Solving $n(n+d)\cdots\big(n+(k-1)d\big) = by^2$ with $P(b) \le Ck$

M. Filaseta
Department of Mathematics
University of South Carolina
Columbia, SC 29208
U.S.A.

S. Laishram
Department of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo, ON, N2l 3G1
Canada

N. Saradha
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400005
India

1 Introduction

With n, d and k positive integers, we set

$$\Delta(n,k,d) = n(n+d)\cdots\big(n+(k-1)d\big).$$

Fix d as above and real numbers $\varepsilon > 0$ and $C \ge d$. We are interested in establishing that the equation

$$\Delta(n,k,d) = by^2 \tag{1}$$

has finitely many solutions in positive integers n, k, b and y with

$$\gcd(n,d) = 1, \quad k \ge 3, \quad n \ge \big(C - d + \varepsilon d\big)k \quad \text{and} \quad P(b) \le Ck, \tag{2}$$

where $P(b)$ denotes the largest prime dividing b. As we shall describe at the beginning of the third section, traditional methods allow one to establish such a result when C is small and, in particular, for $C \le 2$ (and a little beyond). But for large C, these methods fail. In this paper, we describe an improvement on these methods which allows one to handle this problem for larger C as well. Our main result is the following.

Theorem 1. *Fix a positive integer d. Let $\varepsilon\in(0,1)$ and $C\ge d$ be arbitrary. There is a finite effectively computable set $S=S(d,\varepsilon,C)$ of 4-tuples such that if* (1) *and* (2) *hold, then $(n,k,b,y)\in S$.*

The above result is formulated in the way that we will establish it. We note, however, that the condition $\gcd(n,d)=1$ can be dropped and the expression εd appearing in the lower bound for n can be replaced by ε. Indeed, these are not actual improvements on Theorem 1 but rather equivalent formulations of it.

2 Preliminaries

We suppose as we may that b is squarefree. For m a positive integer, we will also make use of the notation $\nu_p(m)=e$ where $p^e\|m$. Observe that it suffices to bound n and k as above for which (1) has a solution satisfying (2). In fact, for each fixed $k\ge 3$, there are finitely many possibilities for squarefree b satisfying the last condition in (2). For each such b and solution to (1), we can consider the product $\Delta(n,3,d)$ which will necessarily be a square times a positive squarefree integer having all of its prime factors $\le Ck$. Thus, we obtain

$$\Delta(n,3,d)=b'y^2,$$

where $P(b')\le Ck$. There are finitely many possibilities then for b', and we deduce (1) has finitely many integer solutions as solutions correspond to integer points on the elliptic curve described by $\Delta(n,3,d)=b'y^2$. That these integer points can be effectively computed is a consequence of, for example, Theorem 4.2 in [1]. Thus, it suffices to show that (1) and (2) imply k is bounded.

Beginning with (1), for $0\le j<k$, we can write

$$n+jd=a_jx_j^2,\quad a_j,x_j\in\mathbb{Z},\quad a_j\ \text{squarefree}.$$

We will want to know that $P\big(\Delta(n,k,d)\big)$ is large, so we state this as a first result.

Theorem 2. *Fix a positive integer d. Let $\varepsilon\in(0,1)$ and $C\ge d$ be arbitrary. There is a finite effectively computable set $S'=S'(d,\varepsilon,C)$ of 2-tuples such that if n and k are positive integers for which*

$$k\ge 2,\quad \gcd(n,d)=1,\quad n\ge(C-d+\varepsilon d)k,\quad (n,k)\notin S',$$

then

$$P\big(\Delta(n,k,d)\big)>Ck.$$

Proof. An asymptotic form of Dirichlet's Theorem implies that there is a k_0 such that if $k\ge k_0$, then there is a prime in the set

$$\{n+(k-\lfloor\varepsilon k\rfloor+1)d,n+(k-\lfloor\varepsilon k\rfloor+2)d,\dots,n+(k-2)d,n+(k-1)d\}$$

for all n satisfying

$$(C-d+\varepsilon d)k\le n\le e^{3C}k.$$

We deduce that if $P\big(\Delta(n,k)\big)\le Ck$, then either $k<k_0$ or $n>e^{3C}k$.

Suppose that $n > e^{3C}k$. We show in this case that $P\big(\Delta(n,k)\big) > Ck$ provided k is sufficiently large. We use an idea of Erdős [3]. For each prime $p \le Ck$, we consider n_p from the set

$$T = \{n, n+d, \dots, n+(k-1)d\} \tag{3}$$

for which $\nu_p(n_p)$ is maximal. If x is the number of integers $< n_p$ in T and y the number $> n_p$, then $x + y = k - 1$. We also have

$$\nu_p\Bigg(\prod_{\substack{m\in T\\ m\ne n_p}} m\Bigg) \le \sum_{j=1}^{\infty}\left(\left\lfloor \frac{x}{p^j}\right\rfloor + \left\lfloor \frac{y}{p^j}\right\rfloor\right) \le \sum_{j=1}^{\infty}\left\lfloor \frac{x+y}{p^j}\right\rfloor = \nu_p\big((k-1)!\big).$$

We deduce that

$$\prod_{\substack{p\le Ck\\ p^e\|\Delta(n,k,d)}} p^e \le (k-1)! \prod_{p\le Ck} n_p \le k^k(n+kd)^{\pi(Ck)} \le k^k(2n)^{\pi(Ck)}.$$

Hence,

$$\log \prod_{\substack{p\le Ck\\ p^e\|\Delta(n,k,d)}} p^e \le k\log k + \pi(Ck)\log(2n) \le k\log k + \frac{2Ck}{\log k}\log(2n).$$

On the other hand,

$$\log \prod_{m\in T} m > \log n^k \ge k\log n.$$

Thus,

$$\prod_{\substack{p\le Ck\\ p^e\|\Delta(n,k,d)}} p^e < \prod_{m\in T} m$$

provided

$$\log n \ge \log k + \frac{2C}{\log k}\log(2n).$$

The latter holds provided

$$\left(1 - \frac{2C}{\log k}\right)\log n \ge \log k + \frac{2C\log 2}{\log k}.$$

Since $n > e^{3C}k$, it suffices here for

$$\left(1 - \frac{2C}{\log k}\right)\big(\log k + 3C\big) \ge \log k + \frac{2C\log 2}{\log k},$$

which is easily seen to hold for $k \ge k_0'$, say.

We are left then with the task of considering finitely many $k < \max\{k_0, k_0'\}$. A result of G. Pólya [8] implies that $P(n(n+d))$ tends to infinity with n. In particular, there is an n_0 such that if $n > n_0$, then for all k satisfying $2 \le k < \max\{k_0, k_0'\}$, one has

$$P\big(\Delta(n,k,d)\big) \ge P\big(n(n+d)\big) \ge C\max\{k_0, k_0'\} > Ck.$$

What remains are pairs (k,n) satisfying $2 \le k < \max\{k_0, k_0'\}$ and $(C - d + \varepsilon d)k \le n \le n_0$. There are finitely many such pairs. The prime distribution results used above are effective, so we deduce that the set S' given in the theorem is effectively computable, and the result follows. $\square$

We note that [9] provides some explicit estimates that can be used for obtaining k_0 for small d, for example $d \le 72$. The use of Pólya's result can be replaced by work in [7] for $d \in \{1,2,4\}$ or a use of estimates on linear forms of logarithms or by use of algorithms for Thue equations as in [2] and [11]. The reader may also want to see [6] for related work on the largest prime factor of a product of consecutive numbers in an arithmetic progression.

Given Theorem 2, one can show effectively that, with the conditions on n in (2), the inequality

$$P\big(\Delta(n,k,d)\big) > Ck$$

holds for all but finitely many pairs (n,k). This is more than enough to allow one to consider the case that the a_j are distinct. This is accomplished as follows. Since some x_j is divisible by a prime $> Ck \ge dk$, we deduce that

$$n + kd > a_j x_j^2 \ge (kd+1)^2 > k^2d^2 + kd.$$

Hence, $n > k^2d^2$. Now, if $a_u = a_v$ with $u \ne v$, then

$$(k-1)d \ge |a_u x_u^2 - a_v x_v^2| = a_u(x_u + x_v)|x_u - x_v| > a_u x_u \ge \sqrt{a_u x_u^2} > \sqrt{k^2d^2} = kd,$$

which is impossible. Note that we also obtain that

$$n + kd > C^2k^2.$$

This implies $kd < n$ so that $n + kd \le 2n$. We also have as a consequence of (1) that $P(a_j) \le Ck$ for each j. Before proceeding, we prefer a stronger lower bound on n or, more precisely, on the numbers x_j. We address that next.

Lemma 1. *If* (1) *and* (2) *hold and k is sufficiently large, then $n > Ck^{2.8}$.*

Proof. Assume $n \le Ck^{2.8}$. We use that $n + kd > C^2k^2$ which easily implies $n > k^2/2$. The basic idea is to find a lower bound on the number of integers in the set T given in (3) that are squarefree. We show that at least 57% of the elements of T are squarefree by making use of the assumption $n \le Ck^{2.8}$. We explain first why this leads us to a contradiction.

Let $t = \lceil 0.57k \rceil$. Suppose there are $\ge t$ elements of T that are squarefree. Then

$$\log \prod_{0 \le j < k} a_j \ge \log \prod_{0 \le j < t} (n + jd) > t \log n \ge 0.57k \log(k^2/2) > 1.1k \log k.$$

On the other hand, each prime $p \le Ck$ divides at most $\lfloor k/p \rfloor + 1$ of the numbers a_j. Recall that the a_j are squarefree and satisfy $P(a_j) \le Ck$. For k sufficiently large, we deduce that

$$\log \prod_{0 \le j < k} a_j \le \sum_{p \le Ck} \left(\left\lfloor \frac{k}{p} \right\rfloor + 1 \right) \log p \le k \sum_{p \le Ck} \frac{\log p}{p} + \sum_{p \le Ck} \log p$$

$$\le 1.05k \log k + 1.1Ck \le 1.1k \log k.$$

Thus, we have a contradiction.

We finish the proof by showing that there are at least t squarefree numbers in T. We consider primes in three different ranges.

Let $z = \log k$. We start with primes $\leq z$. Since $\gcd(n,d) = 1$, the number of multiples of m^2 in T is 0 if m has a prime factor in common with d. Otherwise, the number of multiples of m^2 in T is

$$\left\lfloor \frac{k}{m^2} \right\rfloor + R_m = \frac{k}{m^2} + R'_m$$

where $R_m \in \{0,1\}$ and $R'_m \in (-1,1]$. Let P denote the product of the primes $\leq z$. Note that we are considering k sufficiently large. Then the sieve of Eratosthenes implies that number of elements of T that are not divisible by p^2 for every prime $p \leq z$ is

$$\sum_{m|P} \mu(m)\left(\frac{k}{m^2} + R'_m\right) = \prod_{p \leq z}\left(1 - \frac{1}{p^2}\right)k + E,$$

where

$$|E| \leq 2^{\pi(z)} \leq 2^{\log k} = k^{\log 2} \leq 0.01k.$$

Since

$$\prod_{p \leq z}\left(1 - \frac{1}{p^2}\right) \geq \prod_{p}\left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2} > 0.6,$$

we deduce that there are at least $0.59k$ elements of T that are not divisible by p^2 for every prime $p \leq z$.

Next, we observe that the number of elements of T divisible by p^2 for some prime $p \in (z, kd]$ is bounded by

$$\sum_{z < p \leq kd}\left(\left\lfloor \frac{k}{p^2} \right\rfloor + 1\right) \leq k \sum_{m > z} \frac{1}{m^2} + \pi(kd) \leq \frac{k}{z-1} + \frac{2kd}{\log k} < 0.01k.$$

We deduce that there are at least $0.58k$ elements of T that are not divisible by p^2 for every prime $p \leq kd$.

Finally, we consider the primes $p > kd$ for which p^2 divides some element of T. Observe that necessarily $p < \sqrt{n + kd} \leq \sqrt{2n}$. Recall that we are assuming $n \leq Ck^{2.8}$. As a consequence $k \geq (n/C)^{1/2.8} > n^{0.35}$. Thus, we are interested in primes p for which

$$n^{0.35} < kd < p \leq \sqrt{2n}.$$

Observe that, for each such p, there is at most one multiple of p^2 in T. Furthermore, if ap^2 is such a multiple, then a is a positive integer satisfying

$$a \leq \frac{n + kd}{p^2} \leq \frac{2n}{n^{2 \cdot 0.35}} = 2n^{0.3} < 2k^{0.3/0.35} < k^{0.9}.$$

On the other hand, if we also have a second prime $q \in (kd, \sqrt{2n}]$ for which $aq^2 \in T$, then

$$kd > |ap^2 - aq^2| = a|p+q||p-q| \geq a|p+q| > kd,$$

an impossibility. Thus, the primes $p > kd$ for which there is an element of T divisible by p^2 correspond to distinct positive integer multipliers $a < k^{0.9}$. In particular, we deduce that there are $< k^{0.9}$ such primes. Hence, there are also $< k^{0.9} < 0.01k$ elements of T divisible by the square of a prime exceeding kd. We obtain then that there are at least $0.57k$ elements of T that are not divisible by the square of a prime, and the result follows. □

The main idea behind the proof of Lemma 1 comes from the study of gaps between squarefree numbers. Using [4], the lower bound can easily be sharpened further to obtain that $n \ge k^{5-\varepsilon}$ for any $\varepsilon > 0$. For our purposes, we only need the above weaker version of the lemma. In fact, something considerably weaker would also do. Our interest is in the following result.

Corollary 1. *Let α be such that $0 < \alpha \le k^{0.8}$, and let T be as in* (3)*. For k sufficiently large, the numbers x_j, with $0 \le j < k$, for which $a_j \le \alpha k$ are distinct.*

Proof. By Lemma 1, we have $n > Ck^{2.8}$. If $a_j \le \alpha k$, then we obtain from $a_j x_j^2 \ge n$ that

$$x_j^2 \ge \frac{n}{\alpha k} > \frac{Ck^{2.8}}{\alpha k} = \frac{Ck^{1.8}}{\alpha} \ge dk.$$

We deduce that there can be at most one multiple of x_j^2 in T, and the corollary follows. □

3 The Main Lemma

For the moment, consider the case $d = 2$. Let t_j denote the jth odd squarefree number. The prior approach to obtaining the solutions to (1) given (2) is to combine a lower bound and an upper bound on $\sum_{0\le j<k} \log a_j$. The lower bound is obtained from

$$\sum_{0\le j<k} \log a_j \ge \sum_{0\le j<k} \log t_{j+1} \tag{4}$$

and a fairly precise estimate for this last sum. The upper bound is obtained using an approach of Erdős already used in the proof of Theorem 2. Here, the approach can be described roughly as follows. For each prime $p \le Ck$, one can bound the number of a_j divisible by p by estimating the number of $j \in \{0, 1, \dots, k-1\}$ for which $\nu_p(n + jd)$ is odd. If the number of such j is $s(p)$, then

$$\sum_{0\le j<k} \log a_j \le \sum_{p\le Ck} s(p) \log p. \tag{5}$$

We deduce by combining the above estimates that

$$\sum_{p\le Ck} s(p) \log p \ge \sum_{0\le j<k} \log t_{j+1}.$$

Ideally, we want appropriate estimates for each side of this inequality to lead to a contradiction when k is large. Observe that the right side is independent of C. As a consequence, this approach seemingly is bound to fail when C is large.

We modify the above idea. To understand the modification, it helps to examine $s(p)$ more closely. The condition $\gcd(n,d)=1$ in (2) implies that $s(p)=0$ if $p|d$. We observe that otherwise we have

$$s(p) \le \begin{cases} 1 & \text{if } k \le p \le Ck \\ 2 & \text{if } k/2 \le p < k \\ 3 & \text{if } k/3 \le p < k/2 \\ \vdots & \quad\vdots \end{cases}$$

These bounds on $s(p)$ are in some sense best possible. Although we cannot hope to do better, what we will show is that, as p varies, if $s(p)$ takes many values near the upper bound indicated above, then typically a_j is considerably larger than t_{j+1}. In other words, if the bounds for $s(p)$ are near the upper bounds suggested above, at least on average, then the lower bound for $\sum_{0\le j<k} \log a_j$ given by (4) can be improved.

We elaborate on the details of this idea next. We no longer restrict d to being 2. Our main improvement is based on the following lemma.

Lemma 2. *Let k be a positive integer. Fix positive real numbers α and β, possibly depending on k, with $\beta < 1$. Let*

$$A = [1, \alpha k] \cap \{a_0, a_1, \dots, a_{k-1}\} \quad \textit{and} \quad J = [\beta k, k).$$

Then there are at most

$$\left\lfloor \frac{\alpha}{\beta} \right\rfloor^2 \left\lfloor \frac{2\sqrt{2\alpha}+\beta}{\beta} \right\rfloor \cdot \left\lfloor \frac{4d\alpha+\beta^2}{\beta^2} \right\rfloor$$

different primes p in J with the property that $p|a$ for two or more different $a \in A$.

Proof. We only consider $\alpha < k^{1/2}$ since the result is trivial for larger (and somewhat smaller) values of α. Observe that if $p \in J$ and $p|a$ for some $a \in A$, then

$$1 \le \frac{a}{p} \le \frac{\alpha k}{\beta k} = \alpha/\beta.$$

As a/p is also an integer, there are $\le \alpha/\beta$ possibilities for a/p. Suppose that p and q are primes in J such that

$$\frac{a_i}{p} = \frac{a_u}{q} \quad \text{and} \quad \frac{a_j}{p} = \frac{a_v}{q}, \tag{6}$$

where

$$a_i, a_j, a_u, a_v \in A,\ p|a_i,\ p|a_j,\ q|a_u,\ q|a_v.$$

Since $a_i x_i^2 \ge n$ and $a_i \le \alpha k$, we have that $x_i \ge \sqrt{n/(\alpha k)}$. On the other hand, $a_i x_i^2 \le n + (k-1)d \le 2n$ and $p|a_i$ so that $x_i \le \sqrt{2n/p} \le \sqrt{2n/(\beta k)}$. Similar arguments hold for bounding x_j, x_u and x_v. Hence,

$$\sqrt{n/(\alpha k)} \le x_i, x_j, x_u, x_v \le \sqrt{2n/(\beta k)}. \tag{7}$$

Observe that

$$\left|(a_i/p)x_i^2 - (a_j/p)x_j^2\right| \le (k-1)d/p \le d/\beta$$

$$\big|(a_u/q)x_u^2-(a_v/q)x_v^2\big|\le (k-1)d/q\le d/\beta.$$

Setting

$$X=x_v^2\big((a_i/p)x_i^2-(a_j/p)x_j^2\big)-x_j^2\big((a_u/q)x_u^2-(a_v/q)x_v^2\big),$$

we see that

$$|X|\le \frac{d(x_j^2+x_v^2)}{\beta}\le \frac{4dn}{\beta^2 k}.$$

From (6), we also have

$$|X|=\frac{a_i}{p}\big|x_i^2x_v^2-x_j^2x_u^2\big|\ge \big|x_ix_v+x_jx_u\big|\big|x_ix_v-x_jx_u\big|\ge \frac{2n}{\alpha k}\big|x_ix_v-x_jx_u\big|.$$

Therefore,

$$\big|x_ix_v-x_jx_u\big|\le \frac{2d\alpha}{\beta^2}. \tag{8}$$

For the moment, view p, x_i and x_j as fixed. We bound the number of distinct pairs (x_u,x_v) satisfying (7) and (8). Observe that if $\delta=\gcd(x_i,x_j)$, then $a_ix_i^2$ and $a_jx_j^2$ are both divisible by $p\delta^2$. Two multiples of $p\delta^2$ in the arithmetic progression $n+jd$ with difference d must differ by at least $p\delta^2 d\ge \beta k\delta^2 d$. On the other hand, $a_ix_i^2$ and $a_jx_j^2$ differ by at most $(k-1)d$. Hence, $\delta\le 1/\sqrt{\beta}$. For a fixed integer $t\in[-2d\alpha/\beta^2,2d\alpha/\beta^2]$, if $x_ix_v-x_jx_u=t$, then the integer pairs (x,y) satisfying $x_ix-x_jy=t$ are given by

$$x=x_v+\frac{x_js}{\delta},\ y=x_u+\frac{x_is}{\delta},\qquad \text{where } s\in\mathbb{Z}.$$

Due to (7), we are interested in the case that

$$\frac{x_j}{\delta}\ge\frac{\sqrt{n/(\alpha k)}}{1/\sqrt{\beta}}=\frac{\sqrt{\beta n}}{\sqrt{\alpha k}}\qquad\text{and}\qquad\left|x_v+\frac{x_js}{\delta}\right|\le\frac{\sqrt{2n}}{\sqrt{\beta k}}.$$

We deduce that there can be at most

$$\frac{2\sqrt{2n}}{\sqrt{\beta k}}\times\frac{\sqrt{\alpha k}}{\sqrt{\beta n}}+1=\big(2\sqrt{2\alpha}/\beta\big)+1$$

different values of s. Hence, we have $\big(2\sqrt{2\alpha}/\beta\big)+1$ as an upper bound on the number of possibilities for x_u and x_v satisfying (7) and $x_ix_v-x_jx_u=t$ for a fixed $t\in[-2d\alpha/\beta^2,2d\alpha/\beta^2]$. Letting t vary, we get the upper bound

$$B=\left\lfloor\frac{2\sqrt{2\alpha}+\beta}{\beta}\right\rfloor\cdot\left\lfloor\frac{4d\alpha+\beta^2}{\beta^2}\right\rfloor$$

on the total number of distinct pairs (x_u,x_v) that can satisfy (7) and (8). This includes the solution $x_u=x_i$ and $x_v=x_j$.

Set

$$N=\left\lfloor\frac{\alpha}{\beta}\right\rfloor^2\left\lfloor\frac{2\sqrt{2\alpha}+\beta}{\beta}\right\rfloor\cdot\left\lfloor\frac{4d\alpha+\beta^2}{\beta^2}\right\rfloor+1.$$

Observe that if we consider $\geq N$ pairs (u, v) of positive integers with each of u and v being $\leq \alpha/\beta$, then the pigeon-hole principle implies that there must be some pair that occurs $> B$ times. Assume that there are $\geq N$ different primes p in J with the property that $p|a_p$ and $p|a'_p$ for distinct $a_p, a'_p \in A$. Then some pair $(a_p/p, a'_p/p)$ occurs for $> B$ primes. Let $\mathcal{P}$ be such a set of primes in J so that, in particular,

$$|\mathcal{P}| > B. \tag{9}$$

For $p \in \mathcal{P}$, let x_p and x'_p be such that $a_p x_p^2$ and $a'_p(x'_p)^2$ are among the numbers $n, n+d, \ldots, n+(k-1)d$. Thus, if q is in $\mathcal{P}$, then we have that

$$\frac{a_p}{p} = \frac{a_q}{q}, \quad \frac{a'_p}{p} = \frac{a'_q}{q}, \quad \left|x_p x'_q - x'_p x_q\right| \leq \frac{2d\alpha}{\beta^2}$$

and, furthermore, that there are $\leq B$ distinct possibilities for the pair (x_q, x'_q). From (9), we deduce that some pair (x_q, x'_q) is repeated. Recalling that $\alpha < k^{1/2}$, we obtain a contradiction to Corollary 1. Hence, the proof is complete. □

4 Proof of Theorem 1

We will make use of the following result.

Lemma 3. *Let s_j denote the jth squarefree positive integer. There is an m_0 such that if m is an integer $\geq m_0$, then*

$$\prod_{j=1}^{m} s_j \geq (1.6)^m m!. \tag{10}$$

We note that the above result is an easy consequence of the fact that the squarefree integers have asymptotic density $6/\pi^2$. The reader can consult [5] for details. For the approach below, we can also manage with the weaker and trivial estimate $s_j \geq j$ instead of Lemma 3. Presumably, Lemma 3 will, however, help in obtaining effective results for specific C.

We set $\alpha = k^{0.12}$ and $\beta = e^{-33C}$ in Lemma 2. Let U be the set of $j \in \{0, 1, \ldots, k-1\}$ for which $a_j \leq \alpha k$, and let W be the set of $j \in \{0, 1, \ldots, k-1\}$ for which $a_j > \alpha k$. In particular, we have

$$|U| + |W| = k.$$

It suffices to consider k large and, in particular, $k \geq 2m_0$. We set

$$m = k - \left\lfloor \frac{30Ck}{\log k} \right\rfloor$$

in Lemma 3. We use that either (i) $|U| > m$ or (ii) $|U| \leq m$. In the case of (i), we have

$$\prod_{j \in U} a_j \geq \prod_{j=1}^{m} s_j \geq (1.6)^m m!.$$

We use the simple inequality $m! \geq m^m/e^m$ which follows by observing $e^m = \sum_{j=0}^{\infty} m^j/j! \geq m^m/m!$. Thus, still in the case of (i), we deduce

$$\begin{aligned}\sum_{j\in U}\log a_j \geq \sum_{j=1}^{m}\log s_j &\geq m\log(k/2)+m\big(\log(1.6)-1\big)\\ &\geq m\log k + m\big(\log(1.6)-1-\log 2\big) \geq k\log k-(30C+1.23)k.\end{aligned}$$

Observe that primes $\geq k$ can divide at most one a_j. Hence, Lemma 2 implies that there is a constant C' depending on C and d such that for all but $\leq C'\alpha^{3.5} \leq C'k^{0.5}$ primes $p \in [\beta k, Ck]$, there is at most one $j \in U$ such that $p|a_j$. For each of the $\leq C'k^{0.5}$ primes $p \in [\beta k, Ck]$ for which there is more than one $j \in U$ such that $p|a_j$, we use that there are at most

$$\left\lfloor \frac{k}{p}\right\rfloor + 1 \leq \left\lfloor \frac{k}{\beta k}\right\rfloor + 1 \leq \frac{1}{\beta}+1$$

such j. Observe also that such p are necessarily $\leq k$ so that $\log p \leq \log k$ for such p. For each $p < \beta k$, we simply use the upper bound $\lfloor k/p\rfloor + 1$ on the number of $j \in U$ for which $p|a_j$. We obtain

$$\sum_{j\in U}\log a_j \leq \sum_{p\leq Ck}\log p + \frac{C'k^{0.5}}{\beta}\log k + \sum_{p\leq\beta k}\frac{k\log p}{p}.$$

It is not difficult to estimate these sums, but we note that one can appeal to Theorem 4 and Theorem 6 of [10]. Since k is sufficiently large, we easily deduce that

$$\sum_{j\in U}\log a_j \leq k\log k + (1.5C+\log\beta)k = k\log k - 31.5Ck.$$

This contradicts the lower bound we had for the sum; hence, we are done in the case of (i).

Suppose now that (ii) holds. Then we must have $|W| \geq \lfloor 30Ck/\log k\rfloor$. Since all of the prime divisors of each a_j are $\leq Ck$ and the a_j are squarefree, we deduce

$$\sum_{j=0}^{k-1}\log a_j \leq \sum_{p\leq Ck}\left(\left\lfloor\frac{k}{p}\right\rfloor+1\right)\log p \leq k\sum_{p\leq k}\frac{\log p}{p} + \sum_{p\leq Ck}\log p \leq k\log k + 2Ck,$$

where again we can appeal to Theorem 4 and Theorem 6 of [10] for estimates on the sums. On the other hand,

$$\sum_{j=0}^{k-1}\log a_j = \sum_{j\in U}\log a_j + \sum_{j\in W}\log a_j \geq \sum_{j=1}^{|U|}\log s_j + |W|\log(\alpha k).$$

Since $|U| \leq k$, in the last sum each s_j is easily $\leq 2k$. On the other hand, $\alpha k = k^{1.12} > 2k$. As $|U|+|W| = k$, we can find a lower bound for the right-hand expression above by setting

$$|U| = k - \left\lfloor\frac{30Ck}{\log k}\right\rfloor = m \quad\text{and}\quad |W| = \left\lfloor\frac{30Ck}{\log k}\right\rfloor.$$

We deduce

$$\sum_{j=0}^{k-1}\log a_j \geq \sum_{j=1}^{m}\log s_j + \left\lfloor\frac{30Ck}{\log k}\right\rfloor\log\left(k^{1.12}\right).$$

Appealing to the earlier estimate for this last sum, we deduce

$$\sum_{j=0}^{k-1} \log a_j \geq k \log k - (30C + 1.23)k + 33.6Ck - 1.12 \log k \geq k \log k + 2.3Ck.$$

Thus, in this case, we also obtain a contradiction.

Summarizing, we deduce that if k is sufficiently large, then there are no solutions to (1) and (2). As we have seen, this implies the result stated in the theorem.

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