GRIMM'S CONJECTURE ON CONSECUTIVE INTEGERS
(DEDICATED TO THE MEMORY OF PROFESSOR V. C. DUMIR)

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ABSTRACT. For positive integers n and k, it is possible to choose primes $P_1, P_2, \cdots, P_k$ such that $P_i|(n+i)$ for $1 \leq i \leq k$ whenever $n+1, n+2, \cdots, n+k$ are all composites and $n \leq 1.9 \times 10^{10}$. This provides a numerical verification of Grimm's Conjecture.

Let $n \geq 0$ and $k \geq 1$ be integers. For an integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively, and let $\omega(1) = 0$, $P(1) = 1$. Let p_i denote the $i-$th prime number. We shall always write p for a prime number. Let $N_0 = 8.5 \times 10^8$.

We state a Conjecture of Grimm [2].

Suppose $n+1, \cdots, n+k$ are all composite numbers and there are distinct primes P_i such that $P_i|(n+i)$ for $1 \leq i \leq k$. Then we say that Grimm's Conjecture holds for n and k. Further we say that Grimm's Conjecture holds if there are distinct primes P_i such that $P_i|(n+i)$ for $1 \leq i \leq k$ whenever $n+1, \cdots, n+k$ are all composites.

If $k \geq n$, it is well-known that the interval $[n+1, n+k]$ contains a prime. Thus $k < n$ if $n+1, \cdots, n+k$ are all composite numbers. According to Erdős (see [1]), this conjecture implies

$$p_{i+1} - p_i \leq c_1 p_i^{\frac{1}{2}-\alpha}$$

for some $\alpha > 0$ and an absolute constant c_1. The best known result on Grimm's Conjecture is due to Ramachandra, Shorey and Tijdeman [4]:

There exists an absolute constant $c_2 > 0$ such that for $n \geq 3$ and $g = g(n) = [c_2 \left(\frac{\log n}{\log \log n}\right)^3]$, it is possible to choose distinct primes $P_1, P_2, \cdots, P_g$ such that $P_i|(n+i)$ for $1 \leq i \leq g$.

The constant c_2 turns out to be very small. Therefore the above result is valid only for large values of n. In this paper, we confirm Grimm's Conjecture for $n \leq 1.9 \times 10^{10}$ and for all k.

Theorem 1. *Grimm's Conjecture holds for $n \leq p_{N_0}$ and for all k.*

We observe that $p_{N_0} = 19236701629 > 1.9 \times 10^{10}$. As a consequence of Theorem 1, we have

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Corollary 0.1. *Assume that* $n+1, \cdots, n+k$ *are all composite and* $n \le p_{N_0}$. *Then*

$$\omega((n+1)\cdots(n+k)) \ge k. \tag{1}$$

A weaker version of Grimm's Conjecture states that (1) is valid for all n and k such that $n+1, \cdots, n+k$ are all composite numbers. For results in this direction, see [5]. A Conjecture of Cramer states that

$$p_{N+1} - p_N < (\log p_N)^2 \text{ for } N > 1.$$

We check

Lemma 0.2. *Let* $k(N) = p_{N+1} - p_N - 1$. *Then*

$$k(N) < (\log p_N)^2 \text{ for } N \le N_0. \tag{2}$$

We observe that (2) can be sharpened for several values of N and this is important for the value of N_0 in Theorem 1. For the proof of Theorem 1, it suffices to prove the following.

Theorem 2. *Grimm's Conjecture is valid when* $n = p_N$ *and* $k = k(N) = p_{N+1} - p_N - 1$ *for* $1 < N \le N_0$.

The computations in this paper were done by using **MATHEMATICA**. The computations were carried out in an Intel Xeon 2.40 GHz processor with 2.5 GB RAM and it took around a week. The computations turn out to be very slow for $N > N_0$ and this is the reason that we have stated Theorem 2 for $N \le N_0$.

Proof of Theorem 2: Let $1 < N \le N_0$. We put $n = p_N$ and $k = k(N) = p_{N+1} - p_N - 1$. We check that Theorem 2 is valid for $N \le 9$. Thus we may suppose that $10 \le N \le N_0$. Assume that the assertion of Theorem 2 is not valid. Then we derive from a result of Phillip Hall [3] on distinct representations that there exists $t > 0$ and integers $n < n_0 < n_1 < \cdots < n_t < n + k + 1$ with

$$\omega(n_0 n_1 \cdots n_t) \le t. \tag{3}$$

Let $t = t(N)$ be minimal in the above assertion. Then $P(n_i) < k$ for $0 \le i \le t$ and (3) holds with equality sign. We apply a fundamental argument of Sylvester and Erdős. For every prime divisor p of $n_0 n_1 \cdots n_t$, we take an n_{i_p} such that p does not appear to a higher power in the factorisation of any element of $\{n_0, n_1, \cdots, n_t\} =: S$. By deleting all n_{i_p} with p dividing $n_0 n_1 \cdots n_t$ in S, we are left with at least one $n_{i_0} \in S$. If p^ν is the highest power of a prime p dividing n_{i_0}, then p^ν also divides n_{i_p} and hence it divides $|n_{i_0} - n_{i_p}| < k$. Therefore

$$n < n_{i_0} < k^t \tag{4}$$

since $\omega(n_{i_0}) \le t$. By Lemma 0.2, we get

$$\frac{\log p_N}{\log\log p_N} < 2t(N). \tag{5}$$

We see that the left hand side of (5) is an increasing function of N. For $i \geq 2$, let N_i be the largest integer N such that

$$\frac{\log p_N}{\log \log p_N} < 2i.$$

Then we calculate

$$N_2 = 727, \; N_3 = 1514619, \; N_4 = 8579289335. \tag{6}$$

Let A_r and M_r be defined by

$$A_{2r-1} = \prod_{p^\alpha < 2r-1 \leq p^{\alpha+1}} p^\alpha, \; M_{2r-1} = \pi(A_{2r-1}).$$

Then

Lemma 0.3. *Suppose that Theorem 2 is not valid at N with $N > M_{2r-1}$. Then $k(N) > 2r-1$.*

Proof. Assume that $k(N) = p_{N+1} - p_N - 1 \leq 2r - 1$. Since Theorem 2 is not valid, (3) holds for some t and hence there exists a term $\bar{n}$ such that

$$p_N < \bar{n} \leq A_{2r-1}.$$

This is a contradiction since $N > M_{2r-1}$. □

We compute M_{2r-1} for some values of r :

$$M_{11} = 368, M_{13} = 3022, M_{15} = 30785, M_{17} = 58083, M_{19} = 803484,$$
$$M_{21} = M_{23} = 12787622, M_{25} = 250791570.$$

Let

$$S_N = \{p_N + i : P(p_N + i) < k, 1 \leq i \leq k\}$$

and put $t' = t'(N) = |S_N|$. We see that $t' \geq t + 1$. For the proof of Theorem 2, it suffices to find distinct prime divisors of the elements of S_N since a prime $\geq k$ divides at most one $p_N + i$ with $1 \leq i \leq k$.

First we consider $N \leq N_2$. Let $t = 1$. Then there are $1 \leq j < i \leq k$ and a prime p such that $p_N + i = p^\alpha$ and $p_N + j = p^\beta$. This gives

$$p_N + j = p^\beta \leq p^\beta(p^{\alpha-\beta} - 1) = i - j < k = p_{N+1} - p_N - 1$$

implying $2p_N < p_{N+1} - 1$, a contradiction. Let $t = 2$. Then (4) holds only when $N = 30$. We have $S_{30} = \{120, 121, 125, 126\}$ and we choose $3, 11, 5$ and 7 as distinct prime divisors of $120, 121, 125$ and 126, respectively. Therefore the assertion of Theorem 2 holds for $N = 30$. Thus $t \geq 3$ implying $t' \geq t + 1 \geq 4$. Now, by calculating t', we see that $N = 30, 99, 217, 263, 327, 367, 457, 522, 650$ and we verify the assertion of Theorem 2 as above in each of these values of N.

Hence $N > N_2$. Therefore $t \geq 3$ by the definition of N_2 and thus $t' \geq 4$. Next we consider $N_2 < N \leq N_3$. We divide this interval into the following subintervals:

$$I_{11} = (N_2, M_{13}], I_{13} = (M_{13}, M_{15}], I_{15} = (M_{15}, M_{17}], I_{17} = (M_{17}, M_{19}], I_{19} = (M_{19}, N_3].$$

By Lemma 0.3, we restrict to those N for which $k(N) > 2r - 1$ whenever $N \in I_{2r-1}$ with $6 \le r \le 10$. Let $t = 3$. By (4) and $t' \ge 4$, we find that N is one of the following:

$$757, 1183, 1229, 1315, 1409, 1831, 1879, 2225, 2321, 2700, 2788, 2810, 3302, 3385,$$
$$3427, 3562, 3644, 3732, 3793, 3795, 3861, 4009, 4231, 4260, 4522, 4754, 5349, 5949,$$
$$6104, 6880, 9663, 9872, 10229, 10236, 11214, 11684, 12542, 14357, 14862, 15783,$$
$$16879, 17006, 17625, 18266, 19026, 19724, 23283, 23918, 25248, 28593, 31545, 31592,$$
$$33608, 34215, 38590, 40933, 44903, 47350, 66762, 104071, 118505, 126172, 141334, 149689.$$

Let $P(S_N) = \{P(p_N+i) : p_N+i \in S_N\}$. For the proof of Theorem 2, we may suppose that

$$|P(S_N)| < |S_N|. \tag{7}$$

In view of (7), all above possibilities for N other than the following are excluded:

$$\begin{aligned} &1409, 1831, 2225, 2788, 3302, 3385, 3562, 3644, 4522,\\ &14862, 16879, 17006, 23283, 28593, 34215, 104071. \end{aligned} \tag{8}$$

Let N be given by (8). We check that $|P(S_N)| = |S_N| - 1$. Let (i, j) with $i < j$ be the unique pair satisfying $P(p_N + i) = P(p_N + j)$. We check that $\omega(p_N + i) \ge 2$. Now we take $P_\mu = P(p_N + \mu)$ if $\mu \ne i$ and P_i to be the least prime divisor of $p_N + i$. Thus all the possibilities in (8) are excluded. Therefore $t \ge 4$ implying $t' \ge 5$. If $p_N < k^3$, then N is already excluded. Consequently we suppose that $p_N \ge k^3$. Now we calculate t' to find that N is one of the following:

$$11159, 19213, 30765, 31382, 40026, 42673, 51943, 57626, 65274, 65320, 80413,$$
$$81426, 88602, 106286, 184968, 189747, 192426, 212218, 245862, 256263, 261491,$$
$$271743, 278832, 286090, 325098, 327539, 405705, 415069, 435081, 484897, 491237,$$
$$495297, 524270, 528858, 562831, 566214, 569279, 629489, 631696, 822210, 870819,$$
$$894189, 938452, 1036812, 1150497, 1178800, 1319945, 1394268, 1409075.$$

By (7), it suffices to restrict N to

$$57626, 65320, 80413, 106286, 271743, 415069, 822210.$$

These cases are excluded as in (8).

Thus we may assume that $N > N_3$. Then $t \ge 4$ by the definition of N_3 and $t' \ge 5$. We divide the interval $(N_3, N_0]$ into the following subintervals:

$$J_{19} = (N_3, M_{23}], J_{23} = (M_{23}, N_0].$$

By Lemma 0.3, we restrict to those N for which $k(N) > 2r - 1$ whenever $N \in J_{2r-1}$, $r = 10, 12$. By calculating t', we find that N is one of the following:

$$1515930, 1539264, 1576501, 1664928, 2053917, 2074051, 2219883, 2324140, 2341680, 2342711, 2386432, 2775456, 2886673, 3237613, 3695514, 5687203, 6169832, 6443469, 6860556, 7490660, 7757686, 8720333, 9558616, 10247124, 10600736, 10655462, 11274670, 11645754, 12672264, 13377906, 14079145, 14289335, 18339279, 24356055, 28244961, 33772762, 42211295, 53468932, 64955634, 110678632, 118374763, 231921327, 264993166, 398367036.$$

By (7), it suffices to consider only the following values of N :

$$1539264, 2053917, 2775456, 12672264, 110678632$$

which are excluded as in (8). This completes the proof of Theorem 2. □

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