THE GREATEST PRIME DIVISOR OF A PRODUCT OF CONSECUTIVE INTEGERS

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1. Introduction

Let $k \geq 2$ and $n \geq 1$ be integers. We denote by

$$\Delta(n,k) = n(n+1)\cdots(n+k-1).$$

For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively, and we put $\omega(1) = 0$, $P(1) = 1$.

A well known theorem of Sylvester [7] states that

$$P(\Delta(n,k)) > k \quad \text{if} \quad n > k. \tag{1}$$

We observe that $P(\Delta(1,k)) \leq k$ and therefore, the assumption $n > k$ in (1) cannot be removed. For $n > k$, Moser [5] sharpened (1) to $P(\Delta(n,k)) > \frac{11}{10}k$ and Hanson [3] to $P(\Delta(n,k)) > 1.5k$ unless $(n,k) = (3,2),(8,2),(6,5)$. Further Faulkner [2] proved that $P(\Delta(n,k)) > 2k$ if n is greater than or equal to the least prime exceeding $2k$ and $(n,k) \neq (8,2),(8,3)$. In this paper, we sharpen the results of Hanson and Faulkner. We shall not use these results in the proofs of our improvements. We prove

Theorem 1. *We have*
(a)

$$P(\Delta(n,k)) > 2k \text{ for } n > \max(k+13, \frac{279}{262}k). \tag{2}$$

(b)

$$P(\Delta(n,k)) > 1.97k \text{ for } n > k+13. \tag{3}$$

We observe that 1.97 in (3) cannot be replaced by 2 since there are arbitrary long chains of consecutive composite positive integers. The same reason implies that Theorem 1 **(a)** is not valid under the assumption $n > k+13$. Further the assumption $n > \frac{279}{262}k$ in Theorem 1 **(a)** is necessary since $P(\Delta(279,262)) \leq 2 \times 262$.

2000 Mathematics Subject Classification: Primary 11A41, 11N05, 11N13.
Keywords: Arithmetic Progressions, Primes.

Now we give a lower bound for $P(\Delta(n,k)) > 2k$ which is valid for $n > k > 2$ except for an explicitly given finite set. For this, we need some notation. For a pair (n,k) and a positive integer h, we write $[n,k,h]$ for the set of all pairs $(n,k), \cdots (n+h-1,k)$ and we set $[n,k] = [n,k,1] = \{(n,k)\}$. Let

$A_{10} = \{58\}$; $A_8 = A_{10} \cup \{59\}$; $A_6 = A_8 \cup \{60\}$;

$A_4 = A_6 \cup \{12, 16, 46, 61, 72, 93, 103, 109, 151, 163\}$;

$A_2 = A_4 \cup \{4, 7, 10, 13, 17, 19, 25, 28, 32, 38, 43, 47, 62, 73, 94, 104, 110, 124, 152, 164, 269\}$

and $A_{2i+1} = A_{2i}$ for $1 \le i \le 5$. Further let

$$A_1 = A_2 \cup \{3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 23, 26, 29, 33, 35, 39, 41, 44, 48, 50, 53,$$
$$56, 63, 68, 74, 78, 81, 86, 89, 95, 105, 111, 125, 146, 153, 165, 173, 270\}.$$

Finally we denote B as the union of the sets $[8,3], [5,4,3], [14,13,3]$ and $\{(k+1,k) | k = 3, 5, 8, 11, 14, 18, 63\}$. Then

Theorem 2. *We have*

$$P(\Delta(n,k)) > 1.95k \text{ for } n > k > 2 \tag{4}$$

unless and only unless $(n,k) \in [k+1,k,h]$ *for* $k \in A_h$ *with* $1 \le h \le 11$ *or* $(n,k) = (8,3)$.

If $k = 2$, we observe (see Lemma 7) that $P(\Delta(n,k)) > 2k$ unless $n = 3, 8$ and that $P(\Delta(3,2)) = P(\Delta(8,2)) = 3$. Thus the estimate (4) is valid for $k = 2$ whenever $n \ne 3, 8$. We observe that $P(\Delta(k+1,k)) \le 2k$ and therefore, 1.95 in (4) cannot be replaced by 2. There are few exceptions if 1.95 is replaced by 1.8 in Theorem 2. We derive from Theorem 2 the following result.

Corollary 1. *We have*

$$P(\Delta(n,k)) > 1.8k \quad \text{for } n > k > 2 \tag{5}$$

unless and only unless $(n,k) \in B$.

2. Lemmas

We begin with a well known result due to Levi ben Gerson on a particular case of Catalan equation.

Lemma 1. *The solutions of*

$$2^a - 3^b = \pm 1 \quad \text{in integers } a > 0, b > 0$$

are given by $(a,b) = (1,1), (2,1), (3,2)$.

Next we state a result of Saradha and Shorey [6] on a lower bound for $\omega(\Delta(n,k))$.

Lemma 2. *For $n > k > 2$, we have*

$$\omega(\Delta(n,k)) \geq \pi(k) + \left[\frac{1}{3}\pi(k)\right] + 2$$

unless and only unless (n,k) belongs to the union of sets

$$\begin{cases} [4,3],[6,3,3],[16,3],[6,4],[6,5,4],[12,5],[14,5,3],[23,5,2],\\ [7,6,2],[15,6],[8,7,3],[12,7],[14,7,2],[24,7],[9,8],[14,8],\\ [14,13,3],[18,13],[20,13,2],[24,13],[15,14],[20,14],[20,17]. \end{cases}$$

We shall use Lemma 2 only when $k = 3$ or $5 \leq k \leq 8$. Let p_i denote the $i-$th prime number. Then

Lemma 3. *We have*

(6)

$$p_{i+1} - p_i < \begin{cases} 35 & \text{for } p_i \leq 5591 \\ 15 & \text{for } p_i \leq 1123, p_i \neq 523, 887, 1069 \\ 21 & \text{for } p_i = 523, 887, 1069 \\ 9 & \text{for } p_i \leq 361, p_i \neq 113, 139, 181, 199, 211, 241, 283, 293, 317, 337. \end{cases}$$

Lemma 4. *Let $\mathfrak{N}$ be a positive real number and k_0 a positive integer. Let $I(\mathfrak{N}, k_0) = \{i | p_{i+1} - p_i \geq k_0, p_i \leq \mathfrak{N}\}$. Then*

$$P(n(n+1)\cdots(n+k-1)) > 2k$$

for $2k \leq n < \mathfrak{N}$ and $k \geq k_0$ except possibly when $p_i < n < n+k-1 < p_{i+1}$ for $i \in I(\mathfrak{N}, k_0)$.

Proof. Let $2k \leq n < \mathfrak{N}$ and $k > k_0$. We may suppose that none of $n, n+1, \cdots, n+k-1$ is a prime, otherwise the result follows. Let $p_i < n < n+k-1 < p_{i+1}$. Then $i = \pi(n)$ and $p_{\pi(n)} < n < \mathfrak{N}$. For $\pi(n) \notin I(\mathfrak{N}, k_0)$, we have

$$k - 1 = n + k - 1 - n < p_{\pi(n)+1} - p_{\pi(n)} < k_0$$

which implies $k - 1 < k_0 - 1$, a contradiction. Hence the assertion. $\square$

The following result is on the estimates for primes due to Dusart [1, p.14].

Lemma 5. *For $\nu > 1$, we have*

(*i*) $\pi(\nu) \leq \dfrac{\nu}{\log \nu}\left(1 + \dfrac{1.2762}{\log \nu}\right)$

(*ii*) $\pi(\nu) \geq \dfrac{\nu}{\log \nu - 1}$ for $\nu \geq 5393$.

Lemma 6. *Let $X > 0$ and $0 < \theta < e - 1$ be real numbers. For $l \geq 0$, let*

$$X_0 = \max\left(\frac{5393}{1+\theta}, \exp(\frac{\log(1+\theta)+0.2762}{\theta})\right),$$

$$X_{l+1} = \max\left(\frac{5393}{1+\theta}, \exp(\frac{\log(1+\theta)+0.2762)}{\theta + \frac{1.2762(1-\log(1+\theta))}{\log^2 X_l}})\right).$$

Then we have

$$\pi((1+\theta)X) - \pi(X) > 0$$

for $X > X_l$.

Proof. Let $l \geq 0$ and $X > X_l$. Then $(1+\theta)X \geq 5393$. By Lemma 5, we have

$$\begin{aligned}
\delta := \pi((1+\theta)X) - \pi(X) &\geq \frac{(1+\theta)X}{\log(1+\theta)X - 1} - \frac{X}{\log X}\left(1 + \frac{1.2762}{\log X}\right) \\
&\geq \frac{X}{\log(1+\theta)X - 1}\left\{1 + \theta - \frac{\log(1+\theta)X - 1}{\log X}\left(1 + \frac{1.2762}{\log X}\right)\right\} \\
&\geq \frac{X}{\log(1+\theta)X - 1}\left\{1 + \theta - \left(1 - \frac{1 - \log(1+\theta)}{\log X}\right)\left(1 + \frac{1.2762}{\log X}\right)\right\} \\
&\geq \frac{X}{\log(1+\theta)X - 1}\{F(X) + G(X)\}
\end{aligned}$$

where $F(X) = \theta - \frac{\log(1+\theta)+0.2762}{\log X}$ and $G(X) = \frac{1.2762(1-\log(1+\theta))}{\log^2 X}$. We see that $G(X) > 0$ and decreasing since $0 < \theta < e - 1$. Further we observe that $\{X_i\}$ is a non-increasing sequence. We notice that $\delta > 0$ if $F(X) + G(X) > 0$. But $F(X) + G(X) > F(X) > 0$ for $X > X_0$ by the definition of X_0. Thus $\delta > 0$ for $X > X_0$. Let $X \leq X_0$. Then $F(X) + G(X) \geq F(X) + G(X_0)$ and $F(X) + G(X_0) > 0$ if $X > X_1$ by the definition of X_1. Hence $\delta > 0$ for $X > X_1$. Now we proceed inductively as above to see that $\delta > 0$ for $X > X_l$ with $l \geq 2$. $\square$

Lemma 7. *Let $n > k$ and $k \leq 16$. Then*

$$P(\Delta(n,k)) \leq 2k \tag{7}$$

implies that $(n,k) \in \{(8,2),(8,3)\}$ or $(n,k) \in [k+1,k]$ for $k \in \{2,3,5,6,8,9,11,14,15\}$ or $(n,k) \in [k+1,k,3]$ for $k \in \{4,7,10,13\}$ or $(n,k) \in [k+1,k,5]$ for $k \in \{12,16\}$.

Proof. We apply Lemma 1 to derive that (7) is possible only if $n = 3, 8$ when $k = 2$ and $n = 5, 6, 7$ when $k = 4$. For the latter assertion, we apply Lemma 1 after securing $P((n+i)(n+j)) \leq 3$ with $0 \leq i < j \leq 3$ by deleting the terms divisible by 5 and 7 in $n, n+1, n+2$ and $n+3$. For $k = 3$ and $5 \leq k \leq 8$, the assertion follows from Lemma 2.

Thus we may assume that $k \geq 9$. Assume that (7) holds. Then there are at most $1 + [\frac{k-1}{p}]$ terms divisible by the prime p. After removing all the terms divisible by $p \geq 7$, we are left with at least 4 terms only divisible by $2, 3$ and 5. Further out of these terms, for each prime $2, 3$ and 5, we remove a term in which the prime divides to a maximal power. Then we are left with a term $n + i$ such that $n \leq n + i \leq 8 \times 9 \times 5 = 360$. Let $n \geq 2k$. We now apply Lemma 4 with $\mathfrak{N} = 361, k_0 = 9$ and (6) to get $P(\Delta(n, k)) > 2k$ for $k \geq 9$ except possibly when $p_i < n < n + k - 1 < p_{i+1}$, $p_i = 113, 139, 181, 199, 211, 241, 283, 293, 317, 337$. For these values of n, we check that $P(\Delta(n, k)) > 2k$ is valid for $9 \leq k \leq 16$. Thus it suffices to consider $k < n < 2k$. We calculate $P(\Delta(n, k))$ for (n, k) with $9 \leq k \leq 16$ and $k < n < 2k$. We find that (7) holds only if (n, k) is given in the statement of the Lemma 7. □

3. Proof of Theorem 1 (a)

Let $n > \max(k + 13, \frac{279}{262}k)$. In view of Lemma 7, we may take $k \geq 17$ since $n \leq k + 5$ for the exceptions (n, k) given in Lemma 7. It suffices to prove (2) for k such that $2k - 1$ is prime. Let $k_1 < k_2$ be such that $2k_1 - 1$ and $2k_2 - 1$ are consecutive primes. Suppose (2) holds at k_1. Then for $k_1 < k < k_2$, we have

$$P(n(n+1)\cdots(n+k-1)) \geq P(n\cdots(n+k_1-1)) > 2k_1$$

implying $P(\Delta(n, k)) \geq 2k_2 - 1 > 2k$. Therefore we may suppose that $k \geq 19$ since $2k-1$ with $k = 17, 18$ are composites. We assume from now onward in the proof of Theorem 1 **(a)** that $2k-1$ is prime. We put $x = n+k-1$. Then $\Delta(n, k) = x(x-1)\cdots(x-k+1)$. Let $f_1 < f_2 < \cdots < f_\mu$ be all the integers in $[0, k)$ such that

$$P((x - f_1)\cdots(x - f_\mu)) \leq k. \tag{8}$$

We derive as in the proof of [4, Lemma 4] to get

$$k! > x^{\mu - \pi(k)} \left(1 - \frac{k}{x}\right)^{\mu}. \tag{9}$$

We may suppose $\omega(\Delta(n,k)) \le \pi(2k)$ otherwise (2) follows. Then

$$\mu \ge k - \pi(2k) + \pi(k) \tag{10}$$

which we use as in [4, Lemma 4] to derive from (9) that

$$x < k^{\frac{3}{2}} \text{ for } k \ge 87; \; x < k^{\frac{7}{4}} \text{ for } k \ge 40; \; x < k^2 \text{ for } k \ge 19. \tag{11}$$

If $x \ge 7k$ and $k > 57$, then we derive as in [4, Lemma 7] from (10) that $x \ge k^{\frac{3}{2}}$. Thus we get from (11) that $x < 7k$ for $k \ge 87$. Putting back $n = x - k + 1$, we may assume that $n < 6k + 1$ for $k \ge 87$, $n < k^{\frac{7}{4}} - k + 1$ for $40 \le k < 87$ and $n < k^2 - k + 1$ for $19 \le k < 40$.

Let $k < 87$. Suppose $n \ge 2k$. Then $2k \le n < k^{\frac{7}{4}} - k + 1$ for $40 \le k < 87$ and $2k \le n < k^2 - k + 1$ for $19 \le k < 40$. Thus Lemma 4 with $\mathfrak{N} = 87^{\frac{7}{4}} - 87 + 1, k_0 = 35$ and (6) implies that $P(\Delta(n,k)) > 2k$ for $k \ge 35$. We note here that $2k \le n < \mathfrak{N}$ for $35 \le k < 40$. Let $k < 35$. Taking $\mathfrak{N} = 34^2 - 34 + 1, k_0 = 21$ for $21 \le k \le 34$ and $\mathfrak{N} = 19^2 - 19 + 1, k_0 = 19$ for $k = 19$, we see from Lemma 4 and (6) that $P(\Delta(n,k)) > 2k$ for $k \ge 19$. Here the case $k = 20$ is excluded since $2k - 1$ is composite. Therefore we may assume that $n < 2k$. Further we observe that $\pi(n+k-1) - \pi(2k) \ge \pi(2k+13) - \pi(2k)$ since $n > k + 13$. Next we check that $\pi(2k+13) - \pi(2k) > 0$. This implies that $[2k, n+k-1]$ contains a prime.

Thus we may assume that $k \ge 87$. Then we write $n = \alpha k + 1$ with $\frac{279}{262} - \frac{1}{k} < \alpha \le 6$ if $k \ge 201$ and $1 + \frac{12}{k} < \alpha \le 6$ if $k < 201$. Further we consider $\pi(n+k-1) - \pi(\max(n-1, 2k))$ which is

$$\begin{aligned}
&= \pi((\alpha+1)k) - \pi(\alpha k) \quad \text{for } \alpha \ge 2 \\
&\ge \pi([\frac{541}{262}k]) - \pi(2k) \qquad \text{for } \alpha < 2 \text{ and } k \ge 201 \\
&\ge \pi(2k+13) - \pi(2k) \quad \text{for } \alpha < 2 \text{ and } k < 201.
\end{aligned}$$

We check by using exact values of π function that $\pi(2k+13) - \pi(2k) > 0$ for $k < 201$ and $\pi([\frac{541}{262}k]) - \pi(2k) > 0$ for $201 \le k \le 2616$. Thus we may suppose that $k > 2616$ if $\alpha < 2$. Also $[\frac{541}{262}k] \ge \frac{540}{262}k$ for $k > 2616$. Now we apply Lemma 6 with $X = \alpha k, \theta = \frac{1}{\alpha}, l = 0$ if $\alpha \ge 2$ and $X = 2k, \theta = \frac{4}{131}, l = 1$ if $\alpha < 2$ to get $\pi(n+k-1) - \pi(\max(n-1, 2k)) > 0$ for $X > X_0 = \frac{5393}{1+\frac{1}{\alpha}}$ if $\alpha \ge 2$ and $X > X_1 = \frac{5393}{1+\frac{4}{131}}$ if $\alpha < 2$. Further when $\alpha < 2$, we observe that $X = 2k > X_1$ since $k > 2616$. Thus the assertion follows for $n < 2k$. It remains to consider the case $\alpha \ge 2$ and $X \le 5393(1+\frac{1}{\alpha})^{-1}$. Then $2k \le n < n+k-1 = X(1+\frac{1}{\alpha}) \le 5393$. Now we apply Lemma 4 with $\mathfrak{N} = 5393, k_0 = 35$ and (6) to conclude that $P(\Delta(n,k)) > 2k$. $\square$

4. Proof of Theorem 1 (b)

In view of Lemma 7 and Theorem 1 **(a)**, we may assume that $k \geq 17$ and $k < n \leq \frac{279}{262}k$. Let $X = \frac{279}{262}k, \theta = \frac{245}{279}, l = 0$. Then for $k < n \leq X$, we see from Lemma 6 that

$$\pi(2k) - \pi(n-1) \geq \pi((1+\theta)X) - \pi(X) > 0$$

for $X > X_0 = 5393(1+\theta)^{-1}$ which is satisfied for $k > 2696$ since $(1+\theta)X = 2k$. Thus we may suppose that $k \leq 2696$. Now we check with exact values of π function that $\pi(2k) - \pi(\frac{279}{262}k) > 0$. Therefore $P(\Delta(n,k)) \geq P(n(n+1)\cdots 2k) \geq p_{\pi(2k)}$. Further we apply Lemma 6 with $X = 1.97k$, $\theta = \frac{3}{197}$ and $l = 25$. We calculate that $X_l \leq 284000$. We conclude by Lemma 6 that

$$\pi(2k) - \pi(1.97k) = \pi((1+\theta)X) - \pi(X) > 0$$

for $k > 145000$. Let $k \leq 145000$. Then we check that $\pi(2k) - \pi(1.97k) > 0$ is valid for $k \geq 680$ by using the exact values of π function. Thus

$$p_{\pi(2k)} > 1.97k \text{ for } k \geq 680. \tag{12}$$

Therefore we may suppose that $k < 680$. Now we observe that for $n > k+13$, $\pi(n+k-1) - \pi(1.97k) \geq \pi(2k+13) - \pi(1.97k) > 0$, the latter inequality can be checked by using exact values of π function. Hence the assertion follows since $n < 1.97k$. □

5. Proof of Theorem 2

By Theorem 1 (b), we may assume that $n \leq k+13$. Also we may suppose that $k < 680$ by (12). For $k \leq 16$, we calculate $P(\Delta(n,k))$ for all the pairs (n,k) given in the statement of Lemma 7. We find that either $P(\Delta(n,k)) > 1.95k$ or (n,k) is an exception stated in Theorem 1 (a). Thus we may suppose that $k \geq 17$. Now we check that $\pi(n+k-1) - \pi(1.95k) > 0$ except for $(n,k) \in [k+1, k, h]$ for $k \in A_h$ with $1 \leq h \leq 11$ and the assertion follows. □

6. Proof of Corollary 1

We calculate $P(\Delta(n,k))$ for all (n,k) with $k \leq 270$ and $k+1 \leq n \leq k+11$. This contains the set of exceptions given in Theorem 2. We find that $P(\Delta(n,k)) > 1.8k$ unless $(n,k) \in B$. Hence the assertion (5) follows from Theorem 2. □

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