IRREDUCIBILITY OF GENERALIZED HERMITE-LAGUERRE POLYNOMIALS III

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Abstract. For a positive integer n and a real number α, the generalized Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{j!(n-j)!}.$$

These orthogonal polynomials are solutions to *Laguerre's Differential Equation* which arises in the treatment of the harmonic oscillator in quantum mechanics. Schur studied these Laguerre polynomials for its interesting algebraic properties. He obtained irreducibility results of $L_n^{(\pm\frac{1}{2})}(x)$ and $L_n^{(\pm\frac{1}{2})}(x^2)$ and derived that the Hermite polynomials $H_{2n}(x)$ and $\frac{H_{2n+1}(x)}{x}$ are irreducible for each n. In this article, we extend Schur's result by showing that the family of Laguerre polynomials $L_n^{(q)}(x)$ and $L_n^{(q)}(x^d)$ with $q \in \{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}\}$, where d is the denominator of q, are irreducible for every n except when $q = \frac{1}{4}, n = 2$ where we give the complete factorization. In fact, we derive it from a more general result.

1. Introduction

For a positive integer n and a real number α, the generalized Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{j!(n-j)!}.$$

Let $d > 1$ be an integer and q be a rational number with denominator equal to d written in its reduced form

$$q = u + \frac{\alpha}{d}$$

where $u, \alpha \in \mathbb{Z}$ with $1 \le \alpha < d$ and $\gcd(\alpha, d) = 1$. For integers $a_0, a_1, \cdots a_n$, let

$$\begin{aligned} G(x) := G_q(x) &= \sum_{j=0}^{n} a_j (n+q)(n-1+q)\cdots(j+1+q) d^{n-j} x^j \\ &= \sum_{j=0}^{n} a_j x^j \left(\prod_{i=j+1}^{n} (\alpha + (u+i)d) \right). \end{aligned}$$

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This is an extension of Hermite polynomials and generalized Laguerre polynomials. In fact, when $a_j = (-1)^j\binom{n}{j}$, we obtain $d^n n! L_n^{(q)}(\frac{x}{d})$ and Hermite polynomials are given by

$$H_{2n}(x) = (-1)^n 2^{2n} n! L^{(-\frac{1}{2})}(x^2) \text{ and } H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L^{(\frac{1}{2})}(x^2).$$

Therefore we call $G(x)$ the generalized Hermite-Laguerre polynomial. We have

$$G(x^d) := G_q(x^d) = \sum_{j=0}^{dn} b_j x^j \quad \text{where} \quad b_j = \begin{cases} a_l \prod_{i=l+1}^{n} (\alpha + (u+i)d) & \text{if } j = dl \\ 0 & \text{otherwise.} \end{cases}$$

We observe that the irreducibility of $G_q(x^d)$ implies the irreducibility of $G_q(x)$. There is a slight difference in the notation of this paper from that of [ShTi10], [LaSh12] and [LaSh09]; $G_q(x)$ here is $G_{q+1}(x)$ in the above papers. The first result on the irreducibility of these polynomials is due to Schur. Schur [Sch29] proved that $G_{-\frac{1}{2}}(x^2)$ with $a_n = \pm 1$ and $a_0 = \pm 1$ are irreducible and this implies the irreducibility of Hermite poynomial H_{2n}. Schur [Sch31] also established the irreducibility of $\frac{H_{2n+1}(x)}{x}$ by showing that $G_{\frac{1}{2}}(x^2)$ with $a_n = \pm 1$ and $a_0 = \pm 1$ is irreducible except for $n = 12$ where it may have a quadratic factor. In this paper, we extend Schur's result by proving

Theorem 1. *Let $q \in \{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}\}$. The Laguerre polynomials $L_n^{(q)}(x)$ and $L_n^{(q)}(x^d)$, where d is the denominator of q, are irreducible for every n except when $q = \frac{1}{4}, n = 2$ where*

$$L_2^{(\frac{1}{4})}(x) = \frac{1}{32}(4x-3)(4x-15) \text{ and } L_2^{(\frac{1}{4})}(x^4) = \frac{1}{32}(4x^4-3)(4x^4-15).$$

In fact we derive Theorem 1 from the following general result extending the theorems of [LaSh12] and [LaSh09]. For a non-zero integer m, we denote by $P(m)$ the greatest prime divisor of m with the convention $P(\pm 1) = 1$. Observe that if a polynomial of degree m has a factor of degree $k < m$, then it has a co-factor of degree $m - k$. Therefore when we consider a factor of a polynomial of degree m, we always mean the factor whose degree is $\leq \frac{m}{2}$.

Theorem 2. *Let $q \in \{\pm\frac{1}{3}, \pm\frac{2}{3}\}$. Assume that $P(a_0 a_n) \leq 3$ and further $2 \nmid a_0 a_n$ if $\alpha + 3(n+u)$ is a power of 2. Then the polynomials $G(x)$ and $G(x^3)$ with $q \in \{-\frac{1}{3}, -\frac{2}{3}\}$ are both irreducible except when $q = -\frac{2}{3}, n = 2$ where $G(x)$ may have a linear factor and $G(x^3)$ may have a cubic factor or when $q = -\frac{1}{3}, n = 43$ where $G(x^3)$ may have a factor of degree 5. Further the polynomials $G(x)$ and $G(x^3)$ with $q \in \{\frac{1}{3}, \frac{2}{3}\}$ are both irreducible except possibly when*

(i) *$1 + 3n = 2^a$ where $G_{\frac{1}{3}}(x)$ may have a linear factor and $G_{\frac{1}{3}}(x^3)$ may have a quadratic or a cubic factor.*
(ii) *$2 + 3n = 2^a$ and $n \neq 42$ where $G_{\frac{2}{3}}(x^3)$ may have a quadratic factor.*
(iii) *$2 + 3n = 2^b 5^c, b \geq 0, c > 0$ where $G_{\frac{2}{3}}(x)$ may have a linear factor and $G_{\frac{2}{3}}(x^3)$ may have a cubic factor.*

(iv) $n = 42$ *where* $G_{\frac{2}{3}}(x)$ *may have a quadratic factor and* $G_{\frac{2}{3}}(x^3)$ *may have a factor of degree in* $\{2, 4, 5, 6\}$.

Theorem 3. *Let* $q \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$. *Assume that* $P(a_0a_n) \leq 3$ *and further* $P(a_0a_n) \leq 2$ *if* $\alpha + 4(n+u)$ *is a power of* 3 *when* $q \in \{-\frac{1}{4}, -\frac{3}{4}\}$ *and* $3|(\alpha + 4n)$ *when* $q \in \{\frac{1}{4}, \frac{3}{4}\}$. *Then the polynomials* $G_{-\frac{3}{4}}(x)$ *and* $G_{-\frac{3}{4}}(x^4)$ *are both irreducible. Further* $G_{\pm\frac{1}{4}}(x), G_{\pm\frac{1}{4}}(x^4), G_{\frac{3}{4}}(x)$ *and* $G_{\frac{3}{4}}(x^4)$ *are irreducible except possibly when* $3+4(n-1) = 3^a$ *if* $q = -\frac{1}{4}$; $1 + 4n = 3^b5^c, b, c \geq 0, b + c > 0$ *if* $q = \frac{1}{4}$ *and* $3 + 4n = 7^y$ *if* $q = \frac{3}{4}$ *where* $G_q(x)$ *may have a linear factor and* $G_q(x^4)$ *may have a factor of degree* 4.

It follows from Theorem 3 that if n is a multiple of 3, then $G_q(x^4)$ is irreducible for $q \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$. In Theorem 2, the case $q = -\frac{2}{3}, n = 2$ is necessary since $G_q(x) = (x+2)^2$ and $G_q(x) = (x^3+2)^2$ when $a_0 = a_1 = a_2 = 1$. The assumptions on a_0a_n in Theorems 2 and 3 are satisfied if $|a_0| = |a_n| = 1$; in fact the assumptions of Theorem 3 are satisfied if $P(a_0a_n) \leq 2$. Therefore the assertions of Theorems are valid whenever $|a_0| = |a_n| = 1$ and further for Theorem 3 whenever $P(a_0a_n) \leq 2$. We believe that for suitable choices of a_j's, many of the polynomials $G_q(x)$ with conditions given in Theorems 2 and 3 will have linear factor or $G(x^d)$ will have a factor of degree $\leq d$ but we have not found out examples for the same. It will be interesting to either give such examples or prove irreducibility completely for those cases.

The proofs of Theorems 2 and 3 are given in Sections $5-7$. Further we prove Theorem 1 in Section 8. The following result used in the proof of Theorem 3 is also of independent interest.

Theorem 4. *Let* $k \geq 2, n > 4k$ *and* $2 \nmid n$. *Then*

$$P(n(n+4)\cdots(n+4(k-1))) > 4(k+1) \tag{1}$$

unless $k = 2, n \in \{11, 21, 45, 77, 121\}$ *and* $k = 3, n = 117$.

As an immediate consequence of Theorem 4, we obtain

Corollary 1.1. *Let* $k \geq 2, n > 4k$ *and* $2 \nmid n$. *Then*

$$P(n(n+4)\cdots(n+4(k-1))) > 4k \tag{2}$$

unless $k = 2, n \in \{21, 45\}$.

We give a proof of Theorem 4 in Section 4. In Section 2, we give some preliminaries and in Section 3, we give statements and results on Newton polygons.

The proof of Theorems 1-3 involve combinations of ideas of $p-$adic Newton polygons with estimates on the greatest prime factor of a product of consecutive terms of an arithmetic progression. The new ingredients in the paper are Theorem 4 and the exploitation of arithmetic properties of some special numbers arising out of application of Newton polygon ideas and extending the arguments for $G_q(x)$ to $G_q(x^d)$ where d is the denominator of q.

2. Preliminaries

For positive integers m, d, k, we write

$$\Delta(m,d,k) = m(m+d)\cdots(m+d(k-1)).$$

Recall that for an integer $m > 1$, we denote by $P(m)$ the greatest prime factor of m and we put $P(1) = 1$. The following result is [LaSh12, Theorem 3].

Lemma 2.1. *Let $k \geq 2$ and $d = 3$. Let m and k be positive integers such that $3 \nmid m$ and $m > 3k$. Then*

$$P(\Delta(m,3,k)) > 3k \quad \text{unless} \quad (m,k) = (125,2). \tag{3}$$

For a prime p and a nonzero integer r, we define $\nu(r) = \nu_p(r)$ to be the nonnegative integer such that $p^{\nu(r)} | r$ and $p^{\nu(r)+1} \nmid r$. We define $\nu(0) = +\infty$. The following classical result is due to Legendre. See for example, Hasse [Hasse, Ch. 17, no. 3, p. 263].

Lemma 2.2. *Let p be a prime. For any integer $m \geq 1$, write m in base p as*

$$m = m_t p^t + m_{t-1} p^{t-1} + \cdots + m_1 p + m_0$$

where $0 \leq m_i \leq p-1$ for $0 \leq i \leq t$. Then

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1}$$

where $s_p(m) = m_t + m_{t-1} + \cdots + m_1 + m_0$ is the sum of digits of m in base p. In particular $\nu_p(m!) \leq \frac{m-1}{p-1}$ since $s_p(m) \geq 1$.

The next lemma is on solutions of some equations.

Lemma 2.3. *Let $x > 0, y > 0, z > 0$ be integers. The solutions of the following equations are given by*

	Equation	*Solutions*
(i)	$a^x - b^y = \pm 1, a, b \in \{2,3,5\}$	$3-2=1, 2^2-3=1, 5-2^2=1, 3^2-2^3=1$
(ii)	$2^x + 3^y = 5^z$	$2+3=5, 2^4+3^2=5^2$
(iii)	$2^x + 3^y = 7^z$	$2^2+3=7$
(iv)	$2^x 3^y - 5^z = \pm 1$	$2\cdot 3-5=1, 2^3\cdot 3-5^2=-1$
(v)	$3^x 5^y - 2^z = \pm 1$	$3\cdot 5-2^4=-1$
(vi)	$2^x 5^y - 3^z = \pm 1$	$2\cdot 5-3^2=1, 2^4\cdot 5-3^4=-1$

The assertion (i) is a special case of Catalan's Conjecture, now Mihailescu's Theorem when $x > 1, y > 1$, see [Mih04]. The case $x = 1$ or $y = 1$ is immediate. The assertions (ii) and (iii) are due to Nagell [Nag58]. For assertions $(iv) - (vi)$, see [LaSh06a, Lemma 4].

The next lemma is [LaSh12, Corollary 2.12] together with computations for $X \leq 80$.

Lemma 2.4. *Let $X \geq 1, 3 \nmid X$ and $1 \leq i \leq 7$. Then the solutions of*

$$P(X(X+3i)) = 5 \text{ and } 2|X(X+3i)$$

are given by

$$(i, X) \in \{(1,2),(1,5),(1,125),(2,4),(2,10),(2,250),(3,1),(3,16),(4,8),\\ (4,20),(4,500),(5,5),(5,10),(5,25),(5,625),(6,2),(6,32),(7,4)\}.$$

We also need the following result which is [LaSh12, Corollary 2.3] and [LaSh09, Corollary 4.3].

Lemma 2.5. *Let $d \in \{3,4\}$, $\gcd(n,d) = 1$ and $6450 < n \leq 10.6 \cdot 3k$ if $d = 3$ and $10^6 < n \leq 138 \cdot 4k$ if $d = 4$. Then $P(\Delta(n,d,k)) \geq n$.*

Let $p_{i,\mu,l}$ denote the ith prime congruent to l modulo μ. Let $\delta_\mu(i,l) = p_{i+1,\mu,l} - p_{i,\mu,l}$. The following lemma is a computational result.

Lemma 2.6. (*i*) *Let $l \in \{1,2\}$. Then $\delta_3(i,l) \leq 60$ for $p_{i,3,l} \leq 7348$.*

(*ii*) *Let $l \in \{1,3\}$. Then $\delta_4(i,l) \leq 264$ for $p_{i,4,l} \leq 1.1 \cdot 10^7$ except when $(p_{i,4,l}, p_{i+1,4,l}) \in \{(7856441, 7856713), (10087201, 10087481), (3358151, 3358423), (5927759, 5928031), (9287659, 9287939)\}$.*

3. Newton Polygons

Let $f(x) = \sum_{j=0}^{m} a_j x^j \in \mathbb{Z}[x]$ with $a_0 a_m \neq 0$ and p be a prime. Let S be the following set of points in the extended plane:

$$S = \{(0, \nu(a_m)), (1, \nu(a_{m-1})), (2, \nu(a_{m-2})), \cdots, (m-1, \nu(a_1)), (m, \nu(a_0))\}.$$

Consider the lower edges along the convex hull of these points. The left-most endpoint is $(0, \nu(a_m))$ and the right-most endpoint is $(m, \nu(a_0))$. The endpoints of each edge belong to S and the slopes of the edges increase from left to right. When referring to the edges of a Newton polygon, we shall not allow two different edges to have the same slope. The polygonal path formed by these edges is called the Newton polygon of $f(x)$ with respect to the prime p and we denote it by $NP_p(f)$. The end points of the edges on $NP_p(f)$ are called the *vertices* of $NP_p(f)$. We call the $x-$axis of the vertices to be *breaks* of the Newton polygon and usually write $0 =: x_0 < x_1 < \cdots < x_s := m$ as the breaks where $(x_i, \nu(a_{m-x_i}), 0 \leq i \leq s$ are the vertices of $NP_p(f)$. We define the *Newton function* of f with respect to the prime p as the real function $f_p(x)$ on the interval $[0, m]$ which has the polygonal path formed by these edges as its graph. Hence $f_p(i) = \nu(a_{m-i})$ for $i = 0, m$ and at all points i such that $(i, \nu(a_{m-i}))$ is a vertex of $NP_p(f)$. We need the following result which is a refinement of a lemma due to Filaseta [Fil95, Lemma 2]. This was proved in [ShTi10, Lemma 2.13].

Lemma 3.1. *Let k, m and r be integers with $m \geq 2k > 0$. Let $g(x) = \sum_{j=0}^{m} b_j x^j \in \mathbb{Z}[x]$ and let p be a prime such that $p \nmid b_m$. Denote the Newton function of $g(x)$ with respect to p by $g_p(x)$. Let $a_0, a_1, \ldots, a_m$ be integers with $p \nmid a_0 a_m$. Put $f(x) =$*

$\sum_{j=0}^{m} a_j b_j x^j \in \mathbb{Z}[x]$. *If* $g_p(k) > r$ *and* $g_p(m) - g_p(m-k) < r+1$, *then* $f(x)$ *cannot have a factor of degree* k.

Lemma 3.1 implies the following result of Filaseta [Fil95, Lemma 2] together with a remark just after its proof in [Fil95].

Corollary 3.2. *Let* l, k, m *be integers with* $m \geq 2k > 2l \geq 0$. *Suppose* $g(x) = \sum_{j=0}^{m} b_j x^j \in \mathbb{Z}[x]$ *and* p *be a prime such that* $p \nmid b_m$ *and* $p | b_j$ *for* $0 \leq j \leq m-l-1$ *and the right most edge of the* $NP_p(g)$ *has slope* $< \frac{1}{k}$. *Then for any integers* $a_0, a_1, \ldots, a_m$ *with* $p \nmid a_0 a_m$, *the polynomial* $f(x) = \sum_{j=0}^{m} a_j b_j x^j$ *cannot have a factor with degree in* $[l+1, k]$.

Proof. Since $p | b_j$ for $0 \leq j \leq m-l-1$, we have $g_p(K) > 0$ for $K \in [l+1, k]$. Let $(m_1, g_p(m_1))$ be the starting point of the rightmost edge of $NP_p(g)$. Then

$$\frac{1}{m-m_1} \leq \frac{g_p(m) - g_p(m_1)}{m-m_1} < \frac{1}{k}$$

giving $m_1 < m-k \leq m-K$ for $K \leq k$. Hence for $K \in [l+1, k]$, $(m-K, g_p(m-K))$ lie on the rightmost edge implying $\frac{g_p(m)-g_p(m-K)}{K} < \frac{1}{k} \leq \frac{1}{K}$. Thus $g_p(m) - g_p(m-K) < 1$. Now we apply Lemma 3.1 with $r = 0$ to get the assertion. □

Unless otherwise mentioned, we always take $l = k-1$ while using Corollary 3.2. Next we need the following result generalizing [LaSh09, Lemma 1] where the case $u = -1$ was proved.

Lemma 3.3. *Let* $u \in \{-1, 0\}$ *and* $1 \leq k \leq \frac{n}{2}$. *Suppose there is a prime* p *satisfying*

$$p > d, p > \min(2k, d(d-1)) \text{ and further } p \geq \frac{(k+.5)d}{d-1} \text{ if } u = -1, p \leq 2k \tag{4}$$

and

$$p | \prod_{j=0}^{k-1} (\alpha + (u+n-j)d), \quad p \nmid \prod_{j=1}^{k} (\alpha + (u+j)d), \; p \nmid a_0 a_n.$$

Then $G(x)$ *has no factor of degree* k *and* $G(x^d)$ *does not have a factor of degree in* $[dk-d+1, dk]$. *Further for* n *odd and* $k = \lfloor \frac{n}{2} \rfloor$, $G(x^d)$ *does not have a factor of degree in* $[dk+1, dk+\frac{d}{2}]$.

Proof. We use Corollary 3.2. We take (m, k, l) to be $(n, k, k-1)$ for $G(x)$ having a factor of degree k and $(dn, dk, d(k-1))$ for $G(x^d)$ having a factor of degree in $[dk-d+1, dk]$. Further for n odd and $G(x^d)$ having a factor of degree in $(dn_0, dn_0 + \frac{d}{2}]$ where $n_0 = \lfloor \frac{n}{2} \rfloor$, we take (m, k, l) to be $(dn, dn_0 + \lfloor \frac{d}{2} \rfloor, dn_0)$. We observe that the assumptions of Corollary 3.2 are satisfied. Let

$$\Delta_j = (\alpha + (u+1)d) \cdots (\alpha + (u+j)d).$$

By Corollary 3.2, it suffices to show that

$$\frac{\nu_p(\Delta_j)}{j} < \frac{1}{k+\frac{1}{2}} \quad \text{for } 1 \le j \le n. \tag{5}$$

Let $j_0 \ge 1$ be the minimum j such that $p|(\alpha+(u+j)d)$ and we write $\alpha+(u+j_0)d = pl_0$. Then $j_0 > k$ since $p \nmid \Delta_k$. Note that $j_0 \le p$. Further $1 \le l_0 < d$ otherwise $l_0 \ge d+1$ and $p \le pl_0 - pd = \alpha + (u + j_0 - p)d \le \alpha + ud < d < p$, a contradiction. Also $p(d-1) \ge pl_0 = \alpha + (u + j_0)d \ge \alpha + (u + k + 1)d$. Thus $p(d-1) > (k+1)d$ if $u = 0$. If $u = -1$ and $p > 2k$, we have $p(d-1) \ge (2k+1)(d-1) \ge (k+.5)d$ since $d > 1$. This together with (4) imply

$$p \ge \frac{(k+.5)d}{d-1}. \tag{6}$$

For showing (5), we may restrict to those j such that $\alpha + (u + j)d = pl$ for some l. Then $(j - j_0)d = p(l - l_0)$ implying $d|(l - l_0)$ since $\gcd(p, d) = 1$. Writing $l = l_0 + sd$, we get $j = j_0 + ps$. Note that if $p|(\alpha + (u + i)d)$, then $\alpha + (u + i)d = p(l_0 + rd)$ for some $r \ge 0$. Hence we have

$$\nu_p(\Delta_j) = \nu_p((pl_0)(p(l_0 + d)) \cdots (p(l_0 + sd)) = s + 1 + \nu_p(l_0(l_0 + d) \cdots (l_0 + sd))$$

for some integer $s \ge 0$. Further we may suppose that $s > 0$ otherwise the assertion follows since $p > d > l_0$ and $j_0 > k$. Further from (5), $j = j_0 + ps \ge k + 1 + ps$ and $\frac{k+1+ps}{k+.5} = 1 + \frac{ps+.5}{k+.5}$, it suffices to show

$$\phi_s := s + \nu_p(l_0(l_0 + d) \cdots (l_0 + sd)) < \frac{ps + .5}{k + .5}. \tag{7}$$

We consider two cases.

Case I: Assume that $s < p$. Then p divides at most one term of $\{l_0 + id : 0 \le i \le s\}$ and we obtain from $l_0 + sd < (s + 1)d < p^2$ that $\phi_s \le s + 1$. To show (7), we need to show that $\frac{ps+.5}{k+.5} - 1 > s$ or $s(p - k - \frac{1}{2}) > k$. This is true if $p \ge 2k + 1$. Thus we may suppose that $p \le 2k$. Since $p \ge \frac{(k+.5)d}{d-1}$ by (6), we get $(d - 1)(p - k - .5) > k$. Thus $s(p - k - .5) > k$ is valid for $s \ge d - 1$ and therefore we may now assume $s \le d - 2$. Then $l_0 + sd \le d - 1 + (d - 2)d < p$ and hence $\phi_s = s$. Now (7) is valid since $p \ge k + 1$.

Case II: Let $s \ge p$. Let $r_0 \le s$ be such that $\nu_p(l_0 + r_0d)$ is maximal. Then

$$\phi_s \le s + \nu_p(l_0 + r_0d) + \nu_p(r_0!(s - r_0)!) \le s + \frac{\log(l_0 + sd)}{\log p} + \frac{s - 1}{p - 1}$$

by using Lemma 2.2. We have $p \ge d + 1$. This with $l_0 \le d - 1 < p \le s$ imply $\log(l_0 + sd) \le \log s(d + 1) = \log s + \log(d + 1) \le \log s + \log p$. Hence

$$\phi_s \le s + \frac{s}{p - 1} + \frac{\log s}{\log p} + 1 - \frac{1}{p - 1}.$$

To show (7), it is enough to show that

$$1 + \frac{1}{p - 1} + \frac{\log s}{s \log p} + \frac{1}{s}(1 - \frac{1}{p - 1} - \frac{1}{2k + 1}) < \frac{2p}{2k + 1}.$$

The left hand side of the above inequality is a decreasing function in s. Since $s \geq p$, the left hand side of the above inequality is at most

$$1 + \frac{1}{p-1} + \frac{1}{p} + \frac{1}{p}(1 - \frac{1}{p-1} - \frac{1}{2k+1}) = 1 + \frac{3}{p} - \frac{1}{p(2k+1)}$$

and therefore it suffices to show

$$1 + \frac{3}{p} - \frac{1}{p(2k+1)} < \frac{2p}{2k+1}. \tag{8}$$

Let $p \geq 2k+1$. Then $p \geq 3$ and the left hand side of (8) is at most

$$1 + 1 - \frac{1}{p(2k+1)} < 2 \leq \frac{2p}{2k+1}.$$

Thus we may assume that $p \leq 2k$. Then $p > d(d-1)$. Further $d \geq 3$ since $p(d-1) \geq \alpha + (u+k+1)d$ and $p < 2k$. Therefore the left hand side of (8) is at most

$$1 + \frac{3}{d(d-1)} - \frac{1}{p(2k+1)} < 1 + \frac{1}{d-1} = \frac{d}{d-1} \leq \frac{2p}{2k+1}.$$

by (6). □

The following corollary easily follows from Lemma 3.3.

Corollary 3.4. *Let $u \in \{0, -1\}$ and $n \geq 2k > 0$. Suppose that $P(a_0 a_n) \leq d$ and*

$$P((\alpha + d(u+n-k+1)) \cdots (\alpha + d(u+n))) > d(u+k+1).$$

Then $G_q(x)$ does not have a factor of degree k and $G_q(x^d)$ do not have a factor of degree in $[dk-d+1, dk]$. Further for n odd and $k = \lfloor \frac{n}{2} \rfloor$, $G_q(x^d)$ does not have a factor of degree in $[dk+1, dk+\frac{d}{2}]$.

4. Proof of Theorem 4

Let $k \geq 2, n > 4k$ and $2 \nmid n$. Assume that $P(n(n+4)\cdots(n+4(k-1))) \leq 4(k+1)$. Let

$$S_M = \{m : m \geq 1, m \text{ odd}, P(m(m+4)) \leq M\}.$$

The set S_M for $M \leq 31$ is given in [Leh64] and for $M = 100$ in [Naj10]. In fact, $m = x - 2$ with x listed in the table [Naj10] and $m = N - 4$ for N listed in [Leh64, Table IIIA].

Let $k = 2$. Then $P(n(n+4)) \leq 11$ implying $n \in S_{11}$. Since $n > 8$, we have $n \in \{11, 21, 45, 77, 121\}$.

Let $k = 3$. Then $P(n(n+4)(n+8)) \leq 13$ giving $P(n(n+4)) \leq 13$ and $P((n+4)(n+8)) \leq 13$. Hence both $n \in S_{13}$ and $n+4 \in S_{13}$. Since $n > 12$, we have $n = 117$.

Let $4 \leq k \leq 8$. Since $P(\Delta(n, 4, k)) \leq 4k+4$, we have $P(n(n+4)) \leq 31$, $P((n+4)(n+8)) \leq 31$ and $P((n+8)(n+12)) \leq 31$. Hence $n + 4i \in S_{31}$ for each $0 \leq i \leq 2$.

Then $n \in \{17, 19, 21, 23, 27, 87\}$. For these values n and k such that $n > 4k$, we check that $P(\Delta(n, 4, k)) > 4(k+1)$. Thus $k \geq 9$.

Let $9 \leq k < 67$. Since $P(\Delta(n, 4, k)) \leq 4k+4$, we have $\omega(\Delta(n, 4, k)) \leq \pi(4k+4)$. We check that $k-\pi(4k+4)+\pi(100) > \lceil \frac{k}{2} \rceil$. Hence there is some i_0 with $0 \leq i_0 \leq k-2$ such that $P((n+4i_0)(n+4(i_0+1))) \leq 100$. Then $n+4i_0 = m \in S_{100}$. Suppose $m > 10^7$. We check that $P(\prod_{i=1}^{4}(m-4i)) > 280$ and $P(\prod_{i=1}^{4}(m+4+4i)) > 280$ for each $m \in S_{100}$ and $m > 10^7$. Thus $P(\prod_{i=0}^{k-1}(n+4i) > 280$ implying the assertion when $n+4i_0 > 10^7$. Thus we can assume that $m \leq 10^7$. Then $n \leq n+4i_0 \leq 10^7$. We compute that $P(\prod_{i=0}^{8}(n+4i)) > 280$ except when $n \in \{465, 469, 473, 885, 1513\}$. For these values of n, we see that $P(\displaystyle\prod_{i=0}^{8}(n+4i)) > 52$ which is $> 4(k+1)$ for $9 \leq k \leq 12$. Further for these values of n, we also have $P(\prod_{i=0}^{12}(n+4i)) > 280$ which is $> 4(k+1)$ for $13 \leq k < 67$.

Thus we may suppose that $k \geq 67$. Since $P(\Delta(n, 4, k)) \leq 4k+4 < n+4$, we see that each of $n+4, n+8, \cdots, n+4(k-1)$ are composite and hence there is a prime $p_{i,4,l} \equiv n(\mathrm{mod}\ 4)$ such that $p_{i,4,l} \leq n < n+4 < n+4(k-1) < n+4k \leq p_{i+1,4,l}$. Thus $p_{i+1,4,l} - p_{i,4,l} \geq 4k$. Let $n \leq 1.1 \cdot 10^7$. By Lemma 2.6, we can assume that $k \in \{67, 68, 69, 70\}$ and $p_{i,4,l} \leq n < n+4(k-1) < n+4k \leq p_{i+1,4,l}$ for $(p_{i,4,l}, p_{i+1,4,l})$ listed in Lemma 2.6. For such values of n, we check that that $P(\prod_{i=0}^{k}(n+4i)) > 284$. Hence we can assume that $n > 1.1 \cdot 10^7$.

Let $4k < n < 4k+4$. Since $n > 1.1 \cdot 10^7$, we have $10^6 < n+4 \leq 138 \cdot 4(k-1)$. By Lemma 2.5, we have $P(\Delta(n+4, 4, k-1)) \geq n+4$. Hence $P(\Delta(n, 4, k)) \geq P(\Delta(n+4, 4, k-1)) \geq n+4 > 4k+4$. Thus we can assume that $n > 4k+4$. Further again by Lemma 2.5, we can now assume that $n > 138 \cdot 4k$.

Since $P(\Delta(n, 4, k)) \leq 4k+4$, we have $\omega(\Delta(n, 4, k)) \leq \pi(4k+4)-1$. We continue as in [LaSh09, Section 3] with $d = 4, t = \pi(4k+4)-1$ to obtain

$$n \leq \left((k-1)! \prod_{p \leq p_l} p^{L_0(p)}\right)^{\frac{1}{k+1-\pi(4k+4)}} \tag{9}$$

for every $l \geq 1$ where

$$L_0(p) = \begin{cases} \min(0, h_p(k+1-\pi(4k)) - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor) & \text{if } p \nmid d \\ -\nu_p((k-1)!) & \text{if } p | d \end{cases}$$

with $h_p \geq 0$ such that $[\frac{k-1}{p^{h_p+1}}] \leq k+1-\pi(4k+4) < [\frac{k-1}{p^{h_p}}]$. Taking $l = 3$ in (9), we find that $n < 1.1 \cdot 10^7$ when $k \leq 400$. Thus $k > 400$.

We now write $n = v \cdot 4k$ with a real number $v \geq v_0 := 138$. We continue as in the last paragraph of [LaSh09, pp. 433] to obtain

$$\log(v_0 \cdot 8 \cdot e) < \frac{4\log(v_0 \cdot 4k)}{\log(4k+3)} \left(1 + \frac{1.2762}{\log(4k+3)}\right).$$

The right hand side of the above inequality is a decreasing function of k and the inequality does not hold at $k = 401$. This is a contradiction. $\square$

5. Proof of $G_{u+\frac{\alpha}{3}}(x^3)$ not having a factor of degree ≥ 4

Let $d = 3$, $\alpha \in \{1, 2\}$, $u \in \{0, -1\}$ and $P(a_0a_n) \leq 3$. It suffices to show $G_{u+\frac{\alpha}{3}}(x^3)$ does not have a factor of degree in $\{3k, 3k-1, 3k-2\}$ for $2 \leq k \leq \frac{n}{2}$ and further a factor of degree $\frac{3(n-1)}{2} + 1$ when n is odd. By Corollary 3.4, we may assume that $P(\prod_{i=0}^{k-1}(\alpha + 3(u + n - i))) < 3(u + k + 1)$. Since $n \geq 2k$, by Lemma 2.1, we have $u = 0$ and

$$3k < P(\prod_{j=0}^{k-1}(\alpha + 3(n - j))) < 3(k + 1) \tag{10}$$

except when $k = 2$ and $\alpha + 3(u + n - k + 1) = 125$.

Let $k = 2$ and $\alpha+3(u+n-k+1) = 125$. Then $\alpha = 2$ and $(u, n) \in \{(-1, 43), (0, 42)\}$. We consider the Newton polygon with respect to $p = 2$ of the polynomials $G_{u+\frac{2}{3}}(x^3)$ with all a_j's equal to 1. The breaks of the Newton polygon are $0 < 32 \cdot 3 < 40 \cdot 3 < 43 \cdot 3 = 3n$ when $u = -1, n = 43$ and $0 < 32 \cdot 3 < 40 \cdot 3 < 42 \cdot 3 = 3n$ when $u = 0, n = 42$. Further the minimum slope(slope of the left most edge) is $\frac{1}{3}(1 + \frac{1}{32})$ and the maximum slopes (slope of the right most edge) are $\frac{4}{9}$ and $\frac{1}{2}$ when $(u, n) = (-1, 43), (0, 42)$, respectively. Thus by Lemma 3.1 with $r = \lfloor \frac{t}{3} \rfloor$, $t \in \{4, 5, 6\}$, the polynomials $G_{-1+\frac{2}{3}}(x^3)$ does not have factor of degree $t \in \{4, 6\}$. Hence $G_{-1+\frac{2}{3}}(x^3)$ may have a factor of degree 5 when $n = 43$ and $G_{\frac{2}{3}}(x^3)$ may have factor of degree $t \in \{4, 5, 6\}$ when $n = 42$.

Therefore we now suppose that $\alpha + 3(u+n-k+1) \neq 125$ when $k = 2$. By Lemma 3.3, we may restrict to those k such that $P(\prod_{j=0}^{k-1}(\alpha + 3(n - j))) = \alpha + 3k$. Thus $\alpha = 1$ if k is even and $\alpha = 2$ if k is odd. Let

$$R(k) = \{p : p | \prod_{i=1}^{k}(\alpha + 3i), p \text{ prime}\}$$

where $\alpha = 1$ if k is even and $\alpha = 2$ if k is odd. Again by Lemma 3.3, we may suppose that $p | \prod_{j=0}^{k-1}(\alpha + 3(n - j))$ imply $p \in R(k)$. Thus $\omega(\prod_{j=0}^{k-1}(\alpha + 3(n - j))) \leq |R(k)|$. Since $\alpha + 3k$ is prime, we now have

$$|R(k)| = \begin{cases} \pi_1(3k + 1) + \pi_2(\frac{3k+1}{2}) = \pi_1(3k) + 1 + \pi_2(\frac{3k}{2}) & \text{if } k \text{ is even} \\ \pi_2(3k + 2) + \pi_1(\frac{3k+2}{2}) = \pi_2(3k) + 1 + \pi_1(\frac{3k+1}{2}) & \text{if } k \text{ is odd} \end{cases}$$

where $\pi_l(x) = |\{p \leq x : p \equiv l (\text{mod } 3)\}|$ for $l \in \{1, 2\}$.

Let $k = 2$. Then $p|(1+3n)(1+3n-3)$ imply $p \in \{2, 7\}$. Hence $\{1+3n, 1+3n-3\} = \{2^a, 7^b\}$ for some positive integers a, b. Hence $7^b - 2^a = \pm 3$. If $a \geq 3$, we get a contradiction modulo 8. Hence $a \leq 2$ and we have the only solution $7 - 4 = 3$. Therefore $1 + 3n = 7, 1 + 3n - 3 = 4$ giving $n = 2$. This is not possible since $n \geq 2k$.

Thus $k \geq 3$. Let $k \geq 20$. Let $l \in \{1, 2\}$ and m is congruent to l modulo 3. Then note that if the set $\{m, m+3, \cdots, m+3(k-1)\}$ does not contain a prime, then the difference between two consecutive primes congruent $l \pmod 3$ is at least $(m+3k)-(m-3) = 3k+3 \geq 63$ contradicting Lemma 2.6 (i) if $m \leq 7348$. Therefore the set $\{m, m+3, \cdots, m+3(k-1)\}$ contains a prime if $m \leq 7348$ and hence $P(\prod_{i=0}^{k-1}(m+3i)) \geq m$ if $m \leq 7348$. For $3 \leq k < 20$, we check that $P(\prod_{i=0}^{k-1}(m+3i)) \geq \min(m, 3(k+1))$ for $3k < m \leq 7348, 3 \nmid m$ except when $k=3, m=22$. Thus for $k \geq 3$, we may assume by (10) that either $\alpha+3(n-k+1) > 7348$ or $k=3, \alpha+3(n-k+1) = 22$. Since $\alpha = 2$ when k is odd, we obtain $\alpha+3(n-k+1) > 7348$. Let $3 \leq k \leq 8$. After deleting terms in $\{\alpha+3n, \alpha+3(n-1), \cdots, \alpha+3(n-k+1)\}$ divisible by $p \in R(k), p \geq 7$ we are left with at least 2 indices $0 \leq i_1 < i_2 \leq 7$ such that $p|(\alpha+3(n-i_1))(\alpha+3(n-i_2))$ imply $p \in \{2, 5\}$. By putting $X = \alpha+3(n-i_2)$, we obtain from Lemma 2.4 that $X \leq 625$. But $X = \alpha+3(n-i_2) \geq \alpha+3(n-k+1) > 7348$ which is a contradiction.

Thus we now have $k \geq 9$ and $\alpha+3(n-k+1) > 7348$. Further we may also assume that $\alpha+3(n-k+1) \geq 10.6 \cdot 3k$ by Lemma 2.5 and (10). By taking $m = \alpha+3(n-k+1), t = |R(k)|$ in [LaSh09, (4)], we obtain from [LaSh09, (6)] that $\alpha+3(n-k+1) < 4480$ for $9 \leq k \leq 180$. Thus we may suppose that $k > 180$. We proved in the last para of [LaSh12, Section 3(A), pp. 62] that $\omega(\prod_{i=0}^{k-1}(m+3i)) \geq \pi(3k)$ for $k > 180$ when $m > 3k$ and $3 \nmid m$. Therefore $\omega(\prod_{j=0}^{k-1}(\alpha+3(n-j))) \geq \pi(3k)$. But $\pi(3k) = \pi_1(3k)+\pi_2(3k)+1$ and we will have the contradiction $\pi(3k) > |R(k)|$ if

$$\begin{aligned} \pi_2(3k) &> \pi_2(3k/2) \text{ if } k \text{ is even} \\ \pi_1(3k) &> \pi_1((3k+1)/2) \text{ if } k \text{ is odd.} \end{aligned} \tag{11}$$

We check that it is true when $3k/2 \leq 6450$. Hence we now assume $3k/2 > 6450$. Taking $(m, k_1) = (3k/2+1, k/2)$ if k is even and $(m, k_1) = ((3k+1)/2+3, (k-1)/2)$ if k is odd, we see from Lemma 2.5 that $P := P(\Delta(m, 3, k_1) \geq m$. We note that $m \equiv 1, 2$ modulo 3 according as k is even or k is odd, respectively. Further observe that $2P \geq 2m > m+3(k_1-1)$ and hence P is one of the terms of $m, m+3, \cdots, m+3(k_1-1)$ giving the assertion (11). $\square$

6. Proof of $G_{u+\frac{\alpha}{4}}(x^4)$ not having a factor of degree ≥ 5

Let $d=4, u \in \{0, -1\}$ and $\alpha \in \{1, 3\}$. It suffices to show that $G_{u+\frac{\alpha}{4}}(x^4)$ does not have a factor of degree in $\{4k, 4k-1, 4k-2, 4k-3\}$ for $2 \leq k \leq \frac{n}{2}$ and further a factor of degree in $\{\frac{4(n-1)}{2}+1, \frac{4(n-1)}{2}+2\}$ when n is odd. Suppose this is not true. By Corollary 3.4, we may assume that $P(\prod_{i=0}^{k-1}(\alpha+4(u+n-i)) < 4(u+k+1)$. Then by Theorem 4 and Corollary 1.1, we obtain $u=-1, k=2, \alpha+4(u+n-k+1) \in \{21, 45\}$ or $u=0, k=2, \alpha+4(n-k+1) \in \{11, 21, 45, 77, 121\}$ or $u=0, k=3, \alpha+4(u+n-k+1) = 117$. For the values of u, k, n, α given by these values, we obtain from Lemma 3.3 that $G_{u+\frac{\alpha}{4}}(x^4)$ do not have a factor of degree in $\{4k, 4k-1, 4k-2, 4k-3\}$. When $u=-1$, we have $k=2$ and $(n, \alpha) \in \{(7, 1), (13, 1)\}$ and in both these cases, the prime $p=7$ works in Lemma 3.3. For $u=0, k=2$, we have $(n, \alpha) = (3, 3)$ or $\alpha = 1, n \in \{6, 12, 20, 31\}$. Since $n \geq 2k$, we have $\alpha = 1$ and $n \in \{6, 12, 20, 31\}$ and prime $p=7$ works for

$n \in \{6, 12, 20\}$ and $p = 11$ works for $n = 31$ in Lemma 3.3. For $u = 0, k = 3$, we have $\alpha = 1, n = 31$ and here the prime $p = 11$ works in Lemma 3.3. □

7. Proof of Theorems 2 and 3

We observe that if $G(x^d)$ has no factor of degree $\geq l$ with $l \leq \frac{dn}{2}$, then $G(x)$ has no factor of degree $\geq \frac{l}{d}$. Recall that by a factor, we meant the factor of degree less than or equal to half of total degree and its co factor is the one whose degree is more than half of the total degree. If $G(x^d)$ has a factor of degree d only, then $G(x)$ may have a linear factor but no other factor of degree ≥ 2. Further if $G_{\frac{\alpha}{3}}(x^3)$ has a quadratic factor only or a factor of degree 5 only, then $G_{\frac{\alpha}{3}}(x)$ will be irreducible. Hence if the assertion of Theorems 2 and 3 are proved for $G(x^d)$, then the assertion of Theorems 2 and 3 follow.

Therefore we prove the assertions of Theorems 2 and 3 for $G(x^d)$. From Sections 5 and 6, we may assume that $G(x^d)$ has a factor of degree in $\{1, \ldots, d\}$ except when $q = -\frac{1}{3}, n = 43$ where $G_q(x^3)$ may have a factor of degree 5 and $q = \frac{2}{3}, n = 42$ where $G_q(x^3)$ may have a factor of degree in $\{4, 5, 6\}$. Then by Lemma 3.3, we may suppose that prime divisors of $\alpha + d(u + n)$ are given by

d	u	α	$p \mid \alpha + d(u+n)$	d	u	α	$p \mid \alpha + d(u+n)$
3	-1	1	2	4	-1	1	3
3	-1	2	2	4	-1	3	3
3	0	1	2	4	0	1	3, 5
3	0	2	2, 5	4	0	3	3, 7

7.1. **Proof of Theorem 3:** Let $d = 4$. We take p to be the smallest prime dividing $\alpha + 4(u + n)$. Thus $p = 3$ unless $\alpha + 4(u + n) = 1 + 4n = 5^b$ for some positive integer b where we take $p = 5$ and $\alpha + 4(u + n) = 3 + 4n = 7^c$ for some positive integer c where we take $p = 7$. We use Corollary 3.2. Taking $m = 4n, k \in \{1, 2, 3, 4\}, l = k - 1$, we observe that the conditions of Corollary 3.2 are satisfied. We follow the notations as in the proof of Lemma 3.3. Let

$$\Delta_j = (\alpha + (u + 1)d) \cdots (\alpha + (u + j)d).$$

We show that

$$\phi_j = \frac{\nu_p(\Delta_j)}{j} \leq 1 \quad \text{for } 1 \leq j \leq n \tag{12}$$

and

$$\phi_j < 1 \quad \text{for } 1 \leq j \leq n \quad \text{when } p = 3, (u, \alpha) \in \{(-1, 1), (0, 3)\}. \tag{13}$$

This with Corollary 3.2 with $p = 5$ and $p = 7$ according as $(u, \alpha) = (0, 1)$ and $(u, \alpha) = (0, 3)$ respectively and $p = 3$ if $u = -1$ will imply Theorem 3.

We follow as in the proof of Lemma 3.3. We have j_0, l_0 given by

u	α	p	j_0	l_0	u	α	p	j_0	l_0
-1	1	3	3	3	-1	3	3	1	1
0	1	3	2	3	0	3	3	3	5
0	1	5	1	1	0	3	7	1	1

We find that (12) and (13) are valid for $1 \le j \le 3$. Let $j > 3$ and we now show that $\phi_j < 1$ for $j > 3$. We can restrict to j such that $p|(\alpha + 4(u+j))$ and such j are given by $j = j_0 + ps$ with $s > 0$. As in the proof of Lemma 3.3, it suffices to show

$$\nu_p(\Delta_j) = s + 1 + \nu_p(l_0(l_0+4)\cdots(l_0+4s)) < j_0 + ps.$$

This is true for $1 \le s \le 3$. For $s \ge 4$, we find that the left hand side of the above inequality is at most $s + 1 + \nu_p((l_0+4s)!) - 1$ since there is at least one multiple of p dividing $(l_0+4s)!$ but not dividing $l_0(l_0+4)\cdots(l_0+4s)$. This together with $\nu_p(r!) < \frac{r}{p-1}$, $p \ge 3$ and $\frac{l_0}{2} < j_0$ imply

$$\nu_p(\Delta_j) \le s + \frac{l_0+4s}{p-1} \le s + \frac{l_0+4s}{2} = \frac{l_0}{2} + 3s < j_0 + ps.$$

□

7.2. **Proof of Theorem 2:** Let $d = 3$. First assume that $u = 0, \alpha = 2$ and $5|(2+3n)$. We consider the polynomial $G_{\frac{2}{3}}(x^3)$. We use Corollary 3.2 with $p = 5$ to show that $G_{\frac{2}{3}}(x^3)$ does not have a factor with degree in $\{1, 2\}$. As in the proof of Lemma 3.3, it suffices to show

$$\nu_5(5 \cdot 8 \cdots (2+3j)) < \frac{3j}{2}$$

where we may assume that $j > 1$. We obtain by using Lemma 2.2 that

$$\nu_5(5 \cdot 8 \cdots (2+3j)) \le \nu_5((2+3j)!) \le \frac{1+3j}{4} < \frac{3j}{2}.$$

Hence $G_{\frac{2}{3}}(x^3)$ does not have a factor with degree in $\{1, 2\}$ in this case.

From now on, we may suppose that $5 \nmid (2+3n)$ when $u = 0, \alpha = 2$. Therefore for each $u \in \{0, -1\}$ and for each $\alpha \in \{1, 2\}$, we have $\alpha + 3(u+n) = 2^a$ for some integer $a > 1$. We take $p = 2$ and $\nu = \nu_2$ from now onwards in this section. We may assume by Section 5 that $G(x^3)$ has a factor of degree in $\{1, 2, 3\}$. Let $\eta = 0$ if $\alpha = 1$ and 1 if $\alpha = 2$. From $\alpha + 3(u+n) = 2^a$, we have $a = 2s + \eta$ for some $s > 0$ and $n = -u + 2^\eta(1 + 2^2 + \cdots + 2^{2(s-1)})$. Put $n_0 = 0, n_s = n$ and

$$n_i = 2^\eta(2^{2(s-1)} + 2^{2(s-2)} + \cdots + 2^{2(s-i)}) \text{ for } 1 \le i \le s-1. \tag{14}$$

Then for $1 \le i \le s-1$, we have

$$n_i - 1 = 2^\eta(2^{2(s-1)} + 2^{2(s-2)} + \cdots + 2^{2(s-i+1)}) + \sum_{j=0}^{2(s-i)+\eta-1} 2^j$$

and hence by Lemma 2.2, we have

$$\nu((n_i-1)!) = n_i - 1 - (i - 1 + 2(s-i) + \eta) = n_i - a + i. \tag{15}$$

Also

$$\nu((n-1)!) = \begin{cases} n-s & \text{if } u=0, \alpha=1 \\ n-s-1 & \text{otherwise.} \end{cases} \tag{16}$$

Let $1 \leq j < 2^h$ for some $h > 0$. Write $j-1 = j_0 + 2j_1 + \cdots + 2^{h-1}j_{h-1}$ in base 2 with $0 \leq j_u \leq 1$ for $0 \leq u < h$. Note that $\sum_{u=0}^{h-1} j_u \leq h-1$. Hence by Lemma 2.2, we have

$$\nu((j-1)!) = j-1-\sum_{u=0}^{h-1} j_u \geq j-1-(h-1) = j-h. \tag{17}$$

For $1 \leq i \leq n-1$, if $\alpha + 3(u+i) = 2^r t$ with $2 \nmid t$, then from $3(n-i) = 2^r(2^{a-r}-t)$, we obtain $\nu(\alpha+3(u+i)) = r = \nu(n-i)$. Therefore

$$\nu(\prod_{i=l}^{n}(\alpha+3(u+i))) = a + \nu((n-l)!) \text{ for } 1 \leq l < n-1.$$

We now consider the $G(x^3)$ with all a_j's equal to 1 and call it G^*. Recall that $G = G_{u+\frac{\alpha}{3}}$ with $(u,\alpha) \in \{(-1,1),(-1,2),(0,1),(0,2)\}$. Then the Newton Polygon $NP_2(G^*)$ of G^* with respect to prime 2 is given by the lower edges along the convex hull of the following points

$$\{(0,0),(3,a),(3\cdot 2,a),\cdots,(3l,a+\nu((l-1)!)),\cdots,(3n,a+\nu((n-1)!))\}$$

in the extended plane. Let $a = 2s+\eta \leq 5$. Then $\alpha = 1, (u,n) \in \{(-1,2),(-1,6),(0,5)\}$ or $\alpha = 2, (u,n) \in \{(-1,3),(0,2),(-1,11),(0,10)\}$. For these values of (α,u,n), we check that assertion of the Theorem 2 holds by using Lemma 3.1. For example, when $(\alpha,u,n) = (2,-1,11)$, we find that the breaks of $NP_2(G^*)$ are given by $0 < 3\cdot 8 < 3\cdot 11$ and the minimum slope is $\frac{3}{8}$ and the maximum slope is $\frac{4}{9}$. For $t \in \{1,2,3\}$, taking $r = \lfloor \frac{t}{3} \rfloor$ in Lemma 3.1, we obtain that $G_{\frac{-2}{3}}(x^3)$ does not have a factor of degree t and hence irreducible. Similarly we use Lemma 3.1 to get the assertion of Theorem 2 in the remaining cases.

Hence from now on, we assume that $a \geq 6$. If $(0,0)$ and $(3n, a+\nu((n-1)!))$ are the only lattice points on the Newton Polygon $NP_2(G^*)$, then from (16), the unique slope is

$$\frac{a+\nu((n-1)!)}{3n} \leq \frac{2s+\eta+n-s}{3n} = \frac{1}{3} + \frac{2s+2\eta}{2\cdot 3n} = \frac{1}{3} + \frac{a+\eta}{2\cdot 3n} \leq \frac{5}{4\cdot 3}$$

since $n \geq \frac{2^a-2}{3} \geq 2(a+1) \geq 2(a+\eta)$ for $a \geq 6$. Also the unique slope is $> \frac{1}{3}$. Then by using Lemma 3.1 for $t \in \{1,2,3\}$ with $r = \lfloor \frac{t}{3} \rfloor$, we obtain $G(x^3)$ is irreducible. Hence we may suppose that there is a lattice point of $NP_2(G^*)$ with x co-ordinate lying in $(0,3n)$. We prove that the breaks of $NP_2(G^*)$ are given by $0 = 3n_0 < 3n_1 < 3n_2 < \cdots < 3n_{s-2} < 3n_s = 3n$ if $(u,\alpha) = (-1,1)$ and $0 = 3n_0 < 3n_1 < 3n_2 < \cdots < 3n_{s-1} < 3n_s = 3n$ otherwise.

First we show that $(3n_1, a+\nu((n_1-1)!))$ is a lattice point on $NP_2(G^*)$. It suffices to show

(i) $\frac{a+\nu((i-1)!)}{i} > \frac{a+\nu((n_1-1)!)}{n_1}$ for $1 \le i < n_1$.
(ii) $\frac{a+\nu((n_l-1)!)}{n_l} > \frac{a+\nu((n_1-1)!)}{n_1}$ for $2 \le l < s$.
(iii) $\frac{a+\nu((i-1)!)}{i} > \frac{a+\nu((n_1-l)!)}{n_1}$ for $n_l < i < n_{l+1}, 1 \le l < s$.

(i) : Let $1 \le i < n_1 = 2^{a-2}$. Then from (17) and (15),

$$\begin{aligned} &n_1\{a+\nu((i-1)!)\} - i\{a+\nu((n_1-1)!)\} \\ &\ge n_1\{a+i-a+2\} - i\{a+n_1-a+1\} = 2n_1 - i > 0. \end{aligned}$$

(ii) : For $2 \le l < s$, we have from (17)

$$\frac{a+\nu((n_l-1)!)}{n_l} - \frac{a+\nu((n_1-1)!)}{n_1} = \frac{n_l+l}{n_l} - \frac{n_1+1}{n_1} = \frac{l}{n_l} - \frac{1}{n_1} > 0$$

since $n_l = 2^\eta(2^{2(s-1)} + 2^{2(s-2)} + \cdots + 2^{2(s-l)}) < l2^{\eta+2(s-1)} = ln_1$.

(iii) : Let $1 \le l < s$. Write $i = n_l + j$ with $1 \le j < n_{l+1} - n_l = 2^{a-2l-2}$. Since $\nu(u) = \nu(n_l + u)$ for any $1 \le u < n_{l+1} - n_l$, we get $\nu((i-1)!) = \nu((n_l-1)!) + \nu(n_l) + \nu((j-1)!)$. This with (14), (15) and (17) imply

$$\begin{aligned} &\frac{a+\nu((i-1)!)}{i} - \frac{a+\nu((n_1-1)!)}{n_1} \ge \frac{n_l+l+j+2}{n_l+j} - \frac{n_1+1}{n_1} \\ &= \frac{1}{n_1(n_l+j)}((l+2)n_1 - n_l - j)) > \frac{1}{n_1(n_l+j)}\{(l+2)n_1 - n_{l+1}\} > 0 \end{aligned}$$

since $n_{l+1} = 2^\eta(2^{2(s-1)} + \cdots + 2^{2(s-l)} + 2^{2(s-l-1)}) < (l+1)2^{2s+\eta-2} < (l+2)n_1$. Hence the minimum slope is $\frac{1}{3}(1+\frac{1}{n_1})$.

Let $1 \le l < s-2$. Next we show that if $(3n_l, a+\nu((n_l-1)!))$ is a lattice point on $NP_2(G^*)$, then $(3n_{l+1}, a+\nu((n_{l+1}-1)!))$ is a lattice point on $NP_2(G^*)$. Assume that $(3n_l, a+\nu((n_l-1)!))$ is a point on $NP_2(G^*)$. If $(3n, a+\nu((n-1)!))$ is the next lattice point, then from (14)-(16), we see that slope of the rightmost edge is

$$\frac{\nu((n-1)!) - \nu((n_l-1)!)}{3(n-n_l)} \le \frac{n-s-(n_l-a+l)}{3(n-n_l)} \le \frac{1}{3} + \frac{s+\eta-l}{3(n-n_l)} \le \frac{5}{4 \cdot 3}$$

since $1 \le l < s-2$ and $n - n_l \ge 2^\eta \frac{2^{2(s-l)}-1}{3} \ge 4(\eta+s-l)$ for $s-l \ge 3$. Observe that $n_1 > 3$ and the slope of the leftmost edge is $\frac{1}{3}(1+\frac{1}{n_1})$. We now apply Lemma 3.1 for $t \in \{1,2,3\}$ with $r = \lfloor \frac{t}{3} \rfloor$ to obtain $G(x^3)$ is irreducible. Thus we may suppose that $(3n, a+\nu((n-1)!))$ is not the next lattice point on $NP_2(G^*)$. To show $(3n_{l+1}, a+\nu((n_{l+1}-1)!))$ is the next lattice point on $NP_2(G^*)$, it suffices to show

(iv) $\frac{\nu((n_u-1)!)-\nu((n_l-1)!)}{n_u-n_l} > \frac{\nu((n_{l+1}-1)!)-\nu((n_l-1)!)}{n_{l+1}-n_l}$ for $l+1 < u \le s$.
(v) $\frac{\nu((i-1)!)-\nu((n_l-1)!)}{i-n_l} > \frac{\nu((n_{l+1}-1)!)-\nu((n_l-1)!)}{n_{l+1}-n_l}$ for $n_u < i < n_{u+1}, l \le u < s$.

The assertion (iv) follows from (15) and by observing $(u-l)2^{2(s-l-1)+\eta} > 2^\eta(2^{2(s-l-1)} + \cdots + 2^{2(s-u)})$. The assertion (v) follows like (iii) above by observing that if $i = n_u + j$

with $1 \le j < n_{u+1} - n_u = 2^{a-2u-2}$ and $(u-l+2)2^{2(s-l-1)+\eta} > 2^\eta(2^{2(s-l-1)} + \cdots + 2^{2(s-u-1)}) = n_{u+1} - n_l \ge n_u + j - n_l$.

Thus we need to check for lattice points after $(3n_{s-2}, a+\nu((n_{s-2}-1)!))$ on $NP_2(G^*)$. Recall that $n_{s-2} = n + u - 2^\eta - 2^{2+\eta}$ and $\nu(n-i) = \nu(\alpha + 3(u+i))$ for $i \ge 1$. For $(u, \alpha) = (-1, 1)$, we find that $n_{s-2} = n - 6$ and check using $\nu(n-i) = \nu(\alpha+3(u+i))$ for $i \ge 1$ that $(3n, a+\nu((n-1)!)))$ is the lattice point after $(3n_{s-2}, a+\nu((n_{s-2}-1)!))$ and hence the maximum slope is $\frac{7}{18}$. For $(u, \alpha) = (-1, 2)$, we find that $n_{s-1} = n - 3$ and $(3(n-3), a + \nu((n-3)!))$ and $(3n, a + \nu((n-1)!)))$ are the lattice points after $(3n_{s-2}, a + \nu((n_{s-2} - 1)!))$ and the maximum slope is $\frac{4}{9}$. For $(u, \alpha) = (0, 2)$, we find that $n_{s-1} = n-2$ and $(3(n-2), a+\nu((n-3)!))$ and $(3n, a+\nu((n-1)!)))$ are the lattice points after $(3n_{s-2}, a+\nu((n_{s-2}-1)!))$ and the maximum slope is $\frac{1}{2}$. For $(u, \alpha) = (0, 1)$, we find that $n_{s-1} = n - 1$ and $(3(n-1), a + \nu((n-3)!))$ and $(3n, a + \nu((n-1)!)))$ are the lattice points after $(3n_{s-2}, a + \nu((n_{s-2} - 1)!))$ and the maximum slope is $\frac{2}{3}$. Recall that in all these cases, the slope of the leftmost edge is $\frac{1}{3}(1 + \frac{1}{n_1})$.

We now use Lemma 3.1 for $t \in \{1, 2, 3\}$ with $r = \lfloor \frac{t}{3} \rfloor$ to obtain that $G_{\frac{-1}{3}}(x^3)$ and $G_{\frac{-2}{3}}(x^3)$ are irreducible. Further $G_{\frac{1}{3}}(x^3)$ does not have a factor of degree 1 and $G_{\frac{2}{3}}(x^3)$ do not have a factor of degree 1 or 3. $\square$

8. Proof of Theorem 1

We first check that $L_2^{(\frac{1}{4})}(x)$ and $L_2^{(\frac{1}{4})}(x^4)$ are not irreducible and their factorizations are given in the statement of Theorem 1. Therefore we assume from now on that $n \ne 2$ when $q = \frac{1}{4}$. We observe that the irreducibility of $L_n^{(q)}(x^d)$ implies the irreducibility of $L_n^{(q)}(x)$. Hence we show that $L_n^{(q)}(x^d)$ is irreducible. For $(q, n) \in \{(-\frac{2}{3}, 2), (-\frac{1}{3}, 43), (\frac{2}{3}, 42)\}$, we check that $L_n^{(q)}(x^3)$ are irreducible. Thus from Theorems 2 and 3, we need to consider only the following cases:

$$\begin{aligned} q &= \frac{1}{3},\ 1 + 3n = 2^a \\ q &= \frac{2}{3},\ 2 + 3n = 2^a 5^b, a \ge 0, b \ge 0 \\ q &= -\frac{1}{4},\ 3 + 4(n-1) = 3^a \\ q &= \frac{1}{4},\ 1 + 4n = 3^a 5^b, a \ge 0, b \ge 0 \\ q &= \frac{3}{4},\ 3 + 4n = 7^a \end{aligned} \tag{18}$$

Further it suffices to show that $n!L_n^{(q)}(x^d)$ does not have a factor of degree d and for $q \in \{\frac{1}{3}, \frac{2}{3}\}$, $n!L_n^{(q)}(x^3)$ do not have a quadratic or a cubic factor. In fact we show that it does not have a factor of degree $\le d$. First we prove

Lemma 8.1. *For $n > 1$ given by (18), there is a prime $p|n$ such that*

$$p \nmid d(\alpha + (u-1)d)(\alpha + ud)(\alpha + (u+1)d)$$

except when $q = \frac{2}{3}, n \in \{2, 6, 10, 16\}$ *and* $q = \frac{1}{4}, n \in \{6, 20\}$.

Proof. Let $n > 1$ be given by (18). Suppose that $p|n$ implies $p|d(\alpha + (u-1)d)(\alpha + ud)(\alpha + (u+1)d)$.

Let $q = \frac{1}{3}$. Then $p|n$ implies $p \in \{2, 3\}$. Writing $n = 2^r 3^s$, we have $2^a = 1 + 3n = 1 + 2^r 3^{1+s}$ implying $r = 0$, $2^a - 3^{1+s} = 1$. By Lemma 2.3, we have $2^2 - 3 = 1$ or $2^a = 4$ and $3^{1+s} = 3$ giving $n = 1$ which is not possible.

Let $q = \frac{2}{3}$. Then $p|n$ implies $p \in \{2, 3, 5\}$. Writing $n = 2^r 3^s 5^t$, we have $2^a 5^b = 2 + 3n = 2 + 2^r 3^{1+s} 5^t$. If $a = 0$, then $r = t = 0$ and $5^b = 2 + 3^{1+s}$. By Lemma 2.3, we have $5 = 2 + 3$ giving $n = 1$ which is not possible. Hence $a \neq 0$. If $b = 0$, then $a > 1$ giving $r = 1$, $2^a = 2 + 2 \cdot 3^{1+s} 5^t$ or $2^{a-1} = 1 + 3^{1+s} 5^t$. By Lemma 2.3, we get solutions $2^2 = 1 + 3$ and $2^4 = 1 + 3 \cdot 5$ giving $n \in \{2, 10\}$. Hence assume that $ab \neq 0$. Then $t = 0$ and $2^a 5^b = 2 + 2^r 3^{1+s}$. If $a = 1$, then $2 \cdot 5^b = 2 + 2^r 3^{1+s}$ or $5^b = 1 + 2^{r-1} 3^{1+s}$. By Lemma 2.3, the solution $5^2 = 1 + 2^3 \cdot 3$ gives $n = 16$. Finally let $a > 1$. Then $u = 1$ and we get $2^a 5^b = 2 + 2 \cdot 3^{1+s}$ or $2^{a-1} 5^b = 1 + 3^{1+s}$. By Lemma 2.3, its solution $2 \cdot 5 = 1 + 3^2$ gives $n = 6$.

Let $q = -\frac{1}{4}$. Then $p|n$ implies $p \in \{2, 3, 5\}$. Writing $n = 2^r 3^s 5^t$, we have $3^a = 4n - 1 = 2^{2+r} 3^s 5^t - 1$ implying $v = 0$ and $2^{2+r} 5^t - 3^a = 1$. By Lemma 2.3, its solution is $2^2 - 3^1 = 1$ which gives $n = 1$. This is not possible.

Let $q = \frac{1}{4}$. Then $p|n$ implies $p \in \{2, 3, 5\}$. Writing $n = 2^r 3^s 5^t$, we have $3^a 5^b = 1 + 4n = 1 + 2^{2+r} 3^s 5^t$. Let $a = 0$. Then $t = 0$ and $5^b = 1 + 2^{2+r} 3^s$ and by Lemma 2.3, its solutions $5 = 1 + 2^2$ and $5^2 = 1 + 2^3 \cdot 3$ give $n = 6$ since $n > 1$. Let $b = 0$. Then $s = 0, 3^a = 1 + 2^{2+r} 5^t$ and by Lemma 2.3, its solutions $3^2 = 1 + 2^3$ and $3^4 = 1 + 2^4 \cdot 5$ give $n = 20$ since $n \neq 2$. Finally let $ab \neq 0$. Then $s = t = 0, 3^a 5^b = 1 + 2^{2+r}$ and by Lemma 2.3, there are no solutions.

Let $q = \frac{3}{4}$. Then $p|n$ implies $p \in \{2, 3, 7\}$. Writing $n = 2^r 3^s 7^t$, we have $7^a = 3 + 4n = 3 + 2^{2+r} 3^s 7^t$ implying $s = t = 0$ and $7^a = 3 + 2^{2+r}$. By Lemma 2.3, its solution $7 = 3 + 2^2$ imply $n = 1$ which is not possible. $\square$

For $n \in \{2, 6, 10, 16\}$ if $q = \frac{2}{3}$ and $n \in \{6, 20\}$ if $q = \frac{1}{4}$, we check that $L_n^{(q)}(x^d)$ are irreducible. Hence we may suppose that $n \notin \{2, 6, 10, 16\}$ if $q = \frac{2}{3}$ and $n \notin \{2, 6, 20\}$ if $q = \frac{1}{4}$. Then by Lemma 8.1, we find that there is a prime $p|n$ such that $p \nmid d(\alpha + (u-1)d)(\alpha + ud)(\alpha + (u+1)d)$. Let p be largest with this property. Thus we always have $p \geq 5 > d$. We use Corollary 3.2 with $k = d, l = 0$. Since $p|\binom{n}{j}$ for $1 \leq j < p$ and $p|\prod_{i=1}^{p}(\alpha + (u+i)d)$, the conditions of Corollary 3.2 are satisfied. It suffices to show

$$\nu_p\left(\prod_{i=0}^{j}(\alpha + (u+i)d)\right) - \nu_p\left(\binom{n}{j}\right) < \frac{dj}{d} = j \quad \text{for } 1 \leq j \leq n.$$

Observe that p divides at most one of $\alpha + (u+i)d$ when $1 \leq i < p$ and $\alpha + (u+p-1)d < pd < p^2$. By using $p|\binom{n}{j}$ for $1 \leq j < p$, we obtain that the left hand side of above inequality is ≤ 0 for $1 \leq j < p$ and hence the assertion follows for $1 \leq j < p$. Let $j \geq p$. Then there is at least one multiple of p dividing $(\alpha+(u+j)d)!$ but not dividing $\prod_{i=0}^{j}(\alpha+(u+i)d)$. Therefore by using Lemma 2.2, we obtain

$$\begin{aligned}\nu_p\left(\prod_{i=0}^{j}(\alpha+(u+i)d)\right)-\nu_p\left(\binom{n}{j}\right) &\leq \nu_p((\alpha+(u+j)d)!)-1\\ &\leq \frac{\alpha+(u+j)d-1}{p-1}-1 \leq u+j+\frac{\alpha-1}{p-1}-1<j\end{aligned}$$

by using Lemma 2.2 and since $p > d > \alpha$. □

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