FIBONACCI NUMBERS OF THE FORM $x^a \pm x^b + 1$

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Abstract. In this paper, we show that the diophantine equation $F_n = x^a \pm x^b + 1$ with x composed of two prime divisors has only finitely many positive integer solutions (n, x, a, b) and $\max\{a, b\} \geq 2$.

1. Introduction

The well known Fibonacci sequence denoted by $(F_n)_{n \geq 0}$ is the sequence of integers given by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. This sequence has been well studied.

We consider the equation

$$F_n = x^a \pm x^b + 1 \tag{1}$$

in positive integer variables n, x, a, b with $\max\{a, b\} \geq 2$. When $x = p$, a prime, it was proved by Luca and Szalay [LuSz08] that (1) has only finitely many positive integer solutions (n, p, a, b). We extend their result when x is composed of exactly two primes.

Theorem 1. *Equation* (1) *with x divisible by exactly two primes has only finitely many positive integer solutions* (n, x, a, b). *In fact max*$\{a, b\} < 1.25 \cdot 10^{15}$.

Bennett and Bugeaud [BeBu12] also considered a similar equation (1) with F_n replaced by y^q.

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2. Proof of Theorem 1

For the proof, we need the following result on explicit bounds for linear forms in logarithms due to Matveev [Mat00].

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Lemma 2.1. *Let L be a real number field of degree D and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero elements of L and $b_1, b_2, \ldots, b_n$ be rational integers. Set $B = \max\{b_1, \ldots, b_n\}$ and*

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1.$$

Let h be the absolute logarithmic height and let $A_1, \ldots, A_n$ be real numbers with

$$A_j \geq \max\{Dh(\alpha_j), \log \alpha_j, 0.16\}, \quad 1 \leq j \leq n.$$

Assume $\Lambda \neq 0$. Then

$$\log |\Lambda| \geq -1.4 \cdot 30^{n+3} n^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_n.$$

Proof of Theorem 1: Without loss of generality, we may assume that $a \geq b$. We rewrite (1) as

$$F_n - 1 = x^b(x^{a-b} \pm 1).$$

Since $F_1 = F_2 = 1$, as it follows from [LuSz08, Lemma 2] that

$$F_{\frac{n-\delta}{2}} L_{\frac{n+\delta}{2}} = x^b(x^{a-b} \pm 1). \tag{2}$$

where

$$\delta = \begin{cases} 1 & \text{if } n \equiv 1(4) \\ -1 & \text{if } n \equiv -1(4) \\ 2 & \text{if } n \equiv 2(4) \\ -2 & \text{if } n \equiv 0(4) \end{cases}$$

Since $F_{\frac{n-\delta}{2}} | F_{n-\delta}, L_{\frac{n+\delta}{2}} | F_{n+\delta}$ and $\gcd(F_u, F_v) = F_{(u,v)}$ holds for all positive integers (u, v), we get that

$$\gcd(F_{\frac{n-\delta}{2}}, L_{\frac{n+\delta}{2}}) | \gcd(F_{n-\delta}, F_{n+\delta}) | F_{2\delta|} | F_4 = 3.$$

Hence $\gcd(F_{\frac{n-\delta}{2}}, L_{\frac{n+\delta}{2}}) = 1$ or 3 and it is 1 if $n \equiv 0, \pm 1 (\text{mod } 4)$. Further note that $x^a \pm x^b + 1$ is always odd thereby implying $3 \nmid n$. Then either $3 | \frac{n-\delta}{2}$ or $3 | \frac{n+\delta}{2}$ and we write $\frac{n \pm \delta}{2} = 3k$, respectively. We have

$$F_{3k} = F_k(L_k^2 \pm 1) \ \text{ and } \ L_{3k} = L_k(L_k^2 \pm 3).$$

Hence we have from (2) that

$$x^b(x^{a-b} \pm 1) = \begin{cases} F_{3k} L_{3k-\delta} = F_k(L_k^2 \pm 1) L_{3k-\delta} & \text{if } \frac{n+\delta}{2} = 3k \\ F_{3k+\delta} L_{3k} = F_{3k+\delta} L_k (L_k^2 \pm 3) & \text{if } \frac{n-\delta}{2} = 3k. \end{cases} \tag{3}$$

Write $x^b(x^{a-b} \pm 1) = G_1 G_2 G_3$. We observe here that pairwise gcd of G_1, G_2, G_3 is either 1 or 3.

Let $x = p_1^{e_1} p_2^{e_2}$. Let $a = b$. Then $G_1 G_2 G_3 = 2x^a = 2p^{ae_1} q^{ae_2}$ implying either $G_1 \in \{1, 2\}$ or $G_2 \in \{1, 2\}$ or $G_3 \in \{1, 2\}$ implying $k \leq 2$ or $n < 15$ and we see that there are no solutions. Hence we may assume $a > b$. From (3), we get x^b divides either $3G_1G_2$ or $3G_2G_3$ or $3G_3G_1$. Therefore

$$x^b \leq 18\alpha^{5k} \leq 36\alpha^{\frac{5n}{6}}.$$

Again $2x^a \geq \alpha^n$ implying $x^a \geq \frac{\alpha^n}{2} \geq \frac{1}{2}(\frac{x^b}{36})^{\frac{6}{5}}$. Hence $a \geq \frac{6b}{5}$ or $a-b \geq \frac{a}{6}$ for $a > b > 10$. This with (2) implies

$$|\frac{\alpha^n}{\sqrt{5}} - x^a| = |x^b + \frac{\beta^n}{\sqrt{5}} + 1| < 2x^b$$

giving

$$|\frac{\alpha^n x^{-a}}{\sqrt{5}} - 1| < 2x^{-(a-b)} < 2x^{-\frac{a}{6}}.$$

By linear forms in three logarithms as in [?], a is bounded and hence b is bounded.

We now compute an explicit bound for a. Let $n = aq + r$ with $0 \leq r < a$. Then

$$|\frac{\alpha^{-r}}{\sqrt{5}}(\frac{\alpha^q}{x})^{-a} - 1| < 2x^{-\frac{a}{6}} < \frac{1}{2}.$$

The minimal polynomial of $\frac{\alpha^q}{x}$ is $x^2Y^2 - (\alpha^q + \beta^q)xY + (-1)^q$ and its conjugate is $\frac{\beta^q}{x}$ and

$$\frac{\alpha^q}{x} < (\frac{3\sqrt{5}\alpha^r}{2})^{\frac{1}{b}} \leq (\frac{3\sqrt{5}}{2\alpha})^{\frac{1}{b}}\alpha < 2.$$

Hence the logarithmic height

$$h(\frac{\alpha^q}{x}) = \frac{1}{2}(\log x^2 + \log\frac{\alpha^q}{x}) \leq \log x + \frac{\log 2}{2}.$$

Also

$$h(\sqrt{5}) = \frac{\log 5}{2} \quad \text{and} \quad h(\alpha) = \frac{\log \alpha}{2}.$$

By Lemma 2.1, we obtain

$$\log 2 - \frac{a \log x}{6} > \log|\frac{\alpha^{-r}}{\sqrt{5}}(\frac{\alpha^q}{x})^{-a} - 1| > -c(1 + \log a)(2\log x + \log 2)$$

where

$$c = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4(1 + \log 2)(\log 5)(\log\frac{1+\sqrt{5}}{2}).$$

Hence

$$\frac{a}{\log a} < \frac{6\log 2}{\log a \log x} + 6c(1 + \frac{1}{\log a})(2 + \frac{\log 2}{\log x}) < 1 + 14c < 3.54 \cdot 10^{13}.$$

This give $a < 1.25 \cdot 10^{15}$. Hence $b < a < 1.25 \cdot 10^{15}$.

We may now assume that both a and b are fixed. We are interested in finding n and X such that F_n takes the polynomial value $X^a \pm X^b + 1$. In [NePe86], all polynomials

$P(X) \in \mathbb{Q}[X]$ of degree ≥ 2 such that the diophantine equation $F_n = P(X)$ admits infinitely many integer solutions (n, X) have been completely classified.

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