# IRREDUCIBILITY OF GENERALIZED HERMITE-LAGUERRE POLYNOMIALS II

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Abstract. In this paper, we show that for each $n \geq 1$, the generalised Hermite-Laguerre Polynomials $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$.

## 1. Introduction

Let $n$ and $1 \leq \alpha < d$ be positive integers with $\gcd(\alpha, d) = 1$. Let $q = \frac{\alpha}{d}$ and let

$$(\alpha)_j = \alpha(\alpha + d) \cdots (\alpha + (j-1)d)$$

for non negative integer $j$. We define

$$F(x) := F_q(x) = a_n \frac{d^n x^n}{(\alpha)_n} + a_{n-1} \frac{d^{n-1} x^{n-1}}{(\alpha)_{n-1}} + \cdots + a_1 \frac{dx}{(\alpha)_1} + a_0$$

where $a_0, a_1, \cdots a_n \in \mathbb{Z}$ and $P(|a_0 a_n|) \leq 2$. Here $P(\nu)$ is the maximum prime divisor for $|\nu| > 1$ and $P(1) = P(-1) = 1$. We put

$$\begin{aligned} G(x) := G_q(x) =& (\alpha)_n F_q(\frac{x}{d}) \\ =& a_n x^n + a_{n-1}(\alpha + (n-1)d) x^{n-1} + \cdots + \\ & a_1 \left( \prod_{i=1}^{n-1} (\alpha + id) \right) x + a_0 \left( \prod_{i=0}^{n-1} (\alpha + id) \right). \end{aligned}$$

Schur [Sch29] proved that $G_{\frac{1}{2}}$ with $|a_0| = |a_n| = 1$ is irreducible. Laishram and Shorey [LaiSho] showed that $G_{\frac{1}{3}}$ and $G_{\frac{2}{3}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$ whenever $|a_0| = |a_n| = 1$. For an account of earlier results, we refer to [ShTi] and [FiFiLe]. We prove

**Theorem 1.** *For each $n$, the polynomials $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$.*

For Theorem 1, we prove the following lemma in Section 2.

**Lemma 1.** *Let $1 \leq k \leq \frac{n}{2}$. Suppose there is a prime $p$ satisfying*

$$p > d, p \geq \min(2k, d(d-1))$$

*and*

$$p \Big| \prod_{j=1}^{k} (\alpha + (n-j)d), \quad p \nmid \prod_{j=1}^{k} (\alpha + (j-1)d). \tag{1}$$

*Then $G(x)$ has no factor of degree $k$.*

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We compare Lemma 1 with [ShTi, Lemma 10.1]. The assumption on $p$ in [ShTi, Lemma 10.1] has been relaxed. For any integer $\nu > 1$, we denote by $\omega(\nu)$ the number of distinct prime factors of $\nu$ and $\omega(1) = 0$. In Section 3, we give an upper bound for $m$ when $\omega(\prod_{i=0}^{k-1}(m+id)) \le t$ for some $t$. In Section 4, we give preliminaries for the proof of Theorem 1. In Section 5, we complete the proof.

## 2. Proof of Lemma 1

Let

$$\Delta_j = \alpha(\alpha+d)\cdots(\alpha+(j-1)d).$$

For each $1 \le l < d$ and $\gcd(l,d) = 1$, we observe that $q|\Delta_k$ for all primes $q \equiv l^{-1}\alpha (\text{mod } d)$ and $q \le \frac{kd}{l}$. Since $p > \alpha$ and $p \nmid \Delta_k$, we have $p > \frac{kd}{d-1}$. Let $j_0$ be the minimum $j$ such that $p|(\alpha+(j-1)d)$ and we write $\alpha+(j_0-1)d = pl_0$. Then $j_0 > k$ since $p \nmid \Delta_k$ and we observe that $1 \le l_0 < d$ by the minimality of $j_0$. As in the proof of [ShTi, Corollary 2.1], it suffices to show that

$$\phi_j = \frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k} \quad \text{for } 1 \le j \le n.$$

We may restrict to those $j$ such that $\alpha+(j-1)d = pl$ for some $l$. Then $(j-j_0)d = p(l-l_0)$ implying $d|(l-l_0)$. Writing $l = l_0+sd$, we get $j = j_0+ps$. Note that if $p|(\alpha+(i-1)d)$, then $\alpha+(i-1)d = p(l_0+rd)$ for some $r \ge 0$. Hence we have

(2)
$$\text{ord}_p(\Delta_j) = \text{ord}_p((pl_0)(p(l_0+d))\cdots(p(l_0+sd)) = s+1+\text{ord}_p(l_0(l_0+d)\cdots(l_0+sd))$$

for some integer $s \ge 0$. Further we may suppose that $s > 0$ otherwise the assertion follows since $p > d > l_0$. Let $r_0$ be such that $\text{ord}_p(l_0+r_0d)$ is maximal. We consider two cases.

**Case I:** Assume that $s < p$. Then $p$ divides at most one term of $\{l_0+id : 0 \le i \le s\}$ and we obtain from (2) and $l_0+sd < (s+1)d < p^2$ that $\phi_j \le \frac{s+2}{j_o+ps}$. Thus $\phi_j < \frac{1}{k}$ if $s(p-k) \ge k$ since $j_0-k+s(p-k)-k \ge 1+s(p-k)-k$. If $p \ge 2k$, then $s(p-k) \ge k$. Thus we may suppose that $p < 2k$. Then $p \ge d(d-1)$. Since $p > \frac{kd}{d-1}$, we obtain $s(p-k) \ge k$ if $s \ge d-1$. We may suppose $s \le d-2$. Then $l_0+sd \le d-1+(d-2)d < p$ and therefore $\phi_j = \frac{s+1}{j_0+ps} \le \frac{s+1}{k+1+(k+1)s} < \frac{1}{k}$.

**Case II:** Let $s \ge p$. Then

$$\text{ord}_p(\Delta_j) \le s+1+\text{ord}_p(l_0+r_0d)+\text{ord}_p(s!) \le s+1+\frac{\log(l_0+sd)}{\log p}+\frac{s}{p-1}.$$

We have $p \ge d+1$. This with $l_0 \le d-1 < p \le s$ imply $\log(l_0+sd) \le \log s(d+1) = \log s + \log(d+1) \le \log s + \log p$. Hence

$$\text{ord}_p(\Delta_j) \le s+1+\frac{s}{p-1}+\frac{\log s}{\log p}+1.$$

Since $\frac{j}{k} = \frac{j_0+ps}{k} > 1+\frac{p}{k}s$, it is enough to show that

$$\frac{p}{k} \ge 1+\frac{1}{p-1}+\frac{1}{s}+\frac{\log s}{s\log p}$$

Since $s \geq p$, the right hand side of the above inequality is at most $1 + \frac{1}{p-1} + \frac{2}{p}$ and therefore it suffices to show

$$1 + \frac{1}{p-1} + \frac{2}{p} \leq \frac{p}{k}. \tag{3}$$

Let $p \geq 2k$. Then $p \geq 2k+1 \geq k+2$ and the left hand side of (3) is at most

$$1 + \frac{1}{2k} + \frac{2}{2k+1} \leq 1 + \frac{2}{k} = \frac{k+2}{k} \leq \frac{p}{k}.$$

Thus we may assume that $p < 2k$. Then $p > d(d-1)$ since $p \nmid d$. Further $d \geq 3$ since $p \geq \frac{kd}{d-1}$. Therefore the left hand side of (3) is at most

$$1 + \frac{3}{d(d-1)} \leq 1 + \frac{1}{d-1} = \frac{d}{d-1} \leq \frac{p}{k}.$$

Hence the proof. $\square$

## 3. An upper bound for $m$ when $\omega(\Delta(m,d,k)) \leq t$

Let $m$ and $k$ be positive integers with $m > kd$ and $\gcd(m,d) = 1$. We write

$$\Delta(m,d,k) = m(m+d)\cdots(m+(k-1)d).$$

Assume that

$$\omega(\Delta(m,d,k)) \leq t. \tag{4}$$

for some integer $t$. For every prime $p$ dividing $\Delta$, we delete a term $m + i_p d$ such that $\mathrm{ord}_p(m + i_p d)$ is maximal. Then we have a set $T$ of terms in $\Delta(m,k)$ with

$$|T| = k - t := t_0.$$

We arrange the elements of $T$ as $m + i_1 d < m + i_2 d < \cdots < m + i_{t_0} d$. Let

$$\mathfrak{P} := \prod_{\nu=1}^{t_0} (m + i_\nu d) \geq m^{t_0}. \tag{5}$$

Now we deduce an upper bound for $\mathfrak{P}$. For a prime $p$, let $r$ be the highest power of $p$ such that $p^r \leq k-1$. Let $w_l = \#\{m + id : p^l | (m+i), m+i \in T\}$ for $1 \leq l \leq r$. By Sylvester and Erdős argument, we have $w_l \leq [\frac{i_0}{p^l}] + [\frac{k-1-i_o}{p^l}] \leq [\frac{k-1}{p^l}]$. Let $h_p > 0$ be such that $[\frac{k-1}{p^{h_p+1}}] \leq t_0 < [\frac{k-1}{p^{h_p}}]$. Then $|\{m + id \in T : \mathrm{ord}_p(n + id) \leq h_p\}| \leq t_0 - w_{h_p+1}$. Hence

$$\begin{aligned}
\mathrm{ord}_p(\mathfrak{P}) &\leq r w_r + \sum_{u=h_p+1}^{r-1} u(w_u - w_{u+1}) + h_p(t_0 - w_{h_p+1}) \\
&= w_r + w_{r-1} + \cdots + w_{h_p+1} + h_p t_0 \\
&\leq \sum_{u=1}^{r} \lfloor \frac{k-1}{p^u} \rfloor + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor = \mathrm{ord}_p((k-1)!) + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor.
\end{aligned}$$

It is also easy to see that $\mathrm{ord}_p(\mathfrak{P}) \leq \mathrm{ord}_p(k-1)!)$ if $p \nmid d$ and $\mathrm{ord}_p(\mathfrak{P}) = 0$ if $p|d$. Therefore

$$m^{t_0} \leq \mathfrak{P} \leq (k-1)! \prod_{p \leq k} p^{L_0(p)}$$

where

$$L_0(p) = \begin{cases} \min(0, h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor) & \text{if } p \nmid d \\ -\text{ord}_p((k-1)!) & \text{if } p|d. \end{cases}$$

Observe that

$$m^{t_0} \leq (k-1)! \prod_{p|d} p^{-\text{ord}_p((k-1)!)}. \tag{6}$$

We also note that $L_0(p) \leq 0$ for any prime $p$. Hence for any $l \geq 1$, we have from (5) that

$$m \leq (\mathfrak{P})^{\frac{1}{t_0}} \leq \left( (k-1)! \prod_{p \leq p_l} p^{L_0(p)} \right)^{\frac{1}{t_0}} =: L(k,l). \tag{7}$$

## 4. Preliminaries for Theorems 1

Let $m$ and $k$ be positive integers with $m > kd$ and $\gcd(m,d) = 1$. We write

$$\Delta(m,d,k) = m(m+d)\cdots(m+(k-1)d).$$

For positive integers $\nu, \mu$ and $1 \leq l < \mu$ with $\gcd(l,\mu) = 1$, we write

$$\pi(\nu,\mu,l) = \sum_{\substack{p \leq \nu \\ p \equiv l (\text{mod } \mu)}} 1, \ \pi(\nu) = \pi(\nu,1,1)$$

$$\theta(\nu,\mu,l) = \sum_{\substack{p \leq \nu \\ p \equiv l (\text{mod } \mu)}} \log p.$$

Let $p_{i,\mu,l}$ denote the $i$th prime congruent to $l$ modulo $\mu$. Let $\delta_\mu(i,l) = p_{i+1,\mu,l} - p_{i,\mu,l}$ and $W_\mu(i,l) = (p_{i,\mu,l}, p_{i+1,\mu,l})$. We recall some well-known estimates from prime number theory.

**Lemma 4.1.** *Let $k \in \mathbb{Z}$ and $\nu \in \mathbb{R}$ be positive. We have*

(*i*) $\pi(\nu) \leq \left(1 + \frac{1.2762}{\log \nu}\right)$ *for* $\nu > 1$

(*ii*) $\text{ord}_p(k-1)! \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ *for* $k \geq 2$.

(*iii*) $\sqrt{2\pi k}\ e^{-k} k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\ e^{-k} k^k e^{\frac{1}{12k}}$.

The estimates $(i)$ is due to Dusart([Dus99]. The estimate $(iii)$ is due to Robbins [Rob55, Theorem 6]. For a proof of $(ii)$, see [LaSh04, Lemma 2(i)]. $\square$

The following lemma is due to Ramaré and Rumely [RaMu96, Theorems 1, 2].

**Lemma 4.2.** *Let $d = 4$ and $l \in \{1,3\}$. For $\nu_0 \leq 10^{10}$, we have*

$$\theta(\nu,d,l) \geq \begin{cases} \frac{\nu}{2}(1 - 0.002238) & \text{for } \nu \geq 10^{10} \\ \frac{\nu}{2}\left(1 - \frac{2 \times 1.798158}{\sqrt{\nu_0}}\right) & \text{for } 10^{10} > \nu \geq \nu_0 \end{cases} \tag{8}$$

*and*

$$\theta(\nu,d,l) \leq \begin{cases} \frac{\nu}{2}(1 + 0.002238) & \text{for } \nu \geq 10^{10} \\ \frac{\nu}{2}\left(1 + \frac{2 \times 1.798158}{\sqrt{\nu_0}}\right) & \text{for } 10^{10} > \nu \geq \nu_0. \end{cases} \tag{9}$$

We derive from Lemmas 4.1 and 4.2 the following result.

**Corollary 4.3.** *Let $10^6 < m \leq 138 \times 4k$. Then $P(\Delta(m, 4, k)) \geq m$.*

*Proof.* Let $d = 4$ and $10^6 \leq m \leq 138 \times dk$. Let $l \in \{1, 3\}$ and assume $m \equiv l(\text{mod } d)$. We observe that $P(\Delta(m, d, k) \geq m$ holds if

$$\theta(m + d(k-1), d, l) - \theta(m - d, d, l) = \sum_{\substack{m<p\leq m+(k-1)d \\ p\equiv l(d)}} \log p > 0.$$

From Lemmas 4.1 and 4.2, we have

$$\frac{\theta(m-d, d, l)}{\frac{m-d}{\phi(d)}} < 1 + \frac{2 \times 1.798158}{\sqrt{10^6}}$$

and

$$\frac{\theta(m+(k-1)d, d, l)}{\frac{m-d+dk}{\phi(d)}} > 1 - \frac{2 \times 1.798158}{\sqrt{10^6}}$$

Thus $P(\Delta(m, d, k) \geq m$ holds if

$$(1 - \frac{2 \times 1.798158}{10^3})dk > \frac{4 \times 1.798158}{10^3}(m-d)$$

which is true since

$$\frac{m}{dk} \leq 138 < \frac{10^3}{4 \times 1.798158} - \frac{1}{2}.$$

Hence the assertion. □

The following lemma is a computational result.

**Lemma 4.4.** *Let $l \in \{1, 3\}$. Then $\delta_4(i, l) \leq 24, 32, 60, 200$ according as $p_{i,4,l} \leq 120, 250, 2400, 10^6$, respectively.*

As a consequence, we obtain

**Corollary 4.5.** *Let $d = 4$, $k \geq 6$ and $m$ be such that $m \leq 120, 250, 2400, 10^6$ when $6 \leq k < 8$, $8 \leq k < 15$, $15 \leq k < 50$ and $k \geq 50$ respectively. Then $P(\Delta(m, d, k)) \geq m$.*

*Proof.* We may assume that $p_{i,d,l} < m < m+(k-1)d < p_{i+1,d,l}$ for some $i$ otherwise the assertion follows. Thus $p_{i+1,d,l} \geq d+m+(k-1)d$ and $p_{i,d,l} \leq m-d$. Therefore $\delta_d(i, l) = p_{i+1,d,l} - p_{i,d,l} \geq d + m + (k-1)d - (m-d) = d(k+1) > dk$. Now the assertion follows from Lemma 4.4. □

## 5. Proof of Theorem 1

Let $2 \leq k \leq \frac{n}{2}$ and assume that $G(x)$ has a factor of degree $k$. We take $m = \alpha + 4(n-k)$. Since $n \geq 2k$, we have $m > 4k$. We may assume that $P(\Delta(m, 4, k)) \leq 4k$ otherwise the assertion follows from Lemma 1 since $\alpha + 4(k-1) < 4k$. Thus $P(\Delta(m, 4, k)) \leq 4k < m$.

Let $k \leq 6$. Then $P(\Delta(m, 4, k)) \leq 4k \leq 23$ implying $P(m(m+4)) \leq 24$. Then $m + 4 = N$ where $N$ is given by [Leh64, Table IIA] for $p \leq 23$. For each such $N$ and for each $2 \leq k \leq 6$, we first restrict to those $m = N - 4 > 4k$ such that $P(\Delta(m, 4, k)) \leq 4k$. They are given by $k = 2$, $m \in \{21, 45\}$. Here $P(m(m+4)) = 7$ and since $m \equiv 1$ modulo 4, the assertion follows by taking $p = 7$ in Lemma 1.

Therefore $k \geq 7$. Let $\omega_1(k) := \max_{\alpha\in\{1,3\}} \omega(\Delta(\alpha, 4, k))$. If $\omega(\Delta(m, 4, k)) > \omega_1$, then there is a prime $p$ satisfying (1) implying $p > k \geq 7$. Observe that $11|\Delta(3, 4, k)$ and $11|\Delta(1, 4, k)$ for $k \geq 9$. For $k \in \{7, 8\}$, if $\omega(\Delta(m, 4, k)) > \omega_1$, then there are two primes $p > k$ dividing $\Delta(m, 4, k)$ but $p \nmid \Delta(1, 4, k)$ and hence there is a prime $p > 11$ satisfying (1). Therefore by Lemma 1, we may assume that $\omega(\Delta(m, 4, k)) \leq \omega_1$. Taking $t = \omega_1$, we obtain from (7) with $p_l = 7$ that $m \leq 104, 245, 2353$ according as $k \leq 10, 20, 400$, respectively. This is not possible by Corollary 4.5.

Hence $k > 400$ and further $m > 10^6$ by Corollary 4.5. By Corollary 4.3, we may further suppose that $m \geq v_0 \cdot 4k$ where $v_0 := 138$. Since $P(\Delta(m, d, k)) \leq 4k$, we have $\omega(\Delta(m, d, k)) \leq \pi(4k) - 1$. Taking $t = \pi(4k) - 1$ in (4), we obtain from (6) that

$$(v_0 \cdot 4k)^{k-\pi(4k)+1} \leq (k-1)!2^{-\mathrm{ord}_2((k-1)!)} = \frac{k!}{k}2^{-\mathrm{ord}_2((k-1)!)}.$$

By using estimates of $\mathrm{ord}_p(k-1)!)$ and $k!$ from Lemma 4.1, we obtain

$$(v_0 \cdot 4k)^{k-\pi(4k)} < \frac{1}{k(v_0 \cdot 4k)}(\frac{k}{e})^k \left((2\pi k)^{\frac{1}{2}}\exp(\frac{1}{12k})\right)(2^{-k+2}(k-1)$$

or

$$(v_0 \cdot 4 \cdot e \cdot 2)^k < (v_0 \cdot 4k)^{\pi(4k)}\frac{\left((2\pi)^{\frac{1}{2}}\exp(\frac{1}{12k})\right)}{v_0 \cdot \sqrt{k}} < (v_0 \cdot 4k)^{\pi(4k)}$$

since $k > 400$. By using estimates of $\pi(4k)$ from Lemma 4.1, we get

$$\log(v_0 \cdot 8 \cdot e) < \frac{4\log(v_0 \cdot 4k)}{\log(4k)}\left(1 + \frac{1.2762}{\log(4k)}\right).$$

The right hand side of the above expression is a decreasing function of $k$ and the inequality does not hold at $k = 401$. This is a contradiction. $\square$

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