BAKER'S EXPLICIT ABC-CONJECTURE AND WARING'S PROBLEM

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Abstract. The conjecture of Masser-Oesterlé, popularly known as *abc*-conjecture has many consequences. We show that Waring's problem is a consequence of an explicit version of $abc-$conjecture due to Baker.

1. Introduction

For any positive integer $i > 1$, let $N = N(i) = \prod_{p|i} p$ be the *radical* of i, $P(i)$ be the greatest prime factor of i and $\omega(i)$ be the number of distinct prime factors of i and we put $N(1) = 1, P(1) = 1$ and $\omega(1) = 0$. The well known conjecture of Masser-Oesterlé states that

Conjecture 1.1. abc-conjecture of Masser and Oesterlé: *For any given $\epsilon > 0$ there exists a computable constant $\mathfrak{c}_\epsilon$ depending only on ϵ such that if*

$$a + b = c \tag{1}$$

where a, b and c are coprime positive integers, then

$$c \leq \mathfrak{c}_\epsilon \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

This is popularly known as $abc-$conjecture. The $abc-$conjecture has already become well known for the number of interesting consequences it entails. Many famous conjectures and theorems in number theory would follow immediately from the $abc-$conjecture. An explicit version of this conjecture due to Baker [Bak94] is the following:

Conjecture 1.2. Explicit $abc-$conjecture: *Let a, b and c be pairwise coprime positive integers satisfying* (1). *Then*

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We observe that $N = N(abc) \geq 2$ whenever a, b, c satisfy (1). We shall refer to Conjecture 1.1 as *$abc-$conjecture* and Conjecture 1.2 as *explicit $abc-$conjecture*. We have

Theorem 1. (*Laishram and Shorey* [LaSh12])
Assume Conjecture 1.2. Let a, b and c be pairwise coprime positive integers satisfying (1) *and $N = N(abc)$. Then we have*

$$c < N^{1+\frac{3}{4}}. \tag{2}$$

Further for $0 < \epsilon \leq \frac{3}{4}$, there exists ω_ϵ depending only ϵ such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq p_{\omega_\epsilon}} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon}$$

where

$$\kappa_\epsilon = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \leq \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$$

with $\omega = \omega(N)$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6460
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

Thus $c < N^2$ which was conjectured in Granville and Tucker [GrTu02].

2. Ideal Waring's Theorem

For each integer $k \geq 2$, denote by $g(k)$ the smallest integer g such that any positive integer is the sum of at most g integers of the form x^k. A result of Euler implies that a lower bound for $g(k)$ is $2^k + \lfloor (3/2)^k \rfloor - 2$. The so-called *Ideal Waring's Theorem* is the following conjecture, dating back to 1853:

Conjecture 2.1. *For any $k \geq 2$, the equality $g(k) = 2^k + \lfloor (\frac{3}{2})^k \rfloor - 2$ holds.*

Theorem 2. *Assume Conjecture 1.2. Then Conjecture 2.1 is true.*

This conjecture has a long and interesting history. We refer to Waldschmidt [Mic00, p. 12] for further details. We now prove Theorem 2.

Proof of Theorem 2: We write

$$3^k = 2^k q + r \text{ with } 0 < r < 2^k \text{ and } q = \lfloor (\frac{3}{2})^k \rfloor.$$

L. E. Dickson and S.S. Pillai (see for instance [HaWr54, Chap. XXI] or [Nar86, p. 226 Chap. IV]) proved independently in 1939 that the ideal Waring's Theorem(Conjecture 2.1) holds provided that the remainder $r = 3^k - 2^k q$ satisfies

$$r \leq 2^k - q - 3. \tag{3}$$

The condition (3) is satisfied for $3 \leq k \leq 471600000$ as well as for sufficiently large k, as shown by K. Mahler [Mah57] in 1957 by means of Ridouts extension of the Thue-Siegel-Roth theorem.

Therefore we may now suppose that $k > 471600000$ and further (3) does not hold, i.e.,

$$r \geq 2^k - q - 2 \tag{4}$$

Let $\gcd(3^k, 2^k(q+1)) = 3^v$ and set

$$a = 3^{k-v}, c = 3^{-v}2^k(q+1) \text{ and } b = c - a = 3^{-v}(2^k - r).$$

Then a, b, c are relatively prime positive integers satisfying $a + b = c$ and

$$b = 3^{-v}(2^k - r) \leq 3^{-v}(q+3)$$

by (4). Then

$$N = N(abc) = N(3^{k-v} \cdot \frac{2^k(q+1)}{3^v} \cdot b) \leq \frac{6b(q+1)}{3^v} \leq \frac{6(q+1)(q+3)}{3^{2v}}. \tag{5}$$

First assume that $N < e^{63727}$. Then by (2), we have

$$2^k \leq \frac{2^k(q+1)}{3^v} < N^{\frac{7}{4}} < e^{63727 \cdot \frac{7}{4}}$$

implying

$$k < \frac{63727 \cdot 7}{4 \cdot \log 2} < 160893.$$

This is a contradiction since $k > 471600000$. Therefore we may suppose that $N \geq e^{63727}$. By Theorem 1 with $\epsilon = \frac{1}{3}$ and (5), we have

$$\frac{2^k(q+1)}{3^v} < \frac{6}{5\sqrt{2\pi \cdot 6460}} \left(\frac{6(q+1)(q+3)}{3^{2v}}\right)^{\frac{4}{3}}.$$

implying

$$2^k < \frac{6^{\frac{7}{3}}}{5\sqrt{12920\pi}} q^{\frac{5}{3}} (1 + \frac{3}{q})^{\frac{5}{3}}.$$

Since $3^k > 2^k q$, we have $q < (\frac{3}{2})^k$. Also $1 + \frac{3}{q} < 2$ since $k \geq 3$. Therefore

$$2^k < \frac{6^{\frac{7}{3}} \cdot 2^{\frac{5}{3}}}{5\sqrt{12920\pi}} (\frac{3}{2})^{\frac{5k}{3}} < \left((\frac{3}{2})^{\frac{5}{3}}\right)^k < 2^k.$$

This is a contradiction. Hence the assertion. $\square$

Acknowledgments

The author would like to thank Michel Waldschmidt for pointing out his paper [Mic00, p. 12] to the author which led to this paper. We would like to also thank the referee for careful reading and suggestions and remarks on an earlier draft of this paper.

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