithmetic progression, a prelude

7.1. Introduction

Let n, d, k, b, y be positive integers such that b is square free, $d \geq 1$, $k \geq 2$, $P(b) \leq k$ and $\gcd(n, d) = 1$. We consider the equation

$$\Delta(n, d, k) = n(n+d)\cdots(n+(k-1)d) = by^2. \tag{7.1.1}$$

If $k = 2$, we observe that (7.1.1) has infinitely many solutions. Therefore we always suppose that $k \geq 3$. Let $p \geq k, p|(n+id)$. Then $p \nmid (n+jd)$ for $j \neq i$ otherwise $p|(i-j)$ and $|i-j| < k$, a contradiction. Equating powers of p on both sides of (7.1.1), we see that $\mathrm{ord}_p(n+id)$ is even. From (7.1.1), we have

$$n + id = a_i x_i^2 = A_i X_i^2 \tag{7.1.2}$$

with a_i squarefree and $P(a_i) \leq k$, $P(A_i) \leq k$ and $(X_i, \prod_{p<k} p) = 1$ for $0 \leq i < k$. Since $\gcd(n, d) = 1$, we also have

$$(A_i, d) = (a_i, d) = (X_i, d) = (x_i, d) = 1 \quad \text{for } 0 \leq i < k. \tag{7.1.3}$$

We call $(a_{k-1}, a_{k-2}, \cdots, a_1, a_0)$ as the mirror image of $(a_0, a_1, a_2, \cdots, a_{k-1})$.

Let $d = 1$. We recall that $\Delta(n, 1, k) = \Delta(n, k)$. A conjecture in the folklore says that a product of two or more consecutive positive integers is never a square. Several particular cases have been treated by many mathematicians. We refer to Dickson [**3**] for a history. It is a consequence of some old diophantine results that (7.1.1) with $k = 3$ is possible only when $n = 1, 2, 48$. Let $k \geq 4$. As mentioned in Chapter 1 after (1.0.1), there are infinitely many pairs (n, k) such that $P(\Delta(n, k)) \leq k$. Then (7.1.1) is satisfied with $P(y) \leq k$ for these infinitely many pairs. Therefore we consider (7.1.1) with $P(\Delta(n, k)) > k$. This assumption is satisfied when $n > k$ by (1.0.1). Developing on the earlier work of Erdős [**9**] and Rigge [**36**], it was shown by Erdős and Selfridge [**11**] that (7.1.1) with $n > k^2$ and $P(b) < k$ does not hold. Suppose $P := P(\Delta(n, k)) > k$. Then there is a unique i with $0 \leq i < k$ such that $n + i$ is divisible by P. Hence by (7.1.1), $n + i$ is divisible by P^2 showing that $n + i \geq (k+1)^2$ giving $n > k^2$. Thus it follows from the result of Erdős and Selfridge [**11**] that (7.1.1) with $P > k$ and $P(b) < k$ does not hold. The assumption $P(b) < k$ has been relaxed to $P(b) \leq k$ in Saradha [**40**]. In Section 7.3, we show that (7.1.1) with $P > k$ implies that k is bounded by an absolute constant.

Let $d > 1$. Let $k = 3$. Then for r, s with r, s of opposite parity, $r > s$ and $\gcd(r, s) = 1$, we see that $n = (r^2 - s^2 - 2rs)^2, d = 4rs(r^2 - s^2)$ give infinitely many solutions of (7.1.1). Therefore we assume from now onward that $k \geq 4$. Fermat (see Mordell [**28**, p.21]) showed that there are no four squares in an arithmetic progression. Euler proved a more general result that a product of four terms in arithmetic progression can never be a square. In the next Section we prove this result using elliptic curves. Euler's result was extended to $k = 5$ by Obláth [**31**] and to $6 \leq k \leq 32$ by Hirata-Kohno, Shorey and Tijdeman [**55**]. This was also proved, independently, by Bennett, Győry and Hajdu [**1**] for $6 \leq k \leq 11$. On the other hand, we shall show in Section 7.2 that (7.1.1) with $k = 4$ and $b = 6$ has infinitely many solutions. We state a conjecture in this regard.

Conjecture 7.1.1. *Equation* (7.1.1) *with* $P(b) \leq k$ *implies that* $k = 4$.

Mukhopadhyay and Shorey [**30**] showed that (7.1.1) with $k = 5$ and $P(b) < k$ does not hold. Further Hirata-Kohno, Shorey and Tijdeman [**55**] showed that (7.1.1) with $6 \le k \le 20$ and $P(b) < k$ does not hold except in the cases

$$\begin{aligned} &k = 6,\ (a_0, a_1, \cdots, a_5) = (1, 2, 3, 1, 5, 6)\\ &k = 8,\ (a_0, a_1, \cdots, a_7) = (2, 3, 1, 5, 6, 7, 2, 1), (1, 2, 3, 1, 5, 6, 7, 2)\\ &k = 9,\ (a_0, a_1, \cdots, a_8) = (1, 2, 3, 1, 5, 6, 7, 2, 1)\end{aligned}$$

or their mirror images. A version of the preceding result was proved, independently, by Bennett, Győry and Hajdu [**1**] when $6 \le k \le 11$ and $P(b) \le 5$.

Marszalek [**26**] proved that (7.1.1) with $b = 1$ implies that $k < 2\ \exp\ (d(d+1)^{\frac{1}{2}})$. Thus if d is fixed, then k is bounded by an absolute constant. Equation (7.1.1) was completely solved for $1 < d \le 104$ in Saradha and Shorey [**43**]. For earlier results, see Saradha [**39**] and Filakovszky and Hajdu [**12**]. The result of Marszalek was refined by Shorey and Tijdeman [**53**]. They showed that

$$k < C_1(\omega(d))$$

where $C_1(\omega(d))$ is a computable number depending only on $\omega(d)$. Thus if $\omega(d)$ is fixed, then k is bounded by an absolute constant.

Let $\omega(d) = 1$ i.e, $d = p^\alpha$, p-prime and $\alpha > 0$. It was shown in Saradha and Shorey [**43**] that (7.1.1) with $b = 1$ and $k \ge 4$ has no solution. In fact the condition $\gcd(n, d) = 1$ is not necessary in the preceding result. Thus a product of four or more terms in an arithmetic progression with common difference a prime power can never be a square. This was the first instance where (7.1.1) was completely solved for an infinite set of values of d. Let $b > 1$. Then it follows from the works of Saradha and Shorey [**43**] and Mukhopadhyay and Shorey [**30**] that (7.1.1) implies either $(n, d, k, b, y) = (75, 23, 4, 6, 4620)$ or $k = 5, P(b) = 5$.

We now take $\omega(d) \ge 2$. Our aim in the next chapter is to give an explicit expression for $C_1(\omega(d))$. Let $\kappa_0 = \kappa_0(\omega(d))$ be given by Table 1 and (6). We prove the following result of Laishram [**20**].

THEOREM 7.1.2. *Equation* (7.1.1) *implies that*

$$k < \kappa_0. \tag{7.1.4}$$

7.2. A proof of Euler's result

Let $k = 4$. We show that (7.1.1) with $b = 1$ does not hold. In fact we prove more.

(7.1.1) *with* b=$1, 2, 3$ *does not hold and there are infinitely many solutions with* b=6.

Assume that $n(n + d)(n + 2d)(n + 3d) = by^2$ where $b \in \{1, 2, 3, 6\}$. Then

$$\left(\frac{6b^2y}{n^2}\right)^2 = 6b\left(1 + \frac{d}{n}\right)3b\left(1 + \frac{2d}{n}\right)2b\left(1 + \frac{3d}{n}\right).$$

Putting $X = 2b + \frac{6bd}{n}$ and $Y = \frac{6b^2y}{n^2}$, we obtain the elliptic equation

$$Y^2 = X(X + b)(X + 4b) \ \text{ in } \ X, Y \in \mathbb{Q}.$$

We check using *MAGMA* that the above curves have rank 0 except when $b = 6$ in which case the rank is 1. Let $b \ne 6$. Then the torsion points are given by

$$\begin{aligned} &b = 1 : (X, Y) = (0, 0), (-1, 0), (2, 6), (2, -6), (-2, 2), (-2, -2), (-4, 0),\\ &b = 2 : (X, Y) = (0, 0), (-2, 0), (-8, 0),\\ &b = 3 : (X, Y) = (0, 0), (-3, 0), (-12, 0).\end{aligned}$$

We observe that $X > 0$. Thus it suffices to consider the torsion points $(X, Y) = (2, 6)$ and $(2, -6)$. Then $2 = 2 + 6\frac{d}{n}$ implying $d = 0$. This is a contradiction. Therefore the above torsion points do not give any solution for (7.1.1).

Next we consider $b = 6$ where we refer to Mordell [**28**, p.68] and Tijdeman [**57**]. Suppose (n_0, d_0, y_0) is a solution of (7.1.1). Then

$$X_0 = \frac{n_0}{d_0}, \quad Y_0 = \frac{6y_0}{d_0^2} \tag{7.2.1}$$

is a solution of

$$Y^2 = 6X(X+1)(X+2)(X+3) \quad \text{with} \quad X, Y \in \mathbb{Q}. \tag{7.2.2}$$

Putting $X = x + X_0$ with $x \neq 0$, we consider a new equation

$$z^2 = 6(x + X_0)(x + X_0 + 1)(x + X_0 + 2)(x + X_0 + 3) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{7.2.3}$$

in $x, z \in \mathbb{Q}$. Then $(x, z) = (0, Y_0)$ is a solution of (7.2.3) and $a_4 = Y_0^2$. Let A and B be given by $2Y_0A + B^2 = a_2, 2Y_0B = a_3$. Then we see that

$$x = \frac{a_1 - 2AB}{A^2 - a_0}, \quad z = Ax^2 + Bx + Y_0$$

is a solution of (7.2.3). This implies that

$$x + X_0 = \frac{a_1 - 2AB}{A^2 - a_0} + X_0$$

is a new value of X satisfying (7.2.2). This gives rise to a new solution (n, d, y) of (7.1.1).

Since $(n, d, y) = (1, 1, 2)$ satisfies (7.1.1), we see that $X_0 = 1, Y_0 = 12$ is a solution of (7.2.2). Thus (7.2.3) becomes

$$z^2 = 6(x + 1)(x + 2)(x + 3)(x + 4) = 6x^4 + 60x^3 + 210x^2 + 300x + 144.$$

Hence $A = \frac{215}{96}, B = \frac{25}{2}$ and $x = -\frac{36960}{9071}$. Thus

$$x + X_0 = -\frac{27889}{9071}$$

is a new value of X satisfying (7.2.2). Thus

$$\left(\frac{144041508}{9071^2}\right)^2 = 6\frac{27889 \cdot 18818 \cdot 9747 \cdot 676}{9071^4}$$

giving $n = 676, d = 9071, y = 24006918$ as a solution of (7.1.1). With these value of (n, d, y), we continue as in the case $(n, d, y) = (1, 1, 2)$ to get another new solution (n, d, y) of (7.1.1). We get infinitely many values of n and d satisfying (7.1.1). $\square$

7.3. k is bounded when $d = 1$

Let $d = 1$ and $k \geq 1900$. As mentioned in Section 7.1 above, we may assume that $n > k^2$. First we show that $a_0, a_1, \cdots, a_{k-1}$ are all distinct where a_i's are given by (7.1.2). Let $a_i = a_j$ with $i > j$. Then

$$k > i - j = a_j(x_i^2 - x_j^2) = a_j(x_i - x_j)(x_i + x_j) \geq 2a_jx_j \geq 2(a_jx_j^2)^{\frac{1}{2}} \geq 2n^{\frac{1}{2}} > 2k,$$

a contradiction.

Let s_i denote the $i-$the square free integer. In any set of 36 consecutive integers, after deleting multiples of 4 and 9, we see that there are at most 24 squarefree integers. Thus the number of square free integers $\le s_i$ which is equal to $i \le 24([\frac{s_i}{36}]+1)$ giving $s_i \ge 1.5(i-24)$. Since a_i's are squarefree and distinct, we have

$$a_0 a_1 \cdots a_{k-1} \ge \prod_{i=25}^{k} s_i > (1.5)^{k-24}(k-24)!. \tag{7.3.1}$$

Let $p \le k$. We see that there are at most $[\frac{k-1}{p}]+1$ terms divisible by p. Since a_i's are squarefree, we see that

$$\mathrm{ord}_p(a_0 a_1 \cdots a_{k-1}) \le [\frac{k-1}{p}] + 1 \le \mathrm{ord}_p((k-1)!) + 1.$$

Therefore

$$a_0 a_1 \cdots a_{k-1} | (k-1)! \left(\prod_{p \le k} p \right)$$

Thus using Lemma 2.0.1 (iii), we have

$$a_0 a_1 \cdots a_{k-1} \le (k-1)!(2.7205)^k \tag{7.3.2}$$

This is not sufficient to contradict (7.3.1). We improve (7.3.2) by counting the power of 2 and 3 in $(k-1)!$ and $a_0 a_1 \cdots a_{k-1}$ as follows. We see that $2|a_i$ if and only if 2 divides $n+i$ to an odd power. After removing a term $n+i$ to which 2 appears to a maximum power, the number of terms in the remaining set divisible by 2 to an odd power is at most

$$\begin{aligned} &\left[\frac{k-1}{2}\right] - \left(\left[\frac{k-1}{2^2}\right] - 1\right) + \left[\frac{k-1}{2^3}\right] - \left(\left[\frac{k-1}{2^4}\right] - 1\right) + \cdots \\ &\le \left[\frac{k-1}{2}\right] - \left(\left[\frac{k-1}{2^2}\right] - 1\right) + \left[\frac{k-1}{2^3}\right] - \left(\left[\frac{k-1}{2^4}\right] - 1\right) + \left[\frac{k-1}{2^5}\right] \end{aligned}$$

since the remaining expression is dominated by $[\frac{k-1}{2^5}]$. Further since a_i's are square free, we have

$$\mathrm{ord}_2(\prod_i a_i) \le 1 + \left[\frac{k-1}{2}\right] - \left(\left[\frac{k-1}{2^2}\right] - 1\right) + \left[\frac{k-1}{2^3}\right] - \left(\left[\frac{k-1}{2^4}\right] - 1\right) + \left[\frac{k-1}{2^5}\right]$$

It is known that

$$\mathrm{ord}_2\left((k-1)!\left(\prod_{p \le k} p\right)\right) \ge 1 + \left[\frac{k-1}{2}\right] + \left[\frac{k-1}{2^2}\right] + \cdots + \left[\frac{k-1}{2^5}\right].$$

Thus

$$\mathrm{ord}_2\left(\frac{(k-1)!\left(\prod_{p \le k} p\right)}{a_0 a_1 \cdots a_{k-1}}\right) \ge 2\left[\frac{k-1}{2^2}\right] + 2\left[\frac{k-1}{2^4}\right] - 2 \ge \frac{5}{8}k - 10.$$

Similarly

$$\mathrm{ord}_3\left(\frac{(k-1)!\left(\prod_{p \le k} p\right)}{a_0 a_1 \cdots a_{k-1}}\right) \ge 2\left[\frac{k-1}{3^2}\right] + 2\left[\frac{k-1}{3^4}\right] - 2 \ge \frac{20}{81}k - 10.$$

Therefore

$$\frac{(k-1)!\left(\prod_{p \le k} p\right)}{a_0 a_1 \cdots a_{k-1}} \ge 2^{\frac{5}{8}k} 3^{\frac{20}{81}k} 6^{-10}$$

giving

$$a_0 a_1 \cdots a_{k-1} \leq (k-1)! \left(\prod_{p \leq k} p \right) \left(2^{\frac{5}{8}} 3^{\frac{20}{81}} \right)^k 6^{10}.$$

Now we compare this upperbound with the (7.3.1) and using Lemma 2.0.1 (iii) to get

$$\left(\frac{(1.5) 2^{\frac{5}{8}} 3^{\frac{20}{81}}}{2.7205} \right)^k < k^{23} 6^{10} (1.5)^{24}$$

i.e.,

$$0.1091 < \frac{23 \log k + 27.65}{k}$$

implying $k < 1900$. $\square$

Remark: The argument in the proof of Section 7.3 can be improved considerably. We may use more primes in addition to 2 and 3. See Lemma 8.3.12. Also we may use the exact values of s_i. These improvements enable us to show that $k \leq 14$. The cases $k \leq 14$ are excluded by using a counting argument. For instance, let $k = 14$. Then the number of a_i's composed of only 2 and 3 is at least 5. This is a contradiction since there are only 4 distinct squarefree integers composed of 2 and 3, viz, 1, 2, 3, 6.

CHAPTER 8

An explicit bound for the number of terms of an arithmetic progression whose product is almost square: Proof of Theorem 7.1.2

8.1. Two Propositions

Let κ_0 be given by Table 1 and (6). We prove the following two propositions in this chapter. Theorem 7.1.2 is a direct consequence of the these two propositions.

PROPOSITION 8.1.1. *Let $k \geq \kappa_0$. Then* (7.1.1) *implies that*

$$d < 4c_1(k-1)^2, \tag{8.1.1}$$

$$n < c_1(k-1)^3 \tag{8.1.2}$$

and hence

$$n + (k-1)d < 5c_1(k-1)^3 \tag{8.1.3}$$

where

$$c_1 = \begin{cases} \frac{1}{16} & if\ d \ \ is\ odd \\ \frac{1}{8} & if\ \mathrm{ord}_2(d) = 1 \\ \frac{1}{4} & if\ \mathrm{ord}_2(d) \geq 2. \end{cases}$$

PROPOSITION 8.1.2. *Let $k \geq \kappa_0$. Then* (7.1.1) *implies that*

$$n + (k-1)d \geq 2^{\delta}\frac{5}{16}k^3 \tag{8.1.4}$$

where

$$\delta = \min\{\mathrm{ord}_2(d), 3\}.$$

Since $5c_1 \leq 2^{\delta}\frac{5}{16}$, Theorem 7.1.2 follows immediately from (8.1.3) and (8.1.4). □

In the remaining part of this chapter we shall prove Propositions 8.1.1 and 8.1.2.

8.2. Notations and Preliminaries

First we recall that

$$n + id = a_i x_i^2 = A_i X_i^2 \tag{8.2.1}$$

with a_i squarefree, $P(A_i) \leq k$ and $(X_i, \prod_{p\leq k} p) = 1$ for $0 \leq i < k$. Let

$$T = \{i \ \mid 0 \leq i < k,\ X_i = 1\}, \quad T_1 = \{i \ \mid 0 \leq i < k,\ X_i \neq 1\}.$$

Note that $X_i > k$ for $i \in T_1$. For $0 \leq i < k$, denote

$$\nu(A_i) = |\,\{j \in T_1, A_j = A_i\}\,|. \tag{8.2.2}$$

We always suppose that there exist $i_0 > i_1 > \cdots > i_{\nu(A_i)-1}$ such that $A_{i_0} = A_{i_1} = \cdots = A_{i_{\nu(A_i)-1}}$. Similarly we define

$$R = \{a_i \ \mid 0 \le i < k\}$$

and

$$\nu(a_i) = \big|\{j \mid 0 \le j < k, a_i = a_j\}\big|. \tag{8.2.3}$$

Define

$$\rho := \rho(d) = \begin{cases} 1 \text{ if } 3 \nmid d \\ 3 \text{ if } 3 \mid d. \end{cases} \tag{8.2.4}$$

Let $P_1 < P_2 < \cdots$ be all the odd prime divisors of d. Let $r := r(d) \ge 0$ be the unique integer such that

$$P_1 P_2 \cdots P_r < (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}} \text{ but } P_1 P_2 \cdots P_{r+1} \ge (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}}. \tag{8.2.5}$$

If $r = 0$, we understand that the product $P_1 \cdots P_r = 1$.

Let $d' \mid d$ and $d'' = \frac{d}{d'}$ be such that $\gcd(d', d'') = 1$. We write

$$d'' = d_1 d_2, \ \gcd(d_1, d_2) = \begin{cases} 1 \text{ if } \mathrm{ord}_2(d'') \le 1 \\ 2 \text{ if } \mathrm{ord}_2(d'') \ge 2 \end{cases}$$

and we always suppose that d_1 is odd if $\mathrm{ord}_2(d'') = 1$. We call such pairs (d_1, d_2) as partitions of d''.

We observe that the number of partitions of d'' is $2^{\omega(d'')-\theta_1}$ where

$$\theta_1 := \theta_1(d'') = \begin{cases} 1 \text{ if } \mathrm{ord}_2(d'') = 1, 2 \\ 0 \text{ otherwise} \end{cases}$$

and we write θ for $\theta_1(d)$. In particular, by taking $d' = 1$ and $d'' = d$, the number of partitions of d is $2^{\omega(d)-\theta}$.

Let $A_i = A_j, i > j$. Then from (8.2.1) and (7.1.3), we have

$$(i-j)d = A_i(X_i^2 - X_j^2) = A_i(X_i - X_j)(X_i + X_j) \tag{8.2.6}$$

such that $\gcd(d, X_i - X_j, X_i + X_j) = 1$ if d is odd and 2 if d is even. Hence for any divisor d'' of d, we have a partition (d_1, d_2) of d'' corresponding to $A_i = A_j$ such that $d_1 \mid (X_i - X_j)$ and $d_2 \mid (X_i + X_j)$ and it is the unique partition of d'' corresponding to pair (i, j). Similarly, we have unique partition of d'' corresponding to every pair (i, j) whenever $a_i = a_j$.

8.3. Lemmas

LEMMA 8.3.1. *Let $\pi_d(k) \le \pi(k) - 1$. Then*

$$|T_1| > k - \frac{(k-2)\log\,(k-1)}{\log\,(n+(k-1)d) - \log 2} - \pi(k). \tag{8.3.1}$$

PROOF. We first prove that

$$\begin{aligned} |T_1| &> k - \frac{(k-2)\log\,(k-1)}{\log\,(k-2) + \log\,d} - \pi_d(k) - 1, \\ |T_1| &> k - \frac{(k-2)\log\,(k-2)}{\log\,n} - \pi_d(k) - 1 \text{ for } n \ge 2. \end{aligned} \tag{8.3.2}$$

We have $|T_1| = k - |T|$. We may assume that $|T| > \pi_d(k)$ for a proof of (8.3.2). We follow an argument of Erdős. Let $S_T = \{n + id = A_i | i \in T\}$. For each prime $p \le k$ and $p \nmid d$, we remove a term from S_T such that p does not divide any other term of S_T to a higher power. Let S_1 be the remaining set and we have $|S_1| = |T| - \pi_d(k)$. Then by Lemma 5.2.1, we have

$$\prod_{n+id \in S_1} (n + id) \le n_0 \prod_{p \nmid d} p^{\mathrm{ord}_p((k-2)!)} \le n(k-2)! \prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}. \tag{8.3.3}$$

Again

$$\prod_{n+id \in S_1} (n + id) \ge \prod_{i=0}^{|T|-\pi_d(k)-1} (n + id) = n^{|T|-\pi_d(k)} d^{|T|-\pi_d(k)-1} \prod_{i=1}^{|T|-\pi_d(k)-1} (\alpha + i) \tag{8.3.4}$$

where $\alpha = \frac{n}{d}$. Comparing the upper and lower bounds and using $\prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)} \le 1$, we get

$$d^{|T|-\pi_d(k)-1}(|T| - \pi_d(k) - 1)! \le (k-2)! \tag{8.3.5}$$

and

$$n^{|T|-\pi_d(k)-1} \le (k-2)!. \tag{8.3.6}$$

Therefore

$$\begin{aligned}(|T| - \pi_d(k) - 1) \log d \le& \log((k-2) \cdots (|T| - \pi_d(k))) \\ <& (k - |T| + \pi_d(k) - 1) \log(k-1).\end{aligned}$$

The latter relation holds with strict inequality since $|T| \le k - \pi(2k) + \pi_d(k)$ for $k \ge 4$ by Theorem 3.3.1. This shows that

$$|T| < \frac{(k-2)\log(k-1)}{\log d + \log(k-1)} + \pi_d(k) + 1$$

implying (8.3.2). By (8.3.6), we have

$$|T| < \frac{(k-2)\log(k-2)}{\log n} + \pi_d(k) + 1$$

for $n \ge 2$ which yields (8.3.2).

Now we use $\pi_d(k) \le \pi(k) - 1$. Let $n \ge (k-1)d$. Then $\log n \ge \log(n + (k-1)d) - \log 2$. This gives (8.3.1). For $n < (k-1)d$, we have $\log(k-1) + \log d > \log(n + (k-1)d) - \log 2$ implying (8.3.1) again. □

LEMMA 8.3.2. *Let $d = d'd''$ with $\gcd(d', d'') = 1$. Let $i_0 \in T_1$ be such that $A_{i_0} \ge d'$. Then*

$$\nu(A_{i_0}) \le 2^{\omega(d'') - \theta_1(d'')}. \tag{8.3.7}$$

PROOF. For simplicity, we write $\theta_1 = \theta_1(d'')$. Assume that $\nu(A_{i_0}) > 2^{\omega(d'')-\theta_1}$. Then there exists $i_0 > i_1 > \cdots > i_{2^{\omega(d'')-\theta_1}}$ such that $A_{i_0} = A_{i_1} = \cdots = A_{i_{2^{\omega(d'')-\theta_1}}}$. For each pair $(i_0, i_r), r = 1, 2, \cdots 2^{\omega(d'')-\theta_1}$, we have a unique partition corresponding to the pair. But there are at most $2^{\omega(d'')-\theta_1}$ partitions of d''. Since $(i_0 - i_r)d = A_{i_0}(X_{i_0} - X_{i_r})(X_{i_0} + X_{i_r})$ and $A_{i_0} \ge d'$, we have

$$k > i_0 - i_r = \frac{A_{i_0}}{d'} \left(\frac{X_{i_0} - X_{i_r}}{d_1} \right) \left(\frac{X_{i_0} + X_{i_r}}{d_2} \right) \ge \left(\frac{X_{i_0} - X_{i_r}}{d_1} \right) \left(\frac{X_{i_0} + X_{i_r}}{d_2} \right)$$

where (d_1, d_2) is the partition of d'' corresponding to pair (i_0, i_r). This shows that we cannot have the partition $(\frac{d''}{2^{\theta_1}}, 2^{\theta_1})$ corresponding to any pair. Hence there can be at most $2^{\omega(d'')-\theta_1} - 1$ partitions of d'' with respect to $2^{\omega(d'')-\theta_1}$ pairs of $(i_0, i_r), r = 1, \cdots 2^{\omega(d'')-\theta_1}$. By Box Principle, there exist pairs $(i_0, i_r), (i_0, i_s)$ with $1 \le r < s \le 2^{\omega(d'')-\theta_1}$ and a partition (d_1, d_2) of d'' corresponding to these pairs. Thus

$$d_1 \mid (X_{i_0} - X_{i_r}), d_2 \mid (X_{i_0} + X_{i_r}) \text{ and } d_1 \mid (X_{i_0} - X_{i_s}), d_2 \mid (X_{i_0} + X_{i_s})$$

so that $d_1 | (X_{i_0} - X_{i_s}) - (X_{i_0} - X_{i_r}) = X_{i_r} - X_{i_s}$ and $d_2 | (X_{i_0} + X_{i_r}) - (X_{i_0} + X_{i_s}) = X_{i_r} - X_{i_s}$. Therefore $\mathrm{lcm}(d_1, d_2) \mid (X_{i_r} - X_{i_s})$. Since $A_{i_r} = A_{i_s} = A_{i_0}$ and $\gcd(d_1, d_2) \le 2$, we have

$$k > (i_r - i_s) > (i_r - i_s)\frac{d'}{A_{i_0}} = \frac{(X_{i_r} - X_{i_s})}{\mathrm{lcm}(d_1, d_2)} \frac{(X_{i_r} + X_{i_s})}{\gcd(d_1, d_2)} > \frac{(X_{i_r} + X_{i_s})}{2} > \frac{2k}{2} = k.$$

This is a contradiction. □

By taking $d' = 1$ and $d'' = d$, the following result is immediate from Lemma 8.3.2 since $\theta_1(d) = \theta$.

COROLLARY 8.3.3. *For $i_0 \in T_1$, we have $\nu(A_{i_0}) \le 2^{\omega(d)-\theta}$.*

LEMMA 8.3.4. *Let $k \ge 17$. Suppose $n \ge c_1(k-1)^3$ or $d \ge 4c_1(k-1)^2$. Then for $0 \le i_0 < k$, we have*

$$\nu(a_{i_0}) \le 2^{\omega(d)-\theta}. \tag{8.3.8}$$

PROOF. Suppose that $\nu(a_{i_0}) > 2^{\omega(d)-\theta}$. We note that both $x_i + x_j$ and $x_i - x_j$ are even when d is even. Continuing as in the proof of (8.3.7) with $d'' = d$, we see that there exists i, j with $i > j$ and

$$k > \frac{a_{i_0}(x_i + x_j)}{2}$$

where $\frac{d}{2}\Big|(x_i - x_0)$ if d is even and $d\Big|(x_i - x_0)$ if d is odd. We have $x_i \ge x_j + \frac{d}{2}$ so that $k > \frac{1}{2}a_{i_0}(x_i + x_j) \ge \left(a_j x_j^2\right)^{\frac{1}{2}} + \frac{d}{4} \ge n^{\frac{1}{2}} + \frac{d}{4}$ and hence

$$k > \begin{cases} 1 + c_1(k-1)^2 & \text{if } d \ge 4c_1(k-1)^2, \\ 1 + (c_1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}} & \text{if } n \ge c_1(k-1)^3 \end{cases}$$

which is not true for $k \ge 17$. □

LEMMA 8.3.5. *The equation (7.1.1) implies that either*

$$d \ge 4c_1(k-1)^2$$

or

$$r \ge \left[\frac{\omega(d)}{3}\right].$$

PROOF. If $r + 1 \le [\frac{\omega(d)}{3}]$, then $\omega(d) \ge 3(r+1)$ giving $d \ge 4c_1(k-1)^2$ by (8.2.5). □

LEMMA 8.3.6. *Let $S \subseteq \{A_i | 0 \le i < k\}$ and $\min_{A_h \in S} A_h \ge U$. Let $t \ge 1$. Assume that*

$$|S| > Q_t \left(\frac{P_1 - 1}{2}\right) \cdots \left(\frac{P_t - 1}{2}\right) \tag{8.3.9}$$

where $Q_t \ge 1$ is an integer. Then

$$\max_{A_h \in S} A_h \ge 2^{\delta} Q_t P_1 \cdots P_t + U. \tag{8.3.10}$$

PROOF. For an odd $p|d$, we have

$$\left(\frac{A_h}{p}\right) = \left(\frac{A_h X_h^2}{p}\right) = \left(\frac{n}{p}\right)$$

where $(\frac{\cdot}{\cdot})$ is Legendre symbol. We observe that A_h belongs to at most $\frac{p-1}{2}$ distinct residue classes modulo p for each $0 \le h < k$. If d is even, then A_h also belongs to a unique residue class modulo 2^δ for each $0 \le h < k$. Hence, by Chinese remainder theorem, A_h belongs to at most $\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_j-1}{2}\right)$ distinct residue classes modulo $2^\delta P_1 \cdots P_j$ for each j, $1 \le j \le t$. Assume that (8.3.10) does not hold. Then

$$\max_{A_h \in S} A_h - (U-1) \le 2^\delta Q_t P_1 \cdots P_t.$$

Therefore

$$|S| \le \frac{2^\delta Q_t P_1 \cdots P_t}{2^\delta P_1 \cdots P_t} \left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_t-1}{2}\right)$$

contradicting (8.3.9). □

COROLLARY 8.3.7. *Let S and U be as in Lemma 8.3.6. Let $|S| \ge s > \left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_t-1}{2}\right)$. Then*

$$\max_{A_h \in S} A_h \ge \frac{3}{4} 2^{t+\delta} s + U. \tag{8.3.11}$$

PROOF. Let $(f-1)\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_{t-1}-1}{2}\right) < s - Q_t\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_t-1}{2}\right) \le f\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_{t-1}-1}{2}\right)$ where $Q_t \ge 1$ and $1 \le f \le \frac{P_t-1}{2}$ is an integer. To see this, write $s = Q\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_t-1}{2}\right) + Q'\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_{t-1}-1}{2}\right) + R$ where $0 \le Q' < \frac{P_t-1}{2}$ and $0 \le R < \left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_{t-1}-1}{2}\right)$. If $R > 0$, then take $Q_t = Q$, $f - 1 = Q'$; if $R = 0$ and $Q' > 0$, then take $Q_t = Q$, $f = Q'$; and if $R = Q' = 0$, then take $Q_t = Q - 1$ and $f = \frac{P_t-1}{2}$. We arrange the elements of S in increasing order and let $S' \subseteq S$ be the first $(f-1)\left(\frac{P_1-1}{2}\right) \cdots \left(\frac{P_{t-1}-1}{2}\right) + 1$ elements and S'' consist of the remaining set. Then we see from Lemma 8.3.6 with $t = t-1$ and $Q_t = f - 1$ that

$$\max_{A_h \in S'} A_h \ge 2^\delta (f-1) P_1 P_2 \cdots P_{t-1} + U = U'.$$

Now we apply Lemma 8.3.6 with $U = U'$ in S'' to derive

$$\max_{A_h \in S} A_h \ge 2^\delta Q_t P_1 P_2 \cdots P_t + 2^\delta (f-1) P_1 P_2 \cdots P_{t-1} + U.$$

Hence to derive (8.3.11), it is enough to prove

$$Q_t P_1 \cdots P_t + (f-1) P_1 \cdots P_{t-1} \ge \frac{3}{4}\left\{Q_t(P_1-1)\cdots(P_t-1) + 2f(P_1-1)\cdots(P_{t-1}-1)\right\}.$$

By observing that

$$Q_t(P_1-1)\cdots(P_t-1) \le Q_t P_1 \cdots P_t - Q_t P_1 \cdots P_{t-1},$$
$$2f(P_1-1)\cdots(P_{t-1}-1) \le 2f P_1 \cdots P_{t-1} - 2f P_1 \cdots P_{t-2},$$

it suffices to show that

$$Q_t + \frac{3(Q_t-1) - (2f+1)}{P_t} + \frac{6f}{P_t P_{t-1}} \ge 0$$

which is true since $Q_t \ge 1$ and $1 \le f \le \frac{P_t-1}{2}$. □

Let t_i denote the $i-$th odd squarefree positive integer. We recall here s_i is the i-th squarefree positive integer. The next lemma gives a bound for s_i and t_i.

Lemma 8.3.8. *We have*

$$s_i \geq 1.6i \quad for \quad i \geq 78 \tag{8.3.12}$$

and

$$t_i \geq 2.4i \quad for \quad i \geq 51. \tag{8.3.13}$$

Further we have

$$\prod_{i=1}^{l} s_i \geq (1.6)^l l! \quad for \quad l \geq 286 \tag{8.3.14}$$

and

$$\prod_{i=1}^{l} t_i \geq (2.4)^l l! \quad for \quad l \geq 200. \tag{8.3.15}$$

Proof. The proof is similar to that of [**43**, (6.9)]. For (8.3.12) and (8.3.13), we check that $s_i \geq 1.6i$ for $78 \leq i \leq 286$ and $t_i \geq 2.4i$ for $51 \leq i \leq 132$, respectively. Further we observe that in a given set of 144 consecutive integers, there are at most 90 squarefree integers and at most 60 odd squarefree integers by deleting multiples of $4, 9, 25, 49, 121$ and $2, 9, 25, 49$, respectively. Then we continue as in the proof of [**43**, (6.9)] to get (8.3.12) and (8.3.13). Further we check that (8.3.14) holds at $l = 286$ and (8.3.15) holds at $l = 200$. Then we use (8.3.12) and (8.3.13) to obtain (8.3.14) and (8.3.15), respectively. □

Lemma 8.3.9. *Let $X > 1$ be a positive integer. Then*

$$\sum_{i=1}^{X-1} 2^{\omega(i)} \leq \eta(X) X \log X \tag{8.3.16}$$

where

$$\eta := \eta(X) = \begin{cases} 1 & if \ X = 1 \\ \dfrac{\sum_{i=1}^{X-1} 2^{\omega(i)}}{X \ \log X} & if \ 1 < X < 248 \\ 0.75 & if \ X \geq 248. \end{cases} \tag{8.3.17}$$

Proof. We check that (8.3.16) holds for $1 < X < 11500$. Thus we may assume $X \geq 11500$. Let s_j be the largest squarefree integer $\leq X$. Then $i \geq 78$ and hence by Lemma 8.3.8, we have $1.6j \leq s_j \leq X$ so that $j \leq \left[\frac{X}{1.6}\right]$. We have $2^{\omega(i)} = \sum_{e|i} |\mu(e)|$. Therefore

$$\sum_{i=1}^{X-1} 2^{\omega(i)} = \sum_{i=1}^{X-1} \sum_{e|i} |\mu(e)| \leq \sum_{1 \leq e < X} \left[\frac{X-1}{e}\right] |\mu(e)| \leq (X-1) \sum_{1 \leq e < X} \frac{|\mu(e)|}{e} \leq X \sum_{i=1}^{\left[\frac{X}{1.6}\right]} \frac{1}{s_i}.$$

We check that there are 6990 squarefree integers upto 11500. By using (8.3.12), we have

$$\begin{aligned}\sum_{i=1}^{X-1} 2^{\omega(i)} &\le X\left\{\sum_{i=1}^{6990}\frac{1}{s_i}-\frac{1}{1.6}\sum_{i=1}^{6990}\frac{1}{i}+\frac{1}{1.6}\sum_{i=1}^{[\frac{X}{1.6}]}\frac{1}{i}\right\}\\ &\le X\left\{\sum_{i=1}^{6990}\frac{1}{s_i}-\frac{1}{1.6}\sum_{i=1}^{6990}\frac{1}{i}+\frac{1}{1.6}\left(1+\log\frac{X}{1.6}\right)\right\}\\ &\le \frac{3}{4}X\log X\left\{\frac{4}{3}\frac{1.1658}{\log X}+\frac{4}{3}\frac{1}{1.6}\right\}\end{aligned}$$

implying (8.3.16). □

Lemma 8.3.10. *Let $c>0$ be such that $c2^{\omega(d)-3}>1$, $\mu\ge 2$ and*

$$\mathfrak{C}_\mu=\{A_i\mid \nu(A_i)=\mu,\ A_i>\frac{\rho 2^\delta k}{3c2^{\omega(d)}}\}.$$

Then

$$\mathfrak{C}:=\sum_{\mu\ge 2}\frac{\mu(\mu-1)}{2}|\mathfrak{C}_\mu|\le\frac{c}{8}\eta(c2^{\omega(d)-3})2^{\omega(d)}(2^{\omega(d)-\theta}-1)(\log c2^{\omega(d)-3}). \tag{8.3.18}$$

Proof. Let $i_1>i_2\cdots>i_\mu$ be such that $A_{i_1}=A_{i_2}=\cdots=A_{i_\mu}$. These give rise to $\frac{\mu(\mu-1)}{2}$ pairs of $(i,j), i>j$ with $A_i=A_j$. Therefore the total number of pairs (i,j) with $i>j$ and $A_i=A_j$ is $\mathfrak{C}$.

We know that there is a unique partition of d corresponding to each pair $(i,j), i>j$ such that $A_i=A_j$. Hence by Box Principle, there exists at least $\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1}$ pairs of $(i,j), i>j$ with $A_i=A_j$ and a partition (d_1,d_2) of d corresponding to these pairs. For every such pair (i,j), we write $X_i-X_j=d_1r_{ij},\ X_i+X_j=d_2s_{ij}$. Then $\gcd(X_i-X_j,X_i+X_j)=2$ and $24|(X_i^2-X_j^2)$. Let r'_{ij},s'_{ij} be such that $r'_{ij}|r_{ij}, s'_{ij}|s_{ij}$, $\gcd(r'_{ij},s'_{ij})=1$ and $r_{ij}s_{ij}=\frac{24}{\rho 2^\delta}r'_{ij}s'_{ij}$. Then

$$r'_{ij}s'_{ij}=\frac{\rho 2^\delta}{24}r_{ij}s_{ij}=\frac{\rho 2^\delta}{24}\frac{X_i^2-X_j^2}{d}=\frac{\rho 2^\delta}{24}\frac{i-j}{A_i}<\frac{\rho 2^\delta}{24}\frac{k}{A_i}<c2^{\omega(d)-3}$$

since $A_i>\frac{\rho 2^\delta k}{3c2^{\omega(d)}}$. There are at most $\displaystyle\sum_{i=1}^{c2^{\omega(d)-3}-1}2^{\omega(i)}$ possible pairs of (r'_{ij},s'_{ij}), and hence an equal number of possible pairs of (r_{ij},s_{ij}). By Lemma 8.3.9, we estimate

$$\sum_{i=1}^{c2^{\omega(d)-3}-1}2^{\omega(i)}\le\eta(c2^{\omega(d)-3})c2^{\omega(d)-3}(\log c2^{\omega(d)-3}).$$

Thus if we have

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1}>\eta(c2^{\omega(d)-3})c2^{\omega(d)-3}(\log c2^{\omega(d)-3}),$$

then there exist distinct pairs $(i,j)\neq(g,h), i>j, g>h$ with $A_i=A_j, A_g=A_h$ such that $r_{ij}=r_{gh}, s_{ij}=s_{gh}$ giving

$$X_i-X_j=d_1r_{ij}=X_g-X_h\text{ and }X_i+X_j=d_2s_{ij}=X_g+X_h.$$

Thus $X_i=X_g, X_j=X_h$ implying $(i,j)=(g,h)$, a contradiction. Hence

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1}\le\eta(c2^{\omega(d)-3})c2^{\omega(d)-3}(\log c2^{\omega(d)-3})$$

implying (8.3.18). □

The following Lemma is a refinement of [**53**, Lemma 2].

LEMMA 8.3.11. *Let $i > j, g > h, 0 \le i, j, g, h < k$ be such that*

$$a_i = a_j, \;\; a_g = a_h \tag{8.3.19}$$

and

$$x_i - x_j = d_1 r_1, \;\; x_i + x_j = d_2 r_2, \;\; x_g - x_h = d_1 s_1, \;\; x_g + x_h = d_2 s_2 \tag{8.3.20}$$

where (d_1, d_2) is a partition of d; $r_1 \equiv s_1 (mod\ 2)$, $r_2 \equiv s_2 (mod\ 2)$ when d is even; and either $r_1 \equiv s_1 (mod\ 2)$, $a_i \equiv a_g (mod\ 4)$ or $2|\gcd(r_1, s_1)$ when d is odd. Then we have either

$$a_i = a_g, r_1 = s_1 \text{ or } \;\; a_i = a_g, r_2 = s_2 \tag{8.3.21}$$

or (8.1.1) *and* (8.1.2) *hold.*

PROOF. We follow the proof of [**53**, Lemma 2]. Suppose that (8.3.21) does not hold. Then

$$a_i r_1^2 - a_g s_1^2 \ne 0, \quad a_i r_2^2 - a_g s_2^2 \ne 0. \tag{8.3.22}$$

We proceed as in [**53**, Lemma 2] to conclude from $d \mid (a_i x_i^2 - a_g x_g^2)$ that

$$d_1 d_2 = d \mid \frac{1}{4} \left\{ (a_i r_1^2 - a_g s_1^2) d_1^2 + (a_i r_2^2 - a_g s_2^2) d_2^2 + 2d(a_i r_1 r_2 - a_g s_1 s_2) \right\}. \tag{8.3.23}$$

Thus we have

$$(a_i r_1^2 - a_g s_1^2) d_1^2 = a_i (x_i - x_j)^2 - a_g (x_g - x_h)^2 \ne 0$$

and

$$(a_i r_2^2 - a_g s_2^2) d_2^2 = a_i (x_i + x_j)^2 - a_g (x_g + x_h)^2 \ne 0.$$

Since

$$n \le a_j x_j^2 < a_i x_i x_j < a_i x_i^2 \le n + (k-1)d$$

and

$$n \le a_h x_h^2 < a_g x_g x_h < a_g x_g^2 \le n + (k-1)d,$$

we have

$$\left| a_i x_i x_j - a_g x_g x_h \right| < (k-1)d. \tag{8.3.24}$$

Also

$$\begin{aligned} |a_i x_i^2 - a_g x_g^2| &= |i - g| d \le (k-1)d, \\ |a_j x_j^2 - a_h x_h^2| &= |j - h| d \le (k-1)d \end{aligned} \tag{8.3.25}$$

and

$$n \le \min \left\{ \frac{1}{4} a_i (x_i + x_j)^2, \frac{1}{4} a_g (x_g + x_h)^2 \right\}. \tag{8.3.26}$$

Hence we derive from (8.3.24), (8.3.25) and (8.3.26) that

$$|(a_i r_2^2 - a_g s_2^2) d_2^2| < 4(k-1)d, \tag{8.3.27}$$

$$n |(a_i r_1^2 - a_g s_1^2) d_1^2| < \frac{1}{4} (k-1)^2 d^2 \tag{8.3.28}$$

and further considering the cases $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$, $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$, $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$ and $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$, we derive

$$G(i, g) = |a_i r_1^2 - a_g s_1^2| d_1^2 + |a_i r_2^2 - a_g s_2^2| d_2^2 < 4(k-1)d. \tag{8.3.29}$$

Let $d = d_1 d_2$ be odd, $\gcd(d_1, d_2) = 1$. We have either r_1, s_1 are even and hence r_1, r_2, s_1, s_2 are even, or $a_i \equiv a_g \pmod 4$ and $r_1 \equiv s_1 \pmod 2$ and hence $r_2 \equiv s_2 \pmod 2$. Then reading modulo d_1 and d_2 separately in (8.3.23), we have

$$d_1 \Big| \frac{1}{4}(a_i r_2^2 - a_g s_2^2) \text{ and } d_2 \Big| \frac{1}{4}(a_i r_1^2 - a_g s_1^2). \tag{8.3.30}$$

Therefore

$$4dd_2 = 4d_1 d_2^2 \le |a_i r_2^2 - a_g s_2^2| d_2^2 \tag{8.3.31}$$

and

$$4dd_1 = 4d_1^2 d_2 \le |a_i r_1^2 - a_g s_1^2| d_1^2. \tag{8.3.32}$$

From (8.3.29), we have

$$4d(d_1 + d_2) \le G(i, g) < 4(k-1)d$$

so that

$$d = d_1 d_2 \le \left(\frac{d_1 + d_2}{2}\right)^2 < \frac{(k-1)^2}{4}.$$

This gives (8.1.1). Again from (8.3.32) and (8.3.28), we see that $4ndd_1 < \frac{1}{4}(k-1)^2 d^2$ i.e. $n < \frac{1}{16}(k-1)^2 d_2$. From (8.3.31) and (8.3.27), we have $4dd_2 < 4(k-1)d$ i.e. $d_2 < (k-1)$. Thus (8.1.2) is also valid.

Let $d = d_1 d_2$ be even with $\mathrm{ord}_2(d) = 1$ and d_1 odd. Then x_i's are odd and therefore both r_1 and s_1 are even. We see from (8.3.23) that

$$4d_1 \Big| (a_i r_2^2 - a_g s_2^2) d_2^2 \text{ and } 4d_2 \Big| (a_i r_1^2 - a_g s_1^2) d_1^2. \tag{8.3.33}$$

Since $r_1 \equiv s_1 \pmod 2$, $r_2 \equiv s_2 \pmod 2$, $\gcd(d_1, d_2) = 1$ and d_1 odd, we derive that

$$2d_1 \Big| (a_i r_2^2 - a_g s_2^2), \quad 4d_2 \Big| (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2dd_2 = 2d_1 d_2^2 \le |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 4dd_1 = 4d_1^2 d_2 \le |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (8.1.1) and (8.1.2).

Let $d = d_1 d_2$ be even with $\mathrm{ord}_2(d) \ge 2$, $\gcd(d_1, d_2) = 2$. Then we see from (8.3.23) that (8.3.33) holds. Since $\gcd(d_1, d_2) = 2$, $r_1 \equiv s_1 \pmod 2$ and $r_2 \equiv s_2 \pmod 2$, we derive that

$$2d_1 \Big| (a_i r_2^2 - a_g s_2^2), \quad 2d_2 \Big| (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2dd_2 = 2d_1 d_2^2 \le |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 2dd_1 = 2d_1^2 d_2 \le |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (8.1.1) and (8.1.2). □

LEMMA 8.3.12. *For a prime $p < k$, let*

$$\gamma_p = \mathrm{ord}_p\left(\prod_{a_i \in R} a_i\right), \ \gamma_p' = 1 + \mathrm{ord}_p((k-1)!).$$

Let $\mathfrak{m} > 1$ by any real number. Then

$$\prod_{2 \le p \le \mathfrak{m}} p^{\gamma_p - \gamma_p'} \le k^{1.5\pi(\mathfrak{m})} \left(z_1 \prod_{2 < p \le \mathfrak{m}} p^{\frac{2p}{p^2-1}}\right) \left(z_2 \prod_{2 < p \le \mathfrak{m}} p^{\frac{2}{p^2-1}}\right)^{-k} \tag{8.3.34}$$

where $(z_1, z_2) = (2^{\frac{4}{3}}, 2^{\frac{2}{3}})$ *if* d *is odd and* $(z_1, z_2) = (4, 2)$ *if* d *is even.*

PROOF. The proof is the refinement of inequality [**43**, (6.4)]. Let $p^h \le k-1 < p^{h+1}$ where h is a positive integer. Then

$$\gamma_p' - 1 = \left[\frac{k-1}{p}\right] + \left[\frac{k-1}{p^2}\right] + \cdots + \left[\frac{k-1}{p^h}\right]. \tag{8.3.35}$$

Let $p \nmid d$. Then we see that γ_p is the number of terms in $\{n, n+d, \cdots, n+(k-1)d\}$ divisible by p to an odd power. After removing a term to which p appears to a maximum power, the number of terms in the remaining set divisible by p to an odd power is at most

$$\left[\frac{k-1}{p}\right] - \left(\left[\frac{k-1}{p^2}\right] - 1\right) + \left[\frac{k-1}{p^3}\right] - \left(\left[\frac{k-1}{p^4}\right] - 1\right) + \cdots + (-1)^{\epsilon}\left(\left[\frac{k-1}{p^h}\right] + (-1)^{\epsilon}\right)$$

where $\epsilon = 1$ or 0 according as h is even or odd, respectively. We note that the above expression is always positive. This with (8.3.35) and $[\frac{k-1}{p^i}] \ge \frac{k-1}{p^i} - 1 + \frac{1}{p^i} = \frac{k}{p^i} - 1$, we have

$$\begin{aligned}
\gamma_p - \gamma_p' &\le -2\left\{\left[\frac{k-1}{p^2}\right] + \cdots + \left[\frac{k-1}{p^{h-1+\epsilon}}\right]\right\} + \frac{h-1+\epsilon}{2} \\
&\le -2\left\{\frac{k}{p^2} + \cdots + \frac{k}{p^{h-1+\epsilon}} - \frac{h-1+\epsilon}{2}\right\} + \frac{h-1+\epsilon}{2} \\
&= -\frac{2k}{p^2(1-\frac{1}{p^2})}(1 - \frac{1}{p^{h-1+\epsilon}}) + 1.5(h-1+\epsilon).
\end{aligned}$$

Since $p^h \ge \frac{k}{p}$ and $h < \frac{\log k}{\log p}$, we get

$$\gamma_p - \gamma_p' < -\frac{2k}{p^2-1} + \frac{1.5\log k}{\log p} + \frac{2p^{2-\epsilon}}{p^2-1} + 1.5\epsilon - 1.5 \le -\frac{2k}{p^2-1} + \frac{1.5\log k}{\log p} + \frac{2p}{p^2-1}.$$

When d is even, we have $\gamma_2 - \gamma_2' = -1 - \mathrm{ord}_2(k-1) < -k + \frac{\log k}{\log 2} + 2$ by Lemma 2.0.5. Now (8.3.34) follows immediately. □

LEMMA 8.3.13. *Suppose that* $n \ge c_1(k-1)^3$ *or* $d \ge 4c_1(k-1)^2$. *Let* $1 \le \varrho \le 2^{\omega(d)-\theta}$ *be the greatest integer such that* $R_\varrho = \{a_i | \nu(a_i) = \varrho\} \ne \phi$. *For* $k \ge \kappa_0$, *we have*

$$\mathfrak{r} = \left|\{(i,j) \middle| a_i = a_j, i > j\}\right| \ge g(\varrho) := \begin{cases} 4\varrho(2^{\omega(d)} - 1) & \text{if } d \text{ is odd} \\ 2\varrho(2^{\omega(d)-\theta} - 1) & \text{if } d \text{ is even.} \end{cases}$$

PROOF. We have

$$k = \sum_{\mu=1}^{\varrho} \mu r_\mu \quad \text{and} \quad |R| = \sum_{\mu=1}^{\varrho} r_\mu$$

where $r_\mu = |R_\mu = \{a_i | \nu(a_i) = \mu\}|$. Each R_μ give rise to $\frac{\mu(\mu-1)}{2} r_\mu$ pairs of i, j with $i > j$ such that $a_i = a_j$. Then

$$\mathfrak{r} = \sum_{\mu=1}^{\varrho} \frac{\mu(\mu-1)}{2} r_\mu = k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu.$$

Suppose that the assertion of the Lemma 8.3.13 does not hold. Then $g(\varrho) > k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu$. We have

$$g(\varrho) - \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu \le g(\varrho) - \frac{(\varrho-1)(\varrho-2)}{2} := g_0(\varrho).$$

We see that $g_0(\varrho)$ is an increasing function of ϱ. Since $\varrho \le 2^{\omega(d)-\theta}$, we find that

$$k - |R| < g_0(2^{\omega(d)-\theta}) = (2^{\omega(d)-\theta} - 1)(z_3 2^{\omega(d)-\theta} + 1) := g_1$$

where $z_3 = \frac{7}{2}$ if d is odd and $\frac{3}{2}$ if d is even. Thus $|R| > k - g_1$. Since a_i's are squarefree, we have by Lemma 8.3.8 that

$$\prod_{a_i \in R} a_i \ge z_4^{k-g_1}(k - g_1)!$$

where $z_4 = 1.6$ if d is odd and 2.4 if d is even. Also, we have

$$\prod_{a_i \in R} a_i \,\Big|\, (k-1)! \left(\prod_{p<k} p\right) \prod_{2 \le p \le \mathfrak{m}} p^{\gamma_p - \gamma_p'}$$

where γ_p, γ_p' and $\mathfrak{m}$ are as in Lemma 8.3.12. This with (8.3.34) and Lemma 2.0.2 (iv) gives

$$\prod_{a_i \in R} a_i < k! k^{1.5\pi(\mathfrak{m})-1} \left(z_1 \prod_{2<p\le\mathfrak{m}} p^{\frac{2p}{p^2-1}}\right) \left(\frac{z_2}{2.7205} \prod_{2<p\le\mathfrak{m}} p^{\frac{2}{p^2-1}}\right)^{-k}.$$

Comparing the lower and upper bounds, we have

$$\frac{z_4^{g_1} k!}{(k-g_1)!} > k^{-1.5\pi(\mathfrak{m})+1} \left(z_1 \prod_{2<p\le\mathfrak{m}} p^{\frac{2p}{p^2-1}}\right)^{-1} \left(\frac{z_2 z_4}{2.7205} \prod_{2<p\le\mathfrak{m}} p^{\frac{2}{p^2-1}}\right)^{k}. \tag{8.3.36}$$

By Lemma 2.0.6, we have

$$\frac{z_4^{g_1} k!}{(k-g_1)!} < z_4^{g_1} e^{-g_1} k^{g_1} \left(\frac{k}{k-g_1}\right)^{k-g_1+\frac{1}{2}} \frac{e^{\frac{1}{12k}}}{e^{\frac{1}{12(k-g_1)+1}}}.$$

Since $k \ge \kappa_0$, we find that $g_1 < \frac{k}{z_5}$ for $\omega(d) \ge 12$ where $z_5 = 37, 18$ for d odd and d even, respectively. Thus

$$\frac{z_4^{g_1} k!}{(k-g_1)!} < \begin{cases} \left(\frac{z_4(k-g_1)}{e}\right)^{g_1} \left(\frac{k}{k-g_1}\right)^{k+\frac{1}{2}} & \text{if } \omega(d) \le 11 \\ \left(\frac{z_5}{z_5-1}\right)^{k+\frac{1}{2}} \left(\frac{(z_4(z_5-1)k}{z_5 e}\right)^{g_1} & \text{if } \omega(d) \ge 12. \end{cases}$$

Hence we derive from (8.3.36) that

$$g_1 > \frac{k \log\left(\frac{z_2 z_4}{2.7205} \prod_{2<p\le\mathfrak{m}} p^{\frac{2}{p^2-1}}\right) + (k+\frac{1}{2})\log(1 - \frac{g_1}{k})}{\log(k-g_1) - 1 + \log z_4} - \frac{(1.5\pi(\mathfrak{m})-1)\log k + \log\left(z_1 \prod_{2<p\le\mathfrak{m}} p^{\frac{2p}{p^2-1}}\right)}{\log(k-g_1) - 1 + \log z_4} \tag{8.3.37}$$

for $\omega(d) \le 11$ and

$$g_1 > \frac{k \log\left(\frac{z_5-1}{z_5}\frac{z_2 z_4}{2.7205} \prod_{2<p\le\mathfrak{m}} p^{\frac{2}{p^2-1}}\right) - (1.5\pi(\mathfrak{m})-1)\log k - \log\left(\sqrt{\frac{z_5}{z_5-1}} z_1 \prod_{2<p\le\mathfrak{m}} p^{\frac{2p}{p^2-1}}\right)}{\log k - 1 + \log z_4(z_5-1) - \log z_5} \tag{8.3.38}$$

for $\omega(d) \geq 12$.

Let $\omega(d) \leq 11$. Taking $\mathfrak{m} = \min(1000, \sqrt{\kappa_0})$ in (8.3.37), we observe that the right hand side of (8.3.37) is an increasing function of k and the inequality does not hold at $k = \kappa_0$. Hence (8.3.37) is not valid for all $k \geq \kappa_0$. For instance, when $\omega(d) = 4$, d odd, we have $\kappa_0 = 15700$ and $g_1 = 855$. With these values, we see that the right hand side of (8.3.37) exceeds 855 at $k = 15700$, a contradiction. Hence (8.3.37) is not valid for all $k \geq 16000$.

Let $\omega(d) \geq 12$. Taking $\mathfrak{m} = 1000$ in (8.3.38), we derive that

$$g_1 > \begin{cases} 0.63104\frac{k}{\log k} & \text{if } d \text{ is odd} \\ 1.183\frac{k}{\log k} & \text{if } d \text{ is even.} \end{cases}$$

For d odd, we see that

$$\begin{aligned} 0.63104\frac{k}{\log k} \geq 0.63104\frac{\kappa_0}{\log \kappa_0} &= \frac{0.63104 \times 11\omega(d)4^{\omega(d)}}{\omega(d)\log 4 + \log 11 + \log \omega(d)} \\ &> \frac{7}{2}4^{\omega(d)} > g_1, \end{aligned}$$

a contradiction. Similarly we get a contradiction for d even. □

LEMMA 8.3.14. *Let $k \geq \kappa_0$. Assume that $d < 4c_1(k-1)^2$. Let $T_1 = \{0 \leq i < k | X_i > 1\}$ defined in Section 8.2 be such that*

$$|T_1| > C_1 := \begin{cases} \frac{k}{C_2} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2 \\ \frac{k}{C_2} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3,4,5 \\ \frac{k}{C_2} + \frac{k}{12} + \frac{k}{9} & \text{if } \omega(d) \geq 6 \end{cases}$$

where $C_2 \leq 2k^{\frac{1}{3}}$ and $C_3 = 39, 42, 195, 806$ for $\omega(d) = 2,3,4,5$, respectively. Then

$$\max_{i\in T_1} A_i \geq 2^\delta C_0\frac{k}{C_2} \text{ where } C_0 = C_0(\omega(d)) = \begin{cases} 1 & \text{if } \omega(d) = 2 \\ \frac{3}{4}2^{[\frac{\omega(d)}{3}]} & \text{if } \omega(d) \geq 3. \end{cases} \tag{8.3.39}$$

PROOF. We see that for $\omega(d) \geq 6$,

$$\frac{k}{20 \cdot 2^{\omega(d)}} \geq \left(4c_1(k-1)^2\right)^{\frac{1}{\omega(d)}} > d^{\frac{1}{\omega(d)}}.$$

where c_1 is given by Proposition 8.1.1. Hence there exists a partition $d = d_1d_2$ of d with

$$d_1 < \frac{k}{20 \cdot 2^{\omega(d)}} \text{ with } \omega(d_1) \geq 1 \text{ and } \omega(d_2) \leq \omega(d) - 1.$$

Therefore

$$\nu(A_i) \leq 2^{\omega(d_2)} \leq 2^{\omega(d)-1} \text{ for } A_i \geq \frac{k}{20 \cdot 2^{\omega(d)}} \tag{8.3.40}$$

by Lemma 8.3.2.

Let

$$T_2 = \{i \in T_1 \big| A_i > \frac{2^\delta \rho k}{3c2^{\omega(d)}}\}, \quad T_3 = T_1 - T_2 \tag{8.3.41}$$

where $c = 16$ if $\omega(d) = 2$, $c = 4$ if $\omega(d) = 3,4,5$ and $c = 2$ if $\omega(d) \geq 6$. Further let

$$S_2 = \{A_i \big| i \in T_2\}, \quad S_3 = \{A_i \big| i \in T_3\} \tag{8.3.42}$$

and $|S_3| = s$. Then considering residue classes modulo $2^\delta\rho$, we derive that

$$\frac{2^\delta \rho k}{3c \cdot 2^{\omega(d)}} \geq \max_{A_i \in S_3} A_i \geq 2^\delta \rho(s-1) + 1$$

so that $|S_3| = s \le \frac{k}{3c2^{\omega(d)}} - \frac{1}{\rho} + 1 \le \frac{k}{3c2^{\omega(d)}} + \frac{2}{3}$. We see from Corollary 8.3.3, (8.3.40), (8.3.41) and (8.3.42) that

$$\begin{aligned}|T_3| &\le \frac{k}{20\cdot 2^{\omega(d)}}2^{\omega(d)} + \left(\frac{k}{6\cdot 2^{\omega(d)}} - \frac{k}{20\cdot 2^{\omega(d)}} + \frac{2}{3}\right)2^{\omega(d)-1}\\ &\le \frac{k}{20} + \left(\frac{k}{6} - \frac{k}{20}\right)2^{-1} + \frac{2}{3}2^{\omega(d)-1} \le \frac{k}{12} + \frac{k}{40} + \frac{k}{6\times 2^6} \le \frac{k}{9}\end{aligned}$$

if $\omega(d) \ge 6$ and

$$|T_3| \le \begin{cases} (\frac{k}{48\cdot 2^{\omega(d)}} + \frac{2}{3})2^{\omega(d)} = \frac{k}{48} + \frac{8}{3} & \text{if } \omega(d) = 2\\ (\frac{k}{12\cdot 2^{\omega(d)}} + \frac{2}{3})2^{\omega(d)} = \frac{k}{12} + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3,4,5.\end{cases}$$

Therefore

$$|T_2| > C_1 - |T_3| \ge C_4 := \begin{cases} \frac{k}{C_2} + C_3 & \text{if } \omega(d) = 2,3,4,5\\ \frac{k}{C_2} + \frac{k}{12} & \text{if } \omega(d) \ge 6.\end{cases}$$

Let $\mathfrak{C}$, $\mathfrak{C}_\mu$ be as in Lemma 8.3.10 with $c = 16$ if $\omega(d) = 2$, $c = 4$ if $\omega(d) = 3,4,5$ and $c = 2$ if $\omega(d) \ge 6$. Then $C_4 < |T_2| = |S_2| + \sum_{\mu\ge 2}(\mu-1)|\mathfrak{C}_\mu|$. Now we apply Lemma 8.3.10 and use $k \ge \kappa_0 \ge \eta(2^{\omega(d)-2})(\log 2^{\omega(d)-2})2^{\omega(d)}(2^{\omega(d)-\theta} - 1)$ for $\omega(d) \ge 6$ to get

$$C_4 < \begin{cases} |S_2| + C_3 & \text{if } 2 \le \omega(d) \le 5\\ |S_2| + \frac{k}{12} & \text{if } \omega(d) \ge 6.\end{cases}$$

Thus

$$|S_2| > \frac{k}{C_2}.$$

Let $\omega(d) = 2$. Then considering modulo 2^δ, we see that

$$\max_{A_i\in S_2} A_i \ge 2^\delta[\frac{k}{C_2}] + \frac{2^\delta k}{48\times 4} \ge 2^\delta\frac{k}{C_2}$$

giving (8.3.39). Now we take $\omega(d) \ge 3$. Since $d < 4c_1(k-1)^2$, we have $r \ge [\frac{\omega(d)}{3}]$ by Lemma 8.3.5. By (8.2.5), we have $\frac{k}{C_2} \ge \frac{k^{\frac{2}{3}}}{2} > \frac{1}{2^r}(4c_1(k-1)^2))^{\frac{1}{3}} > \prod_{j=1}^{r}\left(\frac{P_j - 1}{2}\right)$. We now apply Corollary 8.3.7 with $s = [\frac{k}{C_2} + 1]$ and $U = 1$ to get

$$\max_{A_i\in S_2} A_i \ge \frac{3}{4}2^{r+\delta}[\frac{k}{C_2} + 1] \ge \frac{3}{4}2^{[\frac{\omega(d)}{3}]+\delta}\frac{k}{C_2}$$

giving (8.3.39). □

8.4. Proof of Proposition 8.1.1

We assume that either $n \ge c_1(k-1)^3$ or $d \ge 4c_1(k-1)^2$. Then $\nu(a_{i_0}) \le 2^{\omega(d)-\theta}$ for $0 \le i_0 < k$ by Lemma 8.3.4. Let ϱ be as defined in the statement of Lemma 8.3.13. Then $\nu(a_{i_0}) \le \varrho$. By Lemma 8.3.13, there are at least $z\varrho(2^{\omega(d)} - 1)$ distinct pairs (i,j) with $i > j$ and $a_i = a_j$, where $z = 4$ if d is odd and 2 if d is even. Since there can be at most $2^{\omega(d)-\theta} - 1$ possible partitions of d, by Box principle, there exists a partition (d_1, d_2) of d and at least $z\varrho$ pairs of (i,j) with $a_i = a_j, i > j$ corresponding to this partition. We write

$$x_i - x_j = d_1 r_1(i,j) \quad\text{and}\quad x_i + x_j = d_2 r_2(i,j).$$

Let d be odd. Suppose there are at least ϱ distinct pairs $(i_1, j_1), \cdots, (i_\varrho, j_\varrho), \cdots$ with the corresponding $r_1(i,j)$ even. Then $|\{i_1, \cdots, i_\varrho, j_1, \cdots, j_\varrho\}| > \varrho$. Hence we can find $1 \le l, m \le \varrho$ with $(i_l, j_l) \ne (i_m, j_m)$, $a_{i_l} = a_{j_l}, a_{i_m} = a_{i_m}$ and $a_{i_l} \ne a_{i_m}$ from amongst the pairs. Now the result follows by Lemma 8.3.11. Thus we may assume that there are at most $\varrho - 1$ pairs of (i,j) with even $r_1(i,j)$. Then there are at least $3\varrho + 1$ distinct pairs of (i,j) with $r_1(i,j)$ odd. Since $a_i \equiv 1, 2, 3($ mod 4), we can find at least ϱ pairs with $a_i \equiv a_g($ mod 4) for any two such pairs $(i,j), (g,h)$. Then there exists two distinct pairs $(i,j), (g,h)$ with $a_i = a_j, a_g = a_h$ and $a_i \ne a_g$ from these pairs. Also $r_1(i,j) \equiv r_1(g,h)($ mod 2). This gives (8.1.1) and (8.1.2) by Lemma 8.3.11 which is a contradiction.

Let d be even. We observe that $8|(x_i^2 - x_j^2)$ and $\gcd(x_i - x_j, x_i + x_j) = 2$. We claim that there are at least ϱ pairs with $r_1(i,j) \equiv r_1(g,h)(\text{mod } 2)$ and $r_2(i,j) \equiv r_2(g,h)($ mod 2) for any two such distinct pairs (i,j) and (g,h). If the claim is true, then there are two pairs $(i,j) \ne (g,h)$ with $i > j, g > h, a_i = a_j, a_g = a_h$ and $a_i \ne a_g$ from amongst such pairs since $\nu(a_i) \le \varrho$. This implies (8.1.1) and (8.1.2) by Lemma 8.3.11, contradicting our assumption. Let $\mathrm{ord}_2(d) = 1$. Then d_1 is odd implying $r_1(i,j)$ is even. We choose at least ϱ pairs whose r_2's of the same parity. Thus the claim is true in this case. Let $\mathrm{ord}_2(d) \ge 3$. Then we have either $\mathrm{ord}_2(d_1) = 1$ implying all r_1's are odd, or $\mathrm{ord}_2(d_2) = 1$ implying all r_2's are odd. Thus the claim follows. Finally let $\mathrm{ord}_2(d) = 2$. Then $2 \| d_1$ and $2 \| d_2$ so that r_1 and r_2 are of the opposite parity for any pair and hence the claim holds. □

8.5. Proof of Proposition 8.1.2

In this section, we assume that $k \ge \kappa_0$. In view of Proposition 8.1.1, we may assume that $d < 4c_1(k-1)^2$. We may also assume that X_i is a prime for each $i \in T_1$ in the proof of Proposition 8.1.2. Otherwise $n + (k-1)d \ge (k+1)^4$ implying the assertion.

We see that d has at least one prime divisor $\le k$ otherwise $d > k^{\omega(d)} \ge k^2 > 4c_1(k-1)^2$, a contradiction. Thus $\pi_d(k) \le \pi(k) - 1$. Let $n + (k-1)d \ge L$ for some $L > 0$. By Lemma 8.3.1 and Lemma 2.0.2 (i), we have

$$|T_1| > k - \frac{(k-1)\log(k-1)}{\log L - \log 2} - \frac{k}{\log k}\left(1 + \frac{1.5}{\log k}\right). \tag{8.5.1}$$

We see from Theorem 3.3.9 that $n(n+d)\cdots(n+(k-1)d)$ is divisible by at least $\pi(2k) - \pi_d(k) \ge \pi(2k) - \pi(k) + 1$ primes exceeding k. Hence we have $n + (k-1)d \ge 4k^2$. Thus by taking $L = 4k^2$ in (8.5.1), we get

$$|T_1| > k - \frac{(k-1)\log(k-1)}{\log(2k^2)} - \frac{k}{\log k}\left(1 + \frac{1.5}{\log k}\right).$$

The right hand side of the above inequality is an increasing function of k and

$$|T_1| > \begin{cases} \frac{k}{5} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2 \\ \frac{k}{6} + \frac{k}{12} + C_3 + \frac{16}{3} & \text{if } \omega(d) = 3 \\ \frac{5}{24}k + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 4, 5 \\ \frac{5}{48}k + \frac{k}{12} + \frac{k}{9} & \text{if } \omega(d) \ge 6. \end{cases} \tag{8.5.2}$$

Now we see from Lemma 8.3.14 that (8.3.39) holds with

$$C_2 = \begin{cases} 5 & \text{if } \omega(d) = 2 \\ 6 & \text{if } \omega(d) = 3 \\ \frac{24}{5} & \text{if } \omega(d) = 4, 5 \\ \frac{48}{5} & \text{if } \omega(d) \ge 6. \end{cases}$$

This gives $n+(k-1)d \geq \frac{C_0}{C_2}k^3$. Hence (8.1.4) is valid for $\omega(d) \geq 4$. Now we take $\omega(d) = 2, 3$. Putting $L = \frac{C_0}{5}k^3$ in (8.5.1), we derive that

$$|T_1| > \begin{cases} \frac{5k}{16} + \frac{k}{48} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 2 \\ \frac{5k}{24} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3. \end{cases}$$

We apply Lemma 8.3.14 again to get $\max_{i \in T_1} A_i \geq 2^{\delta}\frac{5}{16}k$ so that $n+(k-1)d \geq 2^{\delta}\frac{5}{16}k^3$ implying (8.1.4). This completes the proof. $\square$

CHAPTER 9

Cubes and higher powers in arithmetic progression, a survey

9.1. Introduction

We end the thesis with a survey on cubes and higher powers in arithmetic progression. We consider the equation

$$\Delta = \Delta(n, d, k) = n(n+d)\cdots(n+(k-1)d) = by^{\ell} \tag{9.1.1}$$

in positive integers n, d, k, b, y and ℓ with $d \geq 1, k \geq 2, \ell \geq 3, P(b) \leq k$, $\gcd(n, d) = 1$ and b is $\ell-$th power free. We have already considered (9.1.1) with $\ell = 2$ in Chapter 7. Therefore it suffices to consider (9.1.1) when ℓ is divisible by an odd prime. Except for Section 9.4, we shall always suppose that ℓ is divisible by an odd prime. Further we always suppose that $k \geq 3$ otherwise (9.1.1) has infinitely many solutions. Let $d = 1$. We also assume that $P(\Delta) > k$ which is necessary as explained in Chapter 7. Then (9.1.1) has been completely solved by Erdős and Selfridge [**11**] for $P(b) < k$. Saradha [**40**] extended this result for $P(b) = k$ with $k \geq 4$ and Győry [**15**] completed the result for $P(b) = k$ with $k = 3$.

From now on we assume (9.1.1) with $d > 1$. Then we always suppose that $(n, d, k) \neq (2, 7, 3)$ so that $P(\Delta) > k$ by (3.3.3). Thus $P(\Delta) > k$ is not an assumption in the case $d > 1$. Erdős conjectured that k is bounded by a computable absolute constant whenever (7.1.1) or (9.1.1) holds. We shall call this Erdős conjecture. Marszalek [**26**] showed that

$$k \leq \begin{cases} \max\ (c_1, \frac{3}{2}\ \exp\ (\frac{1}{2}d(d+2)(d+1)^{1/3})) & \text{if } \ell = 3 \\ \max\ (c_1, \frac{1}{4}d(d+2)(d+1)^{1/2}) & \text{if } \ell = 4 \\ \max\ (c_1, \frac{3}{2}(d+1)) & \text{if } \ell \geq 5 \end{cases}$$

where $c_1 = 3 \cdot 10^4$. Thus when d is fixed, the result of Marszalek confirms Erdős conjecture. Shorey [**54**] showed that k is bounded by an effectively computable number depending only on $P(d)$. Further Shorey and Tijdeman [**53**] proved that k is bounded by an effectively computable number depending only on ℓ and $\omega(d)$. They improved Marszalek's result as $d \geq k^{c_1 \log\log k}$ where c_1 is an effectively computable constant. We state here the Oesterlé and Masser's *abc*-conjecture.

Conjecture 9.1.1. **Oesterlé and Masser's abc-conjecture:** *For any given $\epsilon > 0$ there exists a computable constant κ_ϵ depending only on ϵ such that if*

$$a + b = c$$

where a, b and c are coprime positive integers, then

$$c \leq \kappa_\epsilon \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

It has been shown in Elkies [**7**] and Granville and Tucker [**13**, (13)] that *abc*-conjecture is equivalent to the following:

CONJECTURE 9.1.2. *Let $F(x, y) \in \mathbb{Z}[x, y]$ be a homogenous polynomial. Assume that F has pairwise non-proportional linear factors in its factorisation over $\mathbb{C}$. Given $\epsilon > 0$, there exists a computable constant κ'_ϵ depending only on F and ϵ such that if m and n are coprime integers, then*

$$\prod_{p|F(m,n)} p \geq \kappa'_\epsilon \left(\max\{|m|, |n|\}\right)^{\deg(F)-2-\epsilon}.$$

Shorey [**54**] showed that *abc*-conjecture implies Erdős conjecture for $\ell \geq 4$ using $d \geq k^{c_1 \log\log k}$. Granville (unpublished) gave a proof of the preceding result without using the inequality $d \geq k^{c_1 \log\log k}$. Furthermore his proof is also valid for $\ell = 2, 3$. I give his proof in Section 9.4. I thank Professor A. Granville for allowing me to include his proof in the thesis. A stronger conjecture states that

CONJECTURE 9.1.3. *Equation* (9.1.1) *implies that* $(k, \ell) = (3, 3)$.

On the other hand, it is known that (9.1.1) has infinitely many solutions if $(k, \ell) = (3, 3)$, see Tijdeman [**57**] and Mordell [**28**, p.68]. We give the details of this fact in Section 9.2. Saradha [**40**] showed that (9.1.1) does not hold for $d \leq 6, d \neq 5$ and for $k \geq 4$ when $d = 5$. Saradha and Shorey [**42**] extended this result for an infinite set of values of d of the form $2^a 3^b 5^c > 1$ whenever ℓ is an odd prime. Further they proved in [**44**] that (9.1.1) implies that $d > D$ for $\ell \geq 3$ where D is given by

$$D = \begin{cases} 30 & \text{if } \ell = 3 \\ 950 & \text{if } \ell = 4 \\ 5 \cdot 10^4 & \text{if } \ell = 5, 6 \\ 10^8 & \text{if } \ell = 7, 8, 9, 10 \\ 10^{15} & \text{if } \ell \geq 11. \end{cases}$$

The above result confirms Conjecture 9.1.3 for a large number of values of d.

9.2. The case $(k, \ell) = (3, 3)$

We show that

Equation (9.1.1) *with* $(k, \ell) = (3, 3)$ *implies that* $b = 3, 6, 36$
in which cases there are infinitely many solutions.

We consider

$$n(n + d)(n + 2d) = by^3 \tag{9.2.1}$$

where $b \in \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Then

$$\left(\frac{2by}{n}\right)^3 = 4b\left(1 + \frac{d}{n}\right) 2b\left(1 + \frac{2d}{n}\right) = \left(3b + \frac{4bd}{n} + b\right)\left(3b + \frac{4bd}{n} - b\right).$$

Putting $X = \frac{2by}{n}$ and $Y = 3b + \frac{4bd}{n}$, we obtain the elliptic equation

$$Y^2 = X^3 + b^2 \ \text{ in } \ X, Y \in \mathbb{Q}. \tag{9.2.2}$$

We check using *MAGMA* that each of the above elliptic curve has rank 0 except when $b = 3, 6, 36$ where rank is 1. Thus the elliptic equation (9.2.2) has infinitely many solutions when $b = 3, 6, 36$. Let $b \neq 3, 6, 36$. Then the torsion points are given by $(0, 1), (0, -1), (-1, 0), (2, 3), (2, -3)$ for $b = 1$ and $(0, b), (0, -b)$ for $b \neq 1$. The torsion points (X, Y) with $X = 0$ implies that $y = 0$ which is not possible. Also $Y \leq 0$ is not possible since $3n + 4d > 0$. Thus it remains to consider $b = 1, (X, Y) = (2, 3)$. Then $3 = 3 + \frac{4d}{n}$ giving $d = 0$, a contradiction.

Let $b = 3, 6, 36$. Suppose $(X, Y) = (X_0, Y_0)$ with $Y_0 > 0$ be a solution of (9.2.2). Putting $X = x + X_0$, we have from (9.2.2) that

$$Y^2 = x^3 + 3X_0x^2 + 3X_0^2x + Y_0^2$$

since $Y_0^2 = X_0^3 + b^2$. Here $(x, Y) = (0, Y_0)$ is a solution. We now make the substitution $Y = \alpha x + Y_0$ where $x \neq 0$ and α is to be chosen. Then the above equation becomes

$$x^2 + (3X_0 - \alpha^2)x + (3X_0^2 - 2\alpha Y_0) = 0.$$

We take $\alpha = \frac{3X_0^2}{2Y_0}$. Then

$$x + 3X_0 = \alpha^2 = \frac{9X_0^4}{4Y_0^2}.$$

Thus

$$X = x + X_0 = \frac{9X_0^4}{4Y_0^2} - 2X_0, \quad Y = \frac{3X_0^2}{2Y_0}(\frac{9X_0^4}{4Y_0^2} - 3X_0) + Y_0$$

satisfies (9.2.2). We consider the case $Y > 3b$. We choose n and d to be the denominator and numerator of $\frac{(Y-3b)}{4b}$, respectively. The case $Y < 3b$ follows similarly by considering the mirror image $N(N-d)(N-2d) = by^3$ with $N = n + 2d$ of $n(n+d)(n+2d)$. Note that $Y = 3b$ is not possible otherwise $d = 0$.

Let $b = 3$. Since $(X_0, Y_0) = (-2, 1)$ is a rational point on (9.2.2), we derive that $(X, Y) = (40, 253)$ is a new solution of (9.2.2). This gives $(n, d, y) = (3, 61, 20)$ as a solution of (9.2.1). The solution $(40, 253)$ in turn gives $(\frac{639280}{64009}, \frac{513439919}{16194277})$ as a solution of (9.2.1). This gives $(n, d, y) = (48582831, 116214272, 80868920)$ as another solution of (9.2.1). The process is continued indefinitely.

For $b = 6, 36$, we start with $(X_0, Y_0) = (-3, 3)$ and $(-8, 28)$, respectively and we apply the above process to get infinitely many solutions of (9.2.1).

9.3. $k < 500000$ when (9.1.1) with $d = 1$ and $P(\Delta(n,k) > k$ holds

Let $k \geq 500000$. From (9.1.1), we have

$$n + i = a_i x_i^\ell \ \text{ for } \ 0 \leq i < k$$

where a_i is ℓ-th power free and $P(a_i) \leq k$. Then $n + k - 1 \geq (k+1)^\ell$ implying $n > k^\ell$. First we prove that the products $a_ia_j, 0 \leq i, j < k$ are all distinct. Let $a_ia_j = a_ga_h = A$ with $i + j > g + h$. Then

$$(n+i)(n+j) = A(x_ix_j)^\ell, \quad (n+g)(n+h) = A(x_gx_h)^\ell.$$

If $(n+i)(n+j) = (n+g)(n+h)$, then

$$n \leq n(i+j-g-h) = gh - ij < k^2$$

a contradiction. Thus $(n+i)(n+j) \neq (n+g)(n+h)$. Then $(n+i)(n+j) - (n+g)(n+h) = (i+j-g-h)n - (gh-ij) > 0$ since $n > k^\ell > k^2$ and $gh - ij < k^2$. Hence $x_ix_j \geq x_gx_h + 1$ giving

$$\begin{aligned} 2kn &> (n+k-1)^2 - n^2 \geq (n+i)(n+j) - (n+g)(n+h) \geq A((x_gx_h+1)^\ell - (x_gx_h)^\ell) \\ &> \ell A(x_gx_h)^{\ell-1} \geq \ell\left(A(x_gx_h)^\ell\right)^{\frac{\ell-1}{\ell}} \geq \ell(n^2)^{\frac{\ell-1}{\ell}} \geq 3n^{\frac{4}{3}}, \end{aligned}$$

a contradiction since $n > k^\ell$. Thus a_ia_j are all distinct. We now prove a graph theoretic lemma due to Erdős and Selfridge [**11**] which is applied to get a lower bound for a_i's.

9.3.1. A graph theoretic lemma. A *Bipartite Graph* is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

A subgraph of a graph is a *Rectangle* if it is comprised of two parts of vertices with each member of one pair joined to each member of the other pair.

LEMMA 9.3.1. *Let G be a bipartite graph of s white and t black vertices which contain no rectangles. Then the number of edges of G is at most $s + \binom{t}{2}$.*

PROOF. Let s_i be the number of white vertices having i edges, $i \geq 0$. Then $\sum_{i \geq 0} s_i = s$.

Since there are no rectangles, any two black vertices cannot connect more than one white vertex. Hence the number of $V-$diagrams is at most $\binom{t}{2}$. Further from a white vertex of valency i, we can have $\binom{i}{2}$ number of $V-$diagrams. Therefore

$$\text{The total number of } V - \text{diagrams} = \sum_{i \geq 2} s_i \binom{i}{2}.$$

Hence $\sum_{i \geq 2} s_i \binom{i}{2} \leq \binom{t}{2}$. Therefore the number of edges of G is

$$\sum_i is_i = \sum_{i \geq 2}(i-1)s_i + \sum_i s_i \leq \sum_{i \geq 2} s_i \binom{i}{2} + s \leq \binom{t}{2} + s.$$

This proves the Lemma. □

Given x, let $N(x)$ denote the maximum number of integers $1 \leq b_1 < b_2 < \cdots < b_s \leq x$ such that $b_i b_j, 1 \leq i, j \leq s$ are all different. We prove that

LEMMA 9.3.2. *For any real number $x > 1$, we have*

$$N(x) \leq \frac{270}{961}x + 1832. \tag{9.3.1}$$

In fact Erdős [**10**] proved a stronger result when x is sufficiently large. He proved

$$N(x) < \pi(x) + 3x^{\frac{7}{8}} + 2x^{\frac{1}{2}} < \frac{2x}{\log x}$$

whenever $x \geq x_0$ where x_0 is a computable absolute constant.

PROOF. **of Lemma 9.3.2:** Let $U = \{2^a 3^b 5^c | 0 \leq a \leq 4, 0 \leq b \leq 3, 0 \leq c \leq 2\}$. Thus $|U| = 60$. We take V to be the set of all integer $v \leq x$ such that every integer $n \leq x$ can be written as $n = uv$ with $u \in U, v \in V$. We observe that $v \in V$ is of the form $2^{5r}3^{4s}5^{3t}m$ in $r \geq 0, s \geq 0, t \geq 0$, $m \geq 1$ with $\gcd(m, 30) = 1$. Thus

$$\begin{aligned}|V| &\leq \left(x(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})+1\right)\left(1+\frac{1}{2^5}+\frac{1}{2^{10}}+\cdots\right)\left(1+\frac{1}{3^4}+\cdots\right)\left(1+\frac{1}{5^3}+\cdots\right)\\ &\leq \frac{270}{961}x + 2.\end{aligned}$$

We now take (U, V) to be bipartite graph G with black vertices as elements of U and white vertices as elements of V. Let $\{b_1, \cdots, b_{N(x)}\}$ be the set of positive integers $\leq x$ with the property that $b_i b_j$ for $1 \leq i, j \leq N(x)$ are all distinct. We say that there is an edge between an element $u \in U$ and $v \in V$ if $uv = b_i$ for some i. Then the distinctness of $b_i b_j$'s imply that G has no rectangles. Thus by Lemma 9.3.1, we see that $N(x) \leq |V| + \binom{|U|}{2}$. Hence $N(x) \leq \frac{270}{960}x + 2 + \binom{60}{2} \leq \frac{270}{960}x + 1832$. □

9.3.2. Proof of $k < 500000$ (continued). By using (5.1.3), we can find a sequence $0 \le i_1 < i_2 < \cdots < i_t$ with $t \ge k - \pi(k)$ such that

$$\prod_{j=1}^{t} a_{i_j} \le (k-1)! \tag{9.3.2}$$

By arranging these $a_{i_j}, 1 \le j \le t$ in increasing order, we get a sequence $b_1 < b_2 < \cdots < b_t$ such that $b_i b_j$'s are distinct. We put $b_i = x$ and use Lemma 9.3.2 to get

$$i \le \frac{270}{961} b_i + 1832$$

giving $b_i > 3.559(i - 1832)$. Then we have

$$\prod_{j=1}^{t} a_{i_j} = \prod_{i=1833}^{t} b_i > \prod_{i=69}^{t} 3.559(i-1832) > (3.559)^{t-1832}(t-1832)!.$$

Since $t \ge k - \pi(k)$ and $\pi(k) \le \frac{k}{\log k}(1 + \frac{1.2762}{\log k})$ by Lemma 2.0.2 (i), we have

$$(3.559)^{t-1832}(t-1832)! \ge k\frac{(3.559)^k}{(3.559k)^{1832+\pi(k)}} = k!\left(3.559(3.559k)^{-\frac{1832}{k} - \frac{1}{\log k}(1+\frac{1.2762}{\log k})}\right)^k > k!$$

since $3.559(3.559k)^{-\frac{1832}{k} - \frac{1}{\log k}(1+\frac{1.2762}{\log k})}$ is an increasing function of k and is > 1 at $k = 500000$. Thus

$$\prod_{j=1}^{t} a_{i_j} > k!$$

contradicting (9.3.2). $\square$

9.4. abc-conjecture implies Erdős conjecture

Assume (7.1.1) and (9.1.1). We show that k is bounded by a computable absolute constant. Let $k \ge k_0$ where k_0 is a sufficiently large computable absolute constant. Let $\epsilon > 0$. Let $c_1, c_2, \cdots$ be positive computable constants depending only on ϵ. From (7.1.1) and (9.1.1), we write

$$n + id = A_i X_i^{\ell}$$

with $P(A_i) \le k$ and $(X_i, \prod_{p \le k} p) = 1$ for $0 \le i < k$. We may assume that $(n, d, k) \ne (2, 7, 3)$. Then $P(\Delta(n, d, k)) > k$ by (3.3.3). Thus

$$n + (k-1)d > k^{\ell}.$$

For each $p \le k$ with $p \nmid d$, let $n + i_p d$ be the term to which p divides to the maximal power and we put

$$I = \{i_p | p \le k \text{ and } p \nmid d\}.$$

Let $\Phi = \prod\limits_{\substack{i \ge [\frac{k}{2}] \\ i \notin I}} A_i$. Now we refer to Section 5.1. Taking $\mathcal{S} = \{A_i | i \in I \text{ or } [\frac{k}{2}] \le i < k\}$, we get from the first inequality of (5.1.2) that

$$\mathrm{ord}_p(\Phi) \le \mathrm{ord}_p\left(\prod_{\substack{i \ge [\frac{k}{2}] \\ i \notin I}} (i - i_p)\right) \le \begin{cases} \mathrm{ord}_p\left((k - [\frac{k}{2}] - 1 - (i_p - [\frac{k}{2}]))!(i_p - [\frac{k}{2}])!\right) & \text{if } i_p \ge [\frac{k}{2}], \\ \mathrm{ord}_p\left(\binom{k-1-i_p}{k-[\frac{k}{2}]}(k - [\frac{k}{2}])!\right) & \text{otherwise.} \end{cases}$$

Since $\text{ord}_p(r!s!) \le \text{ord}_p((r+s)!)$ and $k - [\frac{k}{2}] = [\frac{k+1}{2}]$, we see that

$$p^{\text{ord}_p(\Phi)} \le p^{\text{ord}_p(\binom{k-1-i_p}{[\frac{k+1}{2}]})} p^{\text{ord}_p([\frac{k+1}{2}]!)} \le (k-1)p^{\text{ord}_p([\frac{k+1}{2}]!)}.$$

The latter inequality follows from Lemma 1.1.3. Therefore we get

$$\Phi \le (k-1)^{\pi_d(k)}([\frac{k+1}{2}])! \le k^{\frac{k}{2}} e^{c_1 k}$$

by using Lemmas 2.0.2 and 2.0.6.

Let D be a fixed positive integer and let

$$J = \left\{ \frac{k-1}{2D} \le j \le \frac{k-1}{D} - 1 : \{Dj+1, Dj+2, \cdots, Dj+D\} \cap I = \phi \right\}.$$

We shall choose $D = 20$. Let $j, j' \in J$ be such that $j \ne j'$. Then $Dj + i \ne Dj' + i'$ for $1 \le i, i' \le D$ otherwise $D(j - j') = (i - i')$ and $|i' - i| < D$. Further we also see that $[\frac{k}{2}] \le Dj + i \le k - 1$ for $1 \le i \le D$ and consequently $|J| \ge \frac{k-1}{2D} - \pi(k)$. For each $j \in J$, let $\Phi_j = \prod_{i=1}^{D} A_{Dj+i}$. Then $\prod_{j \in J} \Phi_j$ divides Φ implying

$$\prod_{j \in J} \Phi_j \le \Phi \le k^{\frac{k}{2}} e^{c_1 k}.$$

Thus there exists $j_0 \in J$ such that

$$\Phi_{j_0} \le \left(k^{\frac{k}{2}} e^{c_1 k})\right)^{\frac{1}{|J|}} \le \left(k^{\frac{k}{2}} e^{c_1 k}\right)^{\frac{1}{\frac{k-1}{2D} - \pi(k)}} \le c_2^D k^D.$$

Let

$$H := \prod_{i=1}^{D} (n + (Dj_0 + i)d).$$

Since $A_{Dj_0+i} X_{Dj_0+i}^{\ell} \le n + (k-1)d$, we have $X_{Dj_0+i} \le (\frac{n+(k-1)d}{A_{Dj_0+i}})^{\frac{1}{\ell}}$. Thus

$$\prod_{\substack{p|H \\ p>k}} p = \prod_{i=1}^{D} X_{Dj_0+i} \le (n + (k-1)d)^{\frac{D}{\ell}} (\Phi_{j_0})^{-\frac{1}{\ell}}$$

Therefore

$$\prod_{p|H} p = \left(\prod_{\substack{p|H \\ p \le k}} p \right) \left(\prod_{\substack{p|H \\ p>k}} p \right) \le \Phi_{j_0} (n + (k-1)d)^{\frac{D}{\ell}} (\Phi_{j_0})^{-\frac{1}{\ell}} \le c_2^{D(1-\frac{1}{\ell})} k^{D(1-\frac{1}{\ell})} (n + (k-1)d)^{\frac{D}{\ell}}.$$

On the other hand, we have $H = F(n + Dj_0 d, d)$ where

$$F(x, y) = \prod_{i=1}^{D} (x + iy)$$

is a binary form in x and y of degree D such that F has distinct linear factors. From Conjecture 9.1.2, we have

$$\prod_{p|H} p \ge c_3 (n + Dj_0 d)^{D-2-\epsilon}.$$

Comparing the lower and upper bounds of $\prod_{p|H} p$ and using $n + Dj_0d > \frac{n+(k-1)d}{2}$, we get

$$k > c_4(n + (k-1)d)^{1-\frac{2+\epsilon}{D(1-\frac{1}{\ell})}}.$$

We now use $n + (k-1)d > k^\ell$ to derive that

$$c_5 > k^{\ell(1-\frac{2+\epsilon}{D(1-\frac{1}{\ell})})-1}.$$

Taking $\epsilon = \frac{1}{2}$ and putting $D = 20$, we get

$$c_6 > k^{\ell-1-\frac{\ell^2}{8(\ell-1)}} \geq k^{\frac{1}{2}}$$

since $\ell \geq 2$. This is a contradiction since $k \geq k_0$ and k_0 is sufficiently large. $\square$

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