THE NUMBER OF PRIME DIVISORS OF A PRODUCT OF CONSECUTIVE INTEGERS

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Abstract. It is shown under Schinzel's Hypothesis that for a given $\ell \geq 1$, there are infinitely many k such that a product of k consecutive integers each exceeding k is divisible by exactly $\pi(2k) - \ell$ prime divisors.

1. Introduction

For $n > 0, k > 0$ integers, we define

$$\Delta(n,k) = n(n+1)(n+2)\cdots(n+k-1). \tag{1}$$

Let $\omega(n)$ denote the number of distinct prime divisors of n and $\pi(x)$ the number of primes $p \leq x$ for any given real number $x > 1$. We write $p_1 = 2, p_2 = 3, \ldots$ and p_r, the r-th prime.

Let $n = k+1$ in (1). Then we have $\Delta(k+1,k) = (k+1)(k+2)\cdots(2k)$. Since $k!$ divides $\Delta(k+1,k)$, clearly, we have

$$\omega(\Delta(k+1,k)) = \pi(k) + \pi(2k) - \pi(k) = \pi(2k). \tag{2}$$

Hence, it is natural to ask the following question.

Question 1: For any given integer $\ell \geq 1$, can we find infinitely many pairs (n,k) with $n > k$ such that

$$\omega(\Delta(n,k)) = \pi(2k) - \ell\ ? \tag{3}$$

First we observe that the answer to Question 1 is true when $\ell = 1$. For this put $n = k+2$ in (1) and consider

$$\Delta(k+2,k) = \Delta(k+1,k)\frac{2k+1}{k+1}.$$

It suffices to find infinitely many values of k satisfying

(i) $k+1$ is a prime and
(ii) $2k+1$ is a composite number.

Let $k+1$ be a prime of the form $3r+2$. Then $2k+1 = 3(2r+1)$ is composite. Since there are infinitely many primes of the form $3r+2$, we see that there are infinitely many k for which $k+1$ is prime and $2k+1$ is composite. Thus Question 1 is true when $\ell = 1$.

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For a given ℓ, a method to construct pairs (n, k) satisfying (3) has been given in [1]. In particular, it has been observed in [1] that (3) holds if

$$\begin{aligned} (n,k) \in&\{(74,57),(284,252),(3943,3880)\} && \text{when } \ell = 2\\ (n,k) \in&\{(3936,3879),(3924,3880),(3939,3880)\} && \text{when } \ell = 3\\ (n,k) \in&\{(1304,1239),(1308,1241),(3932,3879)\} && \text{when } \ell = 4\\ (n,k) \in&\{(3932,3880),(3932,3881),(3932,3882)\} && \text{when } \ell = 5. \end{aligned}$$

Before we state our result, we need the following hypothesis.

Schinzel's Hypothesis. ([2] and [3]) *Let $f_r(x) = a_r x + b_r$ be non-constant polynomials with $a_r > 0$ and b_r are integers for every $r = 1, 2, \ldots, \ell$. If for every prime p, there exists an integer n such that p doesn't divide $f_1(n)\ f_2(n)\ \cdots\ f_\ell(n)$, then, there exist infinitely many integer values, say, x_1, x_2, $\ldots$, satisfying*

$$f_1(x_j) = q_1, f_2(x_j) = q_2, \ldots, f_\ell(x_j) = q_\ell$$

for all $j = 1, 2, \ldots$ where q_i's are prime numbers.

For a given positive integer $\ell \geq 2$, we first let

$$A = \prod_{p \leq \ell} p$$

and we enumerate all the positive integers > 1 which are coprime to A as $a_1 < a_2 < \ldots < a_n < \ldots$. We define

$$\lambda_\ell := \min_j \{a_{j+\ell-1} - a_j \ : \ j = 1, 2, \ldots\}$$

Clearly, from the definition, we have $\lambda_\ell \geq 2(\ell - 1)$ and we put

$$R = R_\ell = \lambda_\ell + 1.$$

We show that Schinzel's Hypothesis confirms Question 1. In fact, we prove

Theorem 1. *Assume Schinzel's Hypothesis and let $\ell \geq 2$ be an integer. Then there are infinitely many values of k such that*

$$\omega(\Delta(k + 2R, k)) = \pi(2k) - \ell. \tag{4}$$

Note. In the statement of Theorem 1, the value $2R$ cannot be replaced by a smaller value. If there is a smaller value $L < 2R$ for which Theorem 1 is true, then it will contradict the minimality of λ_ℓ. This is clear from (4) with $2R$ replaced by $2S$ such that $S < R$ and (2). Further, in view of Theorem 1, it is of interest to compute R_ℓ and we compute R_ℓ for $2 \leq \ell \leq 100$ in Section 3. We thank the referee for his remarks on an earlier draft of this paper.

2. Proof of Theorem 1

For any given positive integer $\ell \geq 2$, let $M = \lambda_\ell$. Therefore, by the definition of M, we get integers $a_j \geq \ell + 1$ and $a_{j+\ell-1}$ such that $a_{j+\ell-1} - a_j = M$ and hence $a_{j+\ell-1} = M + p_j$ for some positive integer j. So, the sequence $a_j, 1 + a_j, \ldots, M + a_j$ contains exactly ℓ integers which are coprime to A. In other words, we have $a + 1 = a_j, a + 2, \ldots, a + M + 1 = a_j + M$

contains ℓ integers which are coprime to A. In this new notation, we denote the set of those ℓ coprime integers to A to be

$$\mathcal{P} = \{a + x(1), a + x(2), \ldots, a + x(\ell)\},$$

where $x(1), x(2), \ldots, x(\ell)$ are some odd integers not exceeding $M + 1 = R$.

We write

$$\Delta(k + 2R, k) = \Delta(k + 1, k) \times 2^{R-1} \times \frac{(2k + 1)(2k + 3)\cdots(2k + 2R - 1)}{(k + R)(k + R + 1)\cdots(k + 2R - 1)}.$$

We put

$$B_0 = \{(k + (R - 1) + 1), (k + (R - 1) + 2), \ldots, (k + (R - 1) + R)\}.$$

Then B_0 contains at most $[(R + 1)/2]$ even integers. We omit these numbers from B_0 and name the remaining set as B_1. Clearly, B_1 contains $k + (R - 1) + x(r)$ with $r = 1, 2, \ldots, \ell$. Let B_2 be the subset of B_1 obtained by deleting these elements. Further we put

$$B = \{x - k - (R - 1) \ : \ x \in B_2\}$$

so that $|B| = |B_2|$. We order the elements of B as $i_1 < i_2 < \ldots < i_{|B_2|}$.

Now, we choose primes P_j, q_j satisfying the conditions

(i) $4R < P_1 < P_3 < \ldots < P_{2R-1}$ and;
(ii) $P_{2R-1} < q_1 < q_2 < \ldots < q_{|B_2|}$;
(iii) We consider the following system of congruences

$$\begin{aligned} 2x + 1 &\equiv 0 \pmod{P_1} \\ 2x + 3 &\equiv 0 \pmod{P_3} \\ &\cdots\cdots \\ 2x + 2R - 1 &\equiv 0 \pmod{P_{2R-1}} \\ x + (R - 1) + i_j &\equiv 0 \pmod{q_j}\ \forall\ i_j \in B. \end{aligned}$$

By the Chinese Remainder Theorem, we have infinitely many common solutions of the form

$$k = b + \lambda Q; \quad \text{for all} \ \ \lambda \in \mathbb{Z} \ \ \text{and} \ \ Q = \prod_{i=1}^{R} P_{2i-1} \prod_{i=1}^{|B|} q_i,$$

for some positive integer b.

Under Schinzel's hypothesis, we shall prove that there are infinitely many choices for λ such that

$$k + R - 1 + x(1), k + R - 1 + x(2), \ldots, k + R - 1 + x(\ell)$$

are prime numbers.

Now, we use Schinzel's hypothesis with the polynomials

$$f_r(X) = QX + b + R - 1 + x(r) \ \ \text{for} \ r = 1, 2, \ldots, \ell.$$

We only need to show that if q is any prime number and

$$\mathfrak{p}(X) = \prod_{r=1}^{\ell} f_r(X) = \prod_{r=1}^{\ell}(QX + b + R - 1 + x(r)),$$

then there exists $\lambda \in \mathbb{Z}$ such that q does not divide $\mathfrak{p}(\lambda)$.

Let q be any prime number. Then we have the following cases.

Case 1. $(q, Q) = 1$.

Subcase (i). $q \leq \ell$.

In this case, we see that $q|A$. Since $(q, Q) = 1$, we choose λ such that $k + R - 1 = \lambda Q + b + R - 1 \equiv a \pmod q$. Therefore, for every $r = 1, 2, \ldots, \ell$, we have

$$k + R - 1 + x(r) \equiv a + x(r) \pmod q.$$

Since $a + x(r)$ is coprime to q, clearly, q cannot divide $\mathfrak{p}(\lambda)$.

Subcase (ii). $q > \ell$.

In this case, clearly, $\{-(b + R - 1 + x(r))\}_{r=1}^{\ell}$ covers only ℓ residue classes modulo q. Since $q > \ell$, there exists a residue class c modulo q which is not covered. Since $(q, Q) = 1$, choose λ such that

$$\lambda Q \equiv c \pmod q.$$

Since c is not one of the $\{-(b + R - 1 + x(r))\}_{r=1}^{\ell}$, we have

$$k + R - 1 + x(r) = \lambda Q + b + R - 1 + x(r) \equiv c + b + R - 1 + x(r) \not\equiv 0 \pmod q$$

for $r = 1, 2, \ldots, \ell$. Therefore q does not divide $\mathfrak{p}(\lambda)$ for this choice of λ.

Case 2. $q|Q$

Suppose $q = q_j$ for some $j = 1, 2, \ldots, |B_2|$. If possible, q divides $\mathfrak{p}(\lambda)$ for all choices of λ. Then

$$k + R - 1 + x(r) \equiv 0 \pmod{q_j} \text{ for some } r.$$

Note that by the definition of q_j, we have,

$$k + R - 1 + i_j \equiv 0 \pmod{q_j}.$$

Hence, we get

$$k + R - 1 + x(r) \equiv k + R - 1 + i_j \pmod{q_j} \implies x(r) \equiv i_j \pmod{q_j}.$$

As $q_j \geq 4R$ and $x(r), i_j \in \{1, 2, \ldots, R\}$, the above congruence implies that

$$x(r) = i_j$$

which is not possible by the definition of B. Hence, q does not divide $\mathfrak{p}(\lambda)$ for some choice of λ.

Suppose $q = P_i$ for some $i = 1, 3, \ldots, 2R - 1$. If possible, we assume that q divides $\mathfrak{p}(\lambda)$ for all $\lambda \in \mathbb{Z}$. Then

$$k + R - 1 + x(r) \equiv 0 \pmod{P_i} \text{ for some } r.$$

By the definition of P_i, we have $2k + m \equiv 0 \pmod{P_i}$ for some odd integer $m \leq 2R - 1$. Combining the above two congruences, we get,

$$2(R - 1 + x(r)) \equiv m \pmod{P_i}.$$

But since $R - 1 + x(r) \leq 2R - 1$, $m \leq 2R - 1$ and $P_i \geq 4R$, the above congruence implies

$$2(R - 1 + x(r)) = m,$$

which is a contradiction because m is an odd integer. Hence, q does not divide $\mathfrak{p}(\lambda)$ for some choice of λ.

In all the cases, if q is any prime, then q does not divide $\mathfrak{p}(\lambda)$ for some choice of λ. Hence, by Schinzel's Hypothesis, we get infinitely many values of k such that

$$k+(R-1)+x(1), k+(R-1)+x(2), \ldots, k+(R-1)+x(r)$$

are all primes. Thus we arrive at

$$\omega(\Delta(k+2R, k)) = \pi(2k) - \ell.$$

This completes the proof of Theorem 1. □

3. Computation of R_ℓ with $2 \le \ell \le 28$

The computation of R_ℓ depends on the following lemmas.

Lemma 3.1. *For each integer $j \ge 1$ and $m \ge 1$, we have*

$$a_j + mA = a_{j+m\phi(A)}.$$

Proof. Let $b_1, b_2, \ldots, b_{\phi(A)}$ be the positive integers which are coprime to A and $1 \le b_i \le A$ for every i. Then, for each integer $m \ge 1$, we have $mA+1 \le b_i + mA \le (m+1)A$ and $mA + b_i$ are coprime to A for every $i = 1, 2, \ldots, \phi(A)$. If a is any integer such that $mA + 1 \le a \le (m+1)A$ and $a \ne b_i + mA$, then, $a = b + mA$ where $b \ne b_i$ for all $i = 1, 2, \ldots, \phi(A)$ and $b \le A$. Therefore, by the definition of b, $(b, A) > 1$ and hence $(a, A) > 1$. Hence, $b_i + mA$ $(i = 1, 2, \ldots, \phi(A))$ are, precisely, those integers which are in the interval $[mA+1, (m+1)A]$ and coprime to A. Thus, we enumerate all the positive integers which are coprime to A as

$$b_1 < b_2 < \ldots < b_{\phi(A)} < b_1 + A < b_2 + A < \ldots < b_{\phi(A)} + A < b_1 + 2A < b_2 + 2A < \ldots$$

Let $(b_i)_{i=\phi(A)+1}^{\infty}$ be given by

$$b_{\phi(A)+1} = b_1 + A, b_{\phi(A)+2} = b_2 + A, \ldots$$

so that the sequence $(b_i)_{i=1}^{\infty}$ satisfies

$$a_i = b_{i+1} \text{ for } i \ge 1.$$

We observe that for $j \ge 1$,

$$b_j + mA = b_{j+m\phi(A)}$$

implying

$$a_j + mA = b_{j+1} + mA = b_{j+1+m\phi(A)} = a_{j+m\phi(A)}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *For each integer $\ell \ge 2$, we have*

$$\lambda_\ell = \min\{a_{j+\ell-1} - a_j \ : \ j = 1, 2, \ldots, \phi(A)\}.$$

Proof. Assume that $j > \phi(A)$. Then we can write $j = m\phi(A) + i$ for some integer $m \ge 1$ and $1 \le i \le \phi(A)$. Therefore, by Lemma 3.1,

$$a_{j+\ell-1} = a_{m\phi(A)+i+\ell-1} = a_{i+\ell-1} + mA$$

and hence

$$a_{j+\ell-1} - a_j = a_{i+\ell-1} + mA - a_i - mA = a_{i+\ell-1} - a_i$$

for some i satisfying $1 \le i \le \phi(A)$. Thus, to find λ_ℓ, it is enough to find the minimum values of $a_{i+\ell-1} - a_i$ for all $i = 1, 2, \ldots, \phi(A)$. □

Case (a)**.** $\ell = 2$**.** In this case, $A = 2$ and hence $\phi(A) = 1$. So, by Lemma 3.2, we see that $\lambda_2 = a_2 - a_1 = 2$ and $R = 3$.

Case (b)**.** $\ell = 3, 4$**.** We have $A = 6$ and hence $\phi(A) = 2$ and $a_1 = 5, a_2 = 7, a_3 = 11, a_4 = 13, a_5 = 17$. Therefore

$$\lambda_3 = \min\{a_3 - a_1, a_4 - a_2\} = \min\{11 - 5, 13 - 7\} = 6, \quad R = 7$$
$$\lambda_4 = \min\{a_4 - a_1, a_5 - a_2\} = \min\{13 - 5, 17 - 7\} = 8, \quad R = 8.$$

Case (c)**.** $\ell = 5, 6$**.** In this case, $A = 30$ and hence $\phi(A) = 8$. We have

$$a_1 = 7, a_2 = 11, a_3 = 13, a_4 = 17, a_5 = 19, a_6 = 23, a_7 = 29,$$
$$a_8 = 31, a_9 = 37, a_{10} = 41, a_{11} = 43, a_{12} = 47, a_{13} = 49.$$

Therefore

$$\lambda_5 = \min\{a_{i+4} - a_i : 1 \le i \le 8\} = 12, \quad R = 13$$
$$\lambda_6 = \min\{a_{i+5} - a_i : 1 \le i \le 8\} = 16, \quad R = 17.$$

Case (d)**.** $\ell \ge 7$**.** Let $\ell_1 \le \ell < \ell_2$ where ℓ_1, ℓ_2 are consecutive primes. Then $A_\ell = A_{\ell_1} = A$. Define $a_0 = 1$,

$$S^0_{\ell_1} = \{a : 1 \le a < A \text{ and } \gcd(a, \prod_{p \le \ell_1} p) = 1\} = \{a_0\} \cup \{a_1, a_2, \ldots, a_{\phi(A)-1}\}$$

and

$$S^1_{\ell_1} = S^0_{\ell_1} \cup \{A + a_i : 0 \le i < \ell_2\} = \{a_0\} \cup \{a_1, a_2, \ldots, a_{\phi(A)-1}, a_{\phi(A)}, \ldots, a_{\phi(A)+\ell_2-1}\}.$$

Note that $a_{\phi(A)} = A + 1$ and if $\gcd(A + a, \prod_{p\le\ell_1} p) = 1$, then $a \in S^0_{\ell_1}$. To compute λ_ℓ for $\ell_1 \le \ell < \ell_2$, we take the subset of $S^1_{\ell_1}$ containing the first $\phi(A) + \ell - 1$ elements and compute

$$\lambda_\ell = \min\{a_{j+\ell-1} - a_j \;:\; j = 1, 2, \ldots, \phi(A)\}.$$

Suppose we have computed $S^0_{\ell_1}, S^1_{\ell_1}$ and we would like to compute $S^0_{\ell_2}, S^1_{\ell_2}$. Divide $A_{\ell_2} = A$ as

$$(0, A] = \cup_{i=1}^{\ell_2}((i-1)\frac{A}{\ell_2}, i\frac{A}{\ell_2}).$$

Note that $\frac{A}{\ell_2} = A_{\ell_1}$. If $\frac{A}{\ell_2} \equiv r(\text{mod } \ell_2)$, then

$$S^0_{\ell_2} = \cup_{i=1}^{\ell_2}\left\{(i-1)\frac{A}{\ell_2} + a_i : a_i \in S^0_{\ell_1} \text{ and } r(i-1) + a_i \not\equiv (\text{mod } \ell_2)\right\}$$
$$= \{a_0 = 1\} \cup \{a_1, a_2, \ldots, a_{\phi(A)}\}.$$

We now take

$$S^1_{\ell_2} = S^0_{\ell_2} \cup \{A + a_i : 0 \le i < \ell_3\} = \{a_0, a_1, a_2, \ldots, a_{\phi(A)-1}, a_{\phi(A)}, \ldots, a_{\phi(A)+\ell_3-1}\}.$$

where $\ell_3 > \ell_2$ is the prime next to ℓ_2. Finally we compute

$$\lambda_\ell = \min\{a_{j+\ell-1} - a_j \;:\; j = 1, 2, \ldots, \phi(A)\}.$$

For $7 \le \ell \le 18$, computing a_i's and λ_ℓ were fast and we list the values in the following table. For $19 \le \ell \le 22$, we start with $\ell_1 = 17, \ell_2 = 19$ to compute λ_ℓ. For $23 \le \ell \le 28$, we take $\ell_1 = 19, \ell_2 = 23$ and compute λ_ℓ. We stop at $\ell = 28$ since computations increase exponentially when we go to the next prime. Here we list the values of $\ell, A, \phi(A), \lambda_\ell, a_1, a_{\phi(A)}$ and $a_{\phi(A)+\ell-1}$ for $6 \le \ell \le 28$.

ℓ	A	$\phi(A)$	λ_ℓ	a_1	$a_{\phi(A)}$	$a_{\phi(A)+\ell-1}$
6	30	8	16	7	31	49
7	210	48	20	11	211	239
8	210	48	26	11	211	241
9	210	48	30	11	211	247
10	210	48	32	11	211	251
11	2310	480	36	13	2311	2357
12	2310	480	42	13	2311	2363
13	30030	5760	48	17	30031	30091
14	30030	5760	50	17	30031	30097
15	30030	5760	56	17	30031	30101
16	30030	5760	60	17	30031	30103
17	510510	92160	66	19	510511	510593
18	510510	92160	70	19	510511	510599
19	9699690	1658880	76	23	9699691	9699791
20	9699690	1658880	80	23	9699691	9699793
21	9699690	1658880	84	23	9699691	9699797
22	9699690	1658880	90	23	9699691	9699799
23	223092870	36495360	94	29	223092871	223092997
24	223092870	36495360	100	29	223092871	223093001
25	223092870	36495360	110	29	223092871	223093007
26	223092870	36495360	114	29	223092871	223093009
27	223092870	36495360	120	29	223092871	223093019
28	223092870	36495360	126	29	223092871	223093021

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