Number of prime divisors in a product of terms of an arithmetic progression

Shanta Laishram and T.N. Shorey

1 Introduction

Let $d \geq 1, k \geq 3, n \geq 1$ be integers with $\gcd(n, d) = 1$. We denote

$$\Delta = \Delta(n, d, k) = n(n + d) \cdots (n + (k - 1)d).$$

For an integer $\nu > 1$, we write $\omega(\nu)$ and $P(\nu)$ for the number of distinct prime divisors of ν and the greatest prime factor of ν, respectively. Further we put $\omega(1) = 0$ and $P(1) = 1$. For l coprime to d, we write $\pi(\nu, d, l)$ for the number of primes $\leq \nu$ and congruent to l modulo d. Further, we denote by $\pi_d(\nu)$ for the number of primes $\leq \nu$ and coprime to d. The letter p always denote a prime number. Let $W(\Delta)$ denote the number of terms in Δ divisible by a prime $> k$. We observe that every prime exceeding k divides at most one term of Δ. Therefore we have

$$W(\Delta) \leq \omega(\Delta) - \pi_d(k). \tag{1}$$

If $\max(n, d) \leq k$, we see that $n + (k - 1)d \leq k^2$ and therefore no term of Δ is divisible by more than one prime exceeding k. Thus

$$W(\Delta) = \omega(\Delta) - \pi_d(k) \text{ if } \max(n, d) \leq k. \tag{2}$$

Sylvester [17] proved that

$$P(\Delta) > k \text{ if } n \geq d + k$$

and Langevin [6] improved it to

$$P(\Delta) > k \text{ if } n > k.$$

Let $d = 1$. Then Erdős gave another proof of Sylvester's result. It has been improved in [5] to

$$\omega(\Delta) \geq \min\left(\pi(k) + \left[\frac{3}{4}\pi(k)\right] - 1 + \delta(k),\ \pi(2k) - 1\right)$$

where

$$\delta(k) = \begin{cases} 2 \text{ if } k \leq 6 \\ 1 \text{ if } 7 \leq k \leq 16 \\ 0 \text{ otherwise.} \end{cases}$$

This sharpens a result of Saradha and Shorey [12]. For a formulation of this result and a more precise version of the results stated above, see [5]. From now onwards, we suppose that $d > 1$. Shorey and Tijdeman [16] showed that

$$P(\Delta) > k \text{ unless } (n, d, k) = (2, 7, 3). \tag{3}$$

Saradha and Shorey [10] showed that for $k \geq 4$, Δ is divisible by at least 2 distinct primes exceeding k except when $(n, d, k) \in \{(1, 5, 4), (2, 7, 4), (3, 5, 4),$ $(1, 2, 5), (2, 7, 5), (4, 7, 5), (4, 23, 5)\}$. As to the number of prime factors of Δ, Shorey and Tijdeman [15] proved that

$$\omega(\Delta) \geq \pi(k). \tag{4}$$

A conjecture of Schinzel, known as Hypothesis H, implies that there are infinitely many d for which both $1 + d$ and $1 + 2d$ are primes. Thus (4) is likely to be best possible when $k = 3$. Moree [8] sharpened (4) to

$$\omega(\Delta) > \pi(k) \text{ if } k \geq 4 \text{ and } (n, d, k) \neq (1, 2, 5). \tag{5}$$

We observe that (5) implies (3) for $k \geq 4$. If $k = 4$ or 5, then as above, Hypothesis H implies that $\omega(\Delta) = \pi(k) + 1$ for infinitely many d. Further Saradha, Shorey and Tijdeman [14, Theorem 1] improved (5) to

$$\omega(\Delta) > \frac{6}{5}\pi(k) + 1 \text{ for } k \geq 6 \tag{6}$$

unless $(n, d, k) \in V_0$ where V_0 is

$$\{(1, 2, 6), (1, 3, 6), (1, 2, 7), (1, 3, 7), (1, 4, 7), (2, 3, 7), (2, 5, 7), (3, 2, 7),$$
$$(1, 2, 8), (1, 2, 11), (1, 3, 11), (1, 2, 13), (3, 2, 13), (1, 2, 14)\}.$$

In fact they derived (6) from

$$W(\Delta) > \frac{6}{5}\pi(k) - \pi_d(k) + 1 \text{ for } k \geq 6 \tag{7}$$

unless $(n, d, k) \in V_0$. It is easy to see that the preceding result is equivalent to [14, Theorem 2]. The estimate (6) has been applied in [13] and [11]. We have no improvement for (7) when $k = 6, 7$ and 8. For $k \geq 9$, we sharpen (7) as

Theorem 1 *Let $k \geq 9$, $d > 1$ and $(n, d, k) \notin V$ where V is given by*

$$\begin{cases} n = 1,\ d = 3,\ k = 9, 10, 11, 12, 19, 22, 24, 31; \\ n = 2,\ d = 3,\ k = 12;\ n = 4,\ d = 3,\ k = 9, 10; \\ n = 2,\ d = 5,\ k = 9, 10; \\ n = 1,\ d = 7,\ k = 10. \end{cases} \tag{8}$$

Then

$$W(\Delta) \geq \pi(2k) - \pi_d(k) - \rho \tag{9}$$

where

$$\rho = \rho(d) = \begin{cases} 1 \text{ if } d = 2, n \leq k \\ 0 \text{ otherwise.} \end{cases} .$$

When $d = 2$ and $n = 1$, we see that

$$\omega(\Delta) = \pi(2k) - 1$$

and

$$W(\Delta) = \pi(2k) - \pi_d(k) - 1$$

by (2). There are infinitely many pairs (n, k) for which the above relation holds. Therefore (9) is best possible when $d = 2$. On the other hand, for a given d, it has been shown in [14] that

$$\omega(\Delta) \leq \frac{k}{\log k}\bar{d} + C_2 \frac{k \log\log k}{(\log k)^2}\bar{d} \text{ for } \frac{n}{k} \leq d \leq \log k,\ k \geq C_1$$

where C_1 and C_2 are effectively computable absolute constants and

$$\bar{d} = \log 2d + 5.2 \log\log 2d + 5.02,$$

see also [1]. We observe that the exceptions stated in Theorem 1 are necessary. Further we see from Theorem 1 and (1) that

$$\omega(\Delta) \geq \pi(2k) - \rho \text{ if } (n,d,k) \notin V. \tag{10}$$

For $(n,d,k) \in V$, we see that $\omega(\Delta) = \pi(2k) - 1$ except at $(n,d,k) = (1,3,10)$. This is also the case for $(n,d,k) \in V_0$ with $k = 6,7,8$. Now, we apply Theorem 1, (6) for $k = 6,7,8$ and (5) for $k = 4,5$. We conclude

Corollary 1 *Let $k \geq 4$. Then*

$$\omega(\Delta) \geq \pi(2k) - 1. \tag{11}$$

except at $(n,d,k) = (1,3,10)$.

This confirms a conjecture of Moree [8]. Now we give a sketch of the proof of Theorem 1. There is no loss of generality in assuming that $2k-1$ is prime unless (n,d,k) belongs to some finite small set, see Lemma 3. The proofs for the case $d = 2,3,4,5$ and 7 for $n \leq 2k$ if $d = 7$ and $n \leq k$ otherwise depends on the estimates on primes in arithmetic progression. We apply these estimates to count the number of terms of Δ which are of the form ap where $1 \leq a < d$, $\gcd(a,d) = 1$ and $p > k$, see Lemma 5. The proofs of the remaining cases depend on the combinatorial arguments of Sylvester and Erdős. In fact we sharpen the fundamental inequality of Sylvester and Erdős, see Lemma 1. We improve bounds on n,d,k and these enable us to treat the remaining cases on a computer. In the proof of Theorem 1, we first assume that (9) does not hold and give a bound on d. We get $d = 4$ or d is a prime ≤ 53, see Lemmas 6 and 7. For a given d, we give an upper bound for k. If $n \leq k$, we show that $1 \leq n < \min(d, k+1)$, see Lemma 4. Let $n > k$. We reduce the upper bound for k when $n > k$, $n \geq 1.5k$ and so on. We also show that $n \leq 3k$ unless $d = 11$ where $n \leq 4k$. Finally we check that (9) holds for the finitely many remaining possibilities.

We shall follow the notation of this section throughout the paper. We use *MATHEMATICA* for the computations in this paper. We thank N. Saradha and R. Tijdeman for their comments on an earlier version of this paper.

2 Lemmas for the proof of Theorem 1

We begin with the following refinement of a fundamental result of Sylvester and Erdős (see [4, Lemma 2] and [10, Lemma 1]).

Lemma 1 *For* $0 \le i < k$, *let*

$$n + id = B_i B_i' \tag{12}$$

where B_i *and* B_i' *are positive integers such that* $P(B_i) \le k$ *and* $\gcd(B_i', \prod_{p \le k} p) = 1$. *Let* $\mathcal{S} \subset \{B_0, \cdots, B_{k-1}\}$. *Let* $p \le k$ *be such that* $\gcd(p, d) = 1$ *and* p *divides at least one element of* $\mathcal{S}$. *Choose* $B_j \in \mathcal{S}$ *such that* p *does not appear to a higher power in the factorisation of any other element of* $\mathcal{S}$. *Let* $\mathcal{S}_1$ *be the subset of* $\mathcal{S}$ *obtained by deleting from* $\mathcal{S}$ *all such* B_j. *Let* $\mathfrak{P}$ *be the product of all the elements of* $\mathcal{S}_1$ *and let* a *be the number of terms in* $\mathcal{S}_1$ *divisible by* 2. *Also we denote*

$$n_0 = \gcd(n, k-1)$$

and

$$\theta = \begin{cases} 1 \text{ if } 2|n_0 \\ 0 \text{ otherwise.} \end{cases} \tag{13}$$

Then

$$\mathfrak{P} \le n_0 \prod_{p \nmid d} p^{\mathrm{ord}_p((k-2)!)}. \tag{14}$$

Further for d *odd, we have*

$$\mathfrak{P} \le 2^{-\theta} n_0 2^{a + \mathrm{ord}_2([\frac{k-2}{2}]!)} \prod_{p \nmid 2d} p^{\mathrm{ord}_p((k-2)!)}. \tag{15}$$

Proof Let $p < k$, $p \nmid d$ be such that p divides at least one element of $\mathcal{S}$. Let $r_p \ge 0$ be the smallest integer such that $p \mid n + r_p d$. Write $n + r_p d = pn_1$. Then

$$n + r_p d, n + r_p d + pd, \cdots, n + r_p d + p[\frac{k-1-r_p}{p}]d$$

are all the terms in Δ divisible by p. Let $B_{r_p+pi_p}$ be such that p does not divide any other term of $\mathcal{S}$ to a higher power. Let a_p be the number of terms in $\mathcal{S}_1$ divisible by p. We note here that $a_p \le [\frac{k-1-r_p}{p}]$. For any $B_{r_p+pi} \in \mathcal{S}_1$, we have $\mathrm{ord}_p(B_{r_p+pi}) = \mathrm{ord}_p(n + r_pd + pid) \le \mathrm{ord}_p((n + r_pd + pid)) - (n + r_pd + pi_pd)) = 1 + \mathrm{ord}_p(i - i_p)$. Therefore

$$\mathrm{ord}_p(\mathfrak{P}) \le a_p + \mathrm{ord}_p\left(\prod_{\substack{i=0\\ i\ne i_p}}^{[\frac{k-1-r_p}{p}]} (i - i_p)\right) \le a_p + \mathrm{ord}_p\left(i_p![\frac{k-1-r_p}{p} - i_p]!\right) \tag{16}$$

Thus

$$\mathrm{ord}_p(\mathfrak{P}) \le a_p + \mathrm{ord}_p([\frac{k-1-r_p}{p}]!). \tag{17}$$

Let $p \nmid n$. Then $r_p \ge 1$ and hence $a_p \le [\frac{k-2}{p}]$. From (17), we have

$$\mathrm{ord}_p(\mathfrak{P}) \le [\frac{k-2}{p}] + \mathrm{ord}_p([\frac{k-2}{p}]!) = \mathrm{ord}_p((k-2)!). \tag{18}$$

Let $p = 2$. Then $a_2 = a$ and

$$\mathrm{ord}_2(\mathfrak{P}) \le a + \mathrm{ord}_2([\frac{k-2}{2}]!). \tag{19}$$

Let $p|n$. Then $r_p = 0$. Assume that $p \nmid k - 1$. Then from (17), we have

$$\mathrm{ord}_p(\mathfrak{P}) \le a_p + \mathrm{ord}_p([\frac{k-2}{p}]!). \tag{20}$$

Assume $p|(k-1)$ and let $i_0 \in \{0, \frac{k-1}{p}\}$ with $i_0 \ne i_p$ be such that $\mathrm{ord}_p(n + pi_0d) = \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1))$. If $\mathrm{ord}_p(n) = \mathrm{ord}_p(k-1)$, we take $i_0 = 0$ if $i_p \ne 0$ and $i_0 = \frac{k-1}{p}$ otherwise. From (16), we have

$$\mathrm{ord}_p(\mathfrak{P}) \le \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + a_p - 1 + \mathrm{ord}_p\left(\prod_{\substack{i=0\\ i\ne i_0, i_p}}^{\frac{k-1}{p}} (i - i_p)\right).$$

Thus

$$\mathrm{ord}_p(\mathfrak{P}) \le \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + a_p - 1 + \mathrm{ord}_p((\frac{k-1-p}{p})!). \tag{21}$$

From (20) and (21), we conclude

$$\mathrm{ord}_p(\mathfrak{P}) \le \min(\mathrm{ord}_p(n), \mathrm{ord}_p(k-1)) + [\frac{k-2}{p}] + \mathrm{ord}_p([\frac{k-2}{p}]!)$$

since $a_p \le [\frac{k-1}{p}]$. Thus

$$\operatorname{ord}_p(\mathfrak{P}) \le \min(\operatorname{ord}_p(n), \operatorname{ord}_p(k-1)) + \operatorname{ord}_p((k-2)!). \tag{22}$$

Now (14) follows from (18) and (22). Let $p = 2$. Then $a_2 = a$. Hence by (20) and (21), we have in case of even n

$$\operatorname{ord}_2(\mathfrak{P}) \le \min(\operatorname{ord}_2(n), \operatorname{ord}_2(k-1)) - \theta + a + \operatorname{ord}_2([\frac{k-2}{2}]!)$$

which together, with (18), (19) and (22), implies (15). □

Let

$$\chi = \chi(n) = \begin{cases} \min\left(1, \frac{k-1}{n}\prod_{p|2d} p^{-\operatorname{ord}_p(k-1)}\right) & \text{if } 2 \nmid n \\ \min\left(2^{\theta-1}, \frac{k-1}{n}\prod_{p|d} p^{-\operatorname{ord}_p(k-1)}\right) & \text{if } 2 \mid n \end{cases}$$

and

$$\chi_1 = \chi_1(n) = \min\left\{1, \frac{k-1}{n}\prod_{p|d} p^{-\operatorname{ord}_p(k-1)}\right\}.$$

We observe that χ is non-increasing function of n even and n odd separately. Further χ_1 is a non-increasing function of n. We also check that

$$\frac{n_0}{n} \le \chi \le \chi_1 \tag{23}$$

and $\chi(1) = 1$, $\chi(2) = 2^{\theta-1}$.

In the next lemma, we present lower bounds for $\operatorname{ord}_p(k-1)!$, $\pi(x, d, l)$ for some values of d and $\pi(2x, 7, l) - \pi(x, 7, l)$.

Lemma 2 *We have*

$$(i)\ \operatorname{ord}_p(k-1)! \ge \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$$

and

$$(ii)\ \pi(x) \le \frac{x}{\log x}\left(1+\frac{1.2762}{\log x}\right) \text{ for } x>1,$$
$$(iii)\ \pi(x,3,l) \ge 0.49585\frac{x}{\log x} \text{ for } x \ge 91807,$$
$$(iv)\ \pi(x,4,l) \ge 0.45402\frac{x}{\log x} \text{ for } x \ge 1500,$$
$$(v)\ \pi(x,5,l) \ge 0.22894\frac{x}{\log x} \text{ for } x \ge 4500,$$
$$(vi)\ \pi(x,7,l) \ge 0.14308\frac{x}{\log x} \text{ for } x \ge 2200,$$
$$(vii)\ \pi(2x,7,l)-\pi(x,7,l) \le 0.22636\frac{x}{\log x} \text{ for } x \ge 2000$$

Proof For (i), let $p^r \le k-1 < p^{r+1}$. Then we have

$$\begin{aligned}\mathrm{ord}_p(k-1)! &= \left[\frac{k-1}{p}\right]+\cdots+\left[\frac{k-1}{p^r}\right]\\ &\ge \sum_{i=1}^{r}\left(\frac{k}{p^i}-1\right) = \frac{k}{p-1}(1-\frac{1}{p^r})-r \ge \frac{k-p}{p-1}-\frac{\log(k-1)}{\log p}\end{aligned}$$

since $\left[\frac{k-1}{p^i}\right] \ge \frac{k-1}{p^i}-\frac{p^i-1}{p^i}=\frac{k}{p^i}-1$ for $i \ge 1$, giving (i). The estimate (ii) on $\pi(x)$ is due to Dusart [2, p.14]. See also [3, p.55]. The estimate (iii) is derived from [7, Theorem 5.3]. The estimates $(iv)-(vii)$ are derived from [9, Theorems 1, 2]. $\square$

Lemma 3 *Let $k_0 \ge 9$ be such that $2k_0-1$ is a prime. Suppose that*

$$W(\Delta(n,d,k)) \ge \pi(2k)-\pi_d(k)-\rho \tag{24}$$

holds for every $k \ge k_0$ such that $2k-1$ is prime. Then

$$W(\Delta(n,d,k)) \ge \pi(2k)-\pi_d(k)-\rho$$

for all $k \ge k_0$.

Proof There exist $k_0 \le k_1 < k_2$ such that $2k_1-1 \le 2k-1 < 2k_2-1$ and $2k_1-1, 2k_2-1$ are consecutive primes. Then

$$W(\Delta(n,d,k)) \ge W(\Delta(n,d,k_1)) \ge \pi(2k_1)-\pi_d(k_1)-\rho \ge \pi(2k)-\pi_d(k)-\rho$$

since $\pi_d(k) \ge \pi_d(k_1)$. $\square$

Lemma 4 *Let* $\max(n, d) \le k$*. Let* $1 \le r < k$ *with* $\gcd(r, d) = 1$ *be such that*

$$W(\Delta(r, d, k)) \ge \pi(2k) - \rho.$$

Then for each n *with* $r < n \le k$ *and* $n \equiv r (\text{mod } d)$*, we have*

$$W(\Delta(n, d, k)) \ge \pi(2k) - \rho.$$

Proof For $r < n \le k$, we write

$$\Delta(n, d, k) = \Delta(r, d, k) \frac{(r + kd) \cdots (n + (k-1)d}{r(r+d) \cdots (n-d)}.$$

We observe that $p \mid \Delta(n, d, k)$ for every prime $p > k$ dividing $\Delta(r, d, k)$. □

Lemma 5 *Let* $d \le k$*. For each* $1 \le r < d$ *with* $\gcd(r, d) = 1$*, let* r' *be such that* $rr' \equiv 1 (\text{mod } d)$*. Then*

(a) For a given n *with* $1 \le n \le k$*, Theorem 1 holds if*

$$\sum_{\substack{1 \le r < d \\ \gcd(r,d)=1}} \pi\left(\frac{n + (k-1)d}{r}, d, nr'\right) - \pi(2k) + \rho \ge 0 \tag{25}$$

is valid.

(b) For a given n *with* $k < n < 1.5k$*, Theorem 1 holds if*

$$\sum_{\substack{1 \le r < d \\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1) - d + 1}{r}, d, nr'\right) - \pi(2k) + \pi(k, d, n) - \pi(1.5k, d, n) \ge 0 \tag{26}$$

is valid.

(c) For a given n *with* $k < n \le 2k$*, Theorem 1 holds if*

$$\sum_{\substack{1 \le r < d \\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1) - d + 1}{r}, d, nr'\right) - \pi(2k) + \pi(k, d, n) - \pi(2k, d, n) \ge 0 \tag{27}$$

is valid.

Proof Let $1 \le r < d \le k$, $\gcd(r, d) = 1$. Then for each prime $p \equiv nr' (\text{mod } d)$ with $\max(k, \frac{n-1}{r}) < p \le \frac{n+(k-1)d}{r}$, there is a term $rp = n + id$ in $\Delta(n, d, k)$. Therefore

$$W(\Delta(n, d, k)) \ge \sum_{\substack{1 \le r < d \\ \gcd(r,d)=1}} \left(\pi\left(\frac{n + (k-1)d}{r}, d, nr'\right) - \pi(\max(k, \frac{n-1}{r}), d, nr')\right). \tag{28}$$

Since

$$(29) \qquad \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi(k,d,nr') = \pi_d(k),$$

it is enough to prove (25) for deriving (9) for $1 \le n \le k$. This gives (a).

Let $k < n < k^{'}$ where $k^{'} = 1.5k$ or $2k+1$. Then from (28) and (29), we have

$$\begin{aligned} W(\Delta(n,d,k)) &\ge \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \left(\pi\left(\frac{k+1+(k-1)d}{r}, d, nr'\right) - \pi(\max(k, \frac{k^{'}-1}{r}), d, nr')\right) \\ &\ge \sum_{\substack{1\le r<d\\ \gcd(r,d)=1}} \pi\left(\frac{k(d+1)-d+1}{r}, d, nr'\right) - \pi(k^{'}-1,d,n) - \pi_d(k) + \pi(k,d,n) \end{aligned}$$

since $r' = 1$ for $r = 1$. Hence it suffices to show (26) for proving (9) for $k < n < 1.5k$ or (27) for proving (9) for $k < n \le 2k$. Hence (b) and (c) is valid. $\square$

3 Proof of Theorem 1

We suppose that $(n,d,k) \notin V$, $k \ge 34$ if $(n,d) = (1,3)$, $k \ge 15$ if $(n,d) = (2,3)$ and $k \ge 12$ if $(n,d) = (4,3), (2,5), (1,7)$. By Lemma 3, we also assume that $2k-1$ is prime. Further we take $n > k$ whenever $d = 2$. Thus $\rho = 0$ always. We assume that (9) is not valid and we shall arrive at a contradiction. Let

$$(30) \qquad R = \pi(2k) - \pi_d(k) - 1.$$

Then $W(\Delta) \le R$. Let S be the set of all terms of Δ composed of primes not exceeding k. Then $|S| \ge k - R$. For every p dividing an element of S, we delete an $f(p) \in S$ such that

$$\mathrm{ord}_p(f(p)) = \max_{s\in S} \mathrm{ord}_p(s).$$

Then we are left with a set T with $1 + t := |T| \ge k - \pi(2k) + 1$ elements of S. We arrange the elements of T as $n + i_0 d < n + i_1 d < \cdots < n + i_t d$. Let

$$\mathcal{P} := \prod_{\nu=0}^{t} (n + i_\nu d) = (n + i_0 d)(\alpha + i_1) \cdots (\alpha + i_t) d^t$$

with $n=\alpha d$. We now apply Lemma 1 with $\mathcal{S}=S$ and $\mathcal{S}_1=T$ so that $\mathfrak{P}=\mathcal{P}$. Thus the estimates (14) and (15) are valid for $\mathcal{P}$. Comparing $\mathcal{P}$ with its upper bound given by (14), we have

$$d^{k-\pi(2k)} \le \frac{n_0}{n} \frac{\prod\limits_{p\nmid d} p^{\mathrm{ord}_p((k-2)!)}}{(\alpha+i_1)\cdots(\alpha+i_{k-\pi(2k)})} \tag{31}$$

and

$$\frac{(n+i_0d)(n+i_1d)\cdots(n+i_td)}{2^a} \le 2^{-\theta} n_0 2^{\mathrm{ord}_2([\frac{k-2}{2}]!)} \prod_{p\nmid 2d} p^{\mathrm{ord}_p((k-2)!)} \text{ for } d \text{ odd} \tag{32}$$

where a is the number of even elements in T. From (31) and (23), we have

$$d^{k-\pi(2k)} \le \chi_1(n) \frac{(k-2)! \prod\limits_{p|d} p^{-\mathrm{ord}_p((k-2)!)}}{(\alpha+1)\cdots(\alpha+k-\pi(2k))} \tag{33}$$

since $n=\alpha d$, which is also same as

$$\prod_{i=1}^{k-\pi(2k)} (n+id) \le \chi_1(n)(k-2)! \prod_{p|d} p^{-\mathrm{ord}_p((k-2)!)}. \tag{34}$$

From (33), we derive

(35)

$$d^{k-\pi(2k)} \le \begin{cases} \chi_1(n)[\alpha]!(k-2)\cdots([\alpha]+k-\pi(2k)+1)\prod\limits_{p|d} p^{-\mathrm{ord}_p(k-2)!} & \text{if } [\alpha] \le \pi(2k)-3 \\ \chi_1(n)[\alpha]!\prod\limits_{p|d} p^{-\mathrm{ord}_p(k-2)!} & \text{if } [\alpha]=\pi(2k)-2 \\ \chi_1(n)\frac{[\alpha]!}{(k-1)k(k+1)\cdots([\alpha]+k-\pi(2k))}\prod\limits_{p|d} p^{-\mathrm{ord}_p(k-2)!} & \text{if } [\alpha] \ge \pi(2k)-1. \end{cases}$$

We observe that the right hand sides of (33), (34) and (35) are non-increasing functions of $n=\alpha d$ when d and k are fixed. Thus (35) and hence (33) and (34) are not valid for $n \ge n_0$ whenever it is not valid at $n_0=\alpha_0 d$ for given d and k. This will be used without reference throughout the paper. We obtain from (33) and $\chi_1 \le 1$ that

$$d^{k-\pi(2k)} \le (k-2)\cdots(k-\pi(2k)+1)\prod_{p|d} p^{-\mathrm{ord}_p(k-2)!} \tag{36}$$

which implies that

$$d^{k-\pi(2k)} \le \begin{cases} (k-2)\cdots(k-\pi(2k)+1)2^{-\mathrm{ord}_2(k-2)!} \text{ if } d \text{ is even} \\ (k-2)\cdots(k-\pi(2k)+1) \text{ if } d \text{ is odd} \end{cases} \tag{37}$$

and

$$d \le (k-2)^{\frac{\pi(2k)-2}{k-\pi(2k)}} \prod_{p|d} p^{\frac{-\mathrm{ord}_p(k-2)!}{k-\pi(2k)}}. \tag{38}$$

Using Lemma 2 (i), (ii), we derive from (38) that

(39)

$$d \le \exp\left[\frac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - \frac{2\log(k-2)}{k}}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] \prod_{p|d} p^{-\max\{0, \left(\frac{k-1-p}{p-1} - \frac{\log(k-2)}{\log p}\right)/\left(k - \frac{2k}{\log 2k}(1+\frac{1.2762}{\log 2k})\right)\}}$$

which implies

$$d \le \begin{cases} \exp\left[\frac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - \frac{2\log(k-2)}{k} - \left((1-\frac{3}{k})\log 2 - \frac{\log(k-2)}{k}\right)}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] \text{ for } d \text{ even} \\ \exp\left[\frac{\frac{2\log(k-2)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - \frac{2\log(k-2)}{k}}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})}\right] \text{ for } d \text{ odd.} \end{cases} \tag{40}$$

We use the inequalities (34)-(40) at several places.

Let d be odd. Then for n even, $2 \mid n+id$ iff i is even and for n odd, $2 \mid n+id$ iff i is odd. Let $b = k-\pi(2k)+1-a$ and $a_0 = \min(k-\pi(2k)+1, [\frac{k-2+\theta}{2}])$. We note here that $a \le [\frac{k-2+\theta}{2}]$ where θ is given by (13). Let n_e, d_e, n_o and d_o be positive integers with n_e even and n_o odd. Let $n \ge n_e$ and $d \le d_e$ for n even, and $n \ge n_o$ and $d \le d_o$ for n odd. Assume (32). The left hand side of (32) is greater than

$$\begin{cases} \frac{n}{2} d^{k-\pi(2k)} \prod_{i=1}^{a-1}\left(\frac{n_e}{2d_e}+i\right)\prod_{j=1}^{b}\left(\frac{n_e}{d_e}+2j-1\right) =: \frac{n}{2}d^{k-\pi(2k)}F(a) \text{ if } n \text{ is even} \\ nd^{k-\pi(2k)}\prod_{i=1}^{a}\left(\frac{n_o}{2d_o}+i-\frac{1}{2}\right)\prod_{j=1}^{b-1}\left(\frac{n_o}{d_o}+2j\right) =: nd^{k-\pi(2k)}G(a) \text{ if } n \text{ is odd.} \end{cases}$$

Let $A_e := \min\left\{a_0, \lceil\frac{2}{3}(k-\pi(2k)) + \frac{n_e}{6d_e} + \frac{1}{3}\rceil\right\}$ and $A_o := \min\left(a_0, \left\lceil\frac{2}{3}(k-\pi(2k)) + \frac{n_o}{6d_o} - \frac{1}{6}\right\rceil\right)$. We see that the functions $F(a)$ and $G(a)$ take minimal values at A_e and A_o, respectively. Thus (32) with (23) implies that

$$d^{k-\pi(2k)}F(A_e) \le 2^{-\theta+1}\chi(n_e)2^{\mathrm{ord}_2([\frac{k-2}{2}]!)}\prod_{p\nmid 2d} p^{\mathrm{ord}_p(k-2)!} \text{ for } n \text{ even} \tag{41}$$

since $\chi(n) \le \chi(n_e)$ and

$$d^{k-\pi(2k)}G(A_o) \le \chi(n_o)2^{\mathrm{ord}_2([\frac{k-2}{2}]!)}\prod_{p\nmid 2d} p^{\mathrm{ord}_p((k-2)!)} \text{ for } n \text{ odd} \tag{42}$$

since $\chi(n) \le \chi(n_o)$.

Lemma 6 *Let d be even. Then (9) holds for every $d > 4$.*

Proof Let d be even. By (40), $d \le 6$ for $k \ge 860$. For $k < 860$, we use (37) to derive that

$$d \le \begin{cases} 6 \text{ for } k > 255 \text{ except at } k = 262, 310, 331, 332, 342 \text{ where } d \le 8 \\ 8 \text{ for } k > 57 \text{ except at } k = 100 \text{ where } d \le 10;\ 12 \text{ for } k \ge 9. \end{cases} \tag{43}$$

Let d be a multiple of 6. Then we see from (39) that $k \le 100$. Also for $k \le 100$, (36) does not hold. Let d be a multiple of 10. Then we see from (43) that $k = 100$ or $k \le 57$. Again, (36) does not hold at these values of k.

Let $d = 8$. By (43), we may assume that $k \le 255$ or $k = 262, 310, 331, 332, 342$. Let $n \le k$. From Lemma 4, it suffices to prove (9) for $n = 1, 3, 5, 7$. This is valid. Let $n > k$. We see see that (34) does not hold with $n_0 = k + 1$. As observed earlier, the right hand side of (34) is a non-increasing function of n, whereas the left hand side is an increasing function of n. Thus (34) does not hold for all $n > k$. Hence the assertion of Lemma 6 follows. □

Lemma 7 *Let d be odd. Then (9) holds for every composite d and all primes $d > 53$.*

Proof Let d be odd. By (40), $d \le 15$ for $k \ge 3630$. For $k < 3630$, we use (37) to derive that $d \le 15$ for $k \ge 2164$, $d \le 59$ for $k \ge 9$ except at $k = 10, 12$, and $d \le 141$ for $k = 10, 12$.

We shall be using (41) with $n_e = 2, \chi(n_e) = 2^{\theta-1}$ and (42) with $n_o = 1, \chi(n_o) = 1$ unless otherwise specified. Let $k < 2164$. We take $d_e = d_o = 59$ when $k \ne 10, 12$ and

$d_e = d_o = 141$ for $k = 10, 12$. We check that (41) is contradicted or

$$(44) \qquad d \leq \begin{cases} 15 \text{ for } k > 957 \text{ unless } k = 1072, 1077, 1081 \\ 17 \text{ for } k > 387 \text{ unless } k = 415, 420, 432, 442, 444 \\ 21 \text{ for } k > 100 \text{ unless } k = 106, 117, 121, 136, 139, 141, 142, 147, 159 \\ 27 \text{ for } k \geq 9 \text{ except at } k = 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57 \\ 57 \text{ for } k = 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57. \end{cases}$$

Further we check that (42) is either contradicted or (44) holds. Thus we may assume (44). Let $d > 3$ be a multiple of 3. Then $k \leq 1600$ by (39) and $k \leq 850$ by (36). Further we apply (41) and (42) with $d_e = d_o = 57$ to conclude that $k \leq 147$ or $k = 157, 159, 232, 234$ and $d = 9$ unless $k = 10$ for which $d \leq 15$. The case $d = 15$ and $k = 10$ is excluded by applying (41) and (42) with $d_e = d_o = 15$. Let $d = 9$. We may suppose that $k \leq 147$ or $k = 157, 159, 232, 234$. Let $n \leq k$. From Lemma 4, it suffices to prove (9) for $1 \leq n < 9$ and $\gcd(n, 3) = 1$. This is valid. Let $n > k$. Taking $n_e = 2\lceil \frac{k+1}{2} \rceil, n_o = 2\lceil \frac{k}{2} \rceil + 1, d_e = d_o = 9$, we see that (41) and (42) are not valid and hence (9) holds for $n > k$.

Let $d > 15$ be a multiple of 5. Then $k \leq 159$ by (44). Now, by taking $d_e = d_o = 55$, we see that (41) and (42) do not hold unless $k = 10, d = 25$ and n is odd. We observe that (42) with $n_o = 3$ and $d_o = 25$ is not valid at $k = 10$. Thus $(n, d, k) = (1, 25, 10)$ and we check that (9) holds. Let $d > 7$ be a multiple of 7. Then, we see from (44) that $d = 49$ and $k = 10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57$ since d is not a multiple of 3 and 5. Taking $d_e = d_o = 49$, we see that both (41) and (42) do not hold. Hence the assertion of Lemma 7 follows. □

Lemma 8 *Let $d = 2, 3, 4, 5$ and 7. Assume that $n \leq 2k$ if $d = 7$, $n \leq k$ otherwise and $(n, d, k) \notin V$. Then (9) holds.*

Proof First, we consider the case $1 \leq n \leq k$. By Lemma 4, it suffices to prove (25) for n with $1 \leq n < d$. Let $d = 2$. Then

$$\pi(1 + 2(k-1), 2, 1) - \pi(2k) + 1 = \pi(2k-1) - 1 - \pi(2k-1) + 1 \geq 0.$$

Hence the assertion follows.

Let $d = 3$. We may assume that $k \neq 9, 10, 11, 12, 19, 22, 24, 31$ otherwise the assertion follows by direct computations. By Lemma 4, it suffices to prove (25) for $n = 1, 2$. By using the bounds for $\pi(x, 3, l)$ and $\pi(x)$ from Lemma 2, we see that the left hand side of (25) is at least

$$k \left\{ \sum_{1}^{2} \frac{0.49585(\frac{3}{i} - \frac{2}{ik})}{\log \frac{1+3k-3}{i}} - \frac{2}{\log 2k} \left(1 + \frac{1.2762}{\log 2k} \right) \right\}$$

which is an increasing function of k and it is non-negative at $k = 150500$. For $k < 150500$, we check using the exact values of $\pi(x, 3, l)$ and $\pi(x)$ that (25) holds. Let $d = 4, 5$ and 7. We may assume that k is different from those given by $(n, d, k) \in V$ otherwise the assertion follows by direct computations. By using the bounds for $\pi(x, d, l)$ and $\pi(x)$ from Lemma 2, we see that (25) holds for $k \geq 1900$ for $d = 4$, $k \geq 4500$ for $d = 5$ and $k \geq 2200$ for $d = 7$. Therefore we conclude from Lemma 5 that k is less than $1900, 4500$ and 2200 according as $d = 4, 5$ and 7, respectively. For these values of k, we check that (9) is valid.

Let $k < n \leq 2k$, $d = 7$. By Lemma 5, it suffices to prove (27). By using the bounds for $\pi(x, 7, l), \pi(2x, 7, l) - \pi(x, 7, l)$ and $\pi(k)$ from Lemma 2, we see that (27) is valid for $k \geq 2000$. Thus $k < 2000$. Taking $n_e = 2\lceil \frac{k+1}{2} \rceil, n_o = 2\lceil \frac{k}{2} \rceil + 1, d_e = d_o = 7$, we see that (41) and (42) do not hold for $k > 342$. Let $k \leq 342$. Taking $n_e = 2\lceil \frac{1.5k}{2} \rceil, n_o = 2\lceil \frac{1.5k-1}{2} \rceil + 1, d_e = d_o = 7$, we see that (41) and (42) do not hold except at $k = 10, 12, 24, 37, 40, 42, 54, 55, 57, 100$. For these values of k, we now check that (9) holds at $1.5k \leq n \leq 2k$. Further we check that (9) holds at $k < n < 1.5k$ for $9 \leq k \leq 342$. Hence the assertion follows. □

Lemma 9 *Let $d = 2, 3, 4, 5$ and 7. Assume $n > k$ if $d \neq 7$ and $n > 2k$ if $d = 7$. Then either (9) holds or*

$k \leq 5266$ when $d = 2$

$k \leq 3226$ or $k = 3501, 3510, 3522$ when $d = 3$

$k = 12, 16, 22, 24, 31, 37, 40, 42, 52, 54, 55, 57, 100, 142$ when $d = 4$

$k \leq 901$ or $k = 940$ when $d = 5$.

Proof Assume $n > k$ and $d \neq 7$. Let $d = 2$. Then we take $\alpha_0 = \frac{k+1}{2}$ so that $n \geq k + 1 = \alpha_0 d$. Further we observe that $\alpha_0 \geq \pi(2k) - 1$ for $k \geq 43$. By induction, we observe that $k! \le \left(\frac{499!}{500^{499}}k^{500}\right)^{\frac{k}{500}} \le (\frac{k}{2.6964})^k$ for $k \ge 500$. Now we apply the preceding inequality and Lemma 2 (i), (ii) in (35). We derive that

$$2 \le \ \exp\left[\frac{\frac{2\log(k+1)}{\log 2k}(1+\frac{1.2762}{\log 2k}) - \frac{\log 5.3928}{2} - \frac{\log k+\log(k+1)}{k}}{1-\frac{2}{\log 2k}(1+\frac{1.2762}{\log 2k})+(1-\frac{2}{k})-\frac{\log(k-1)}{k\log 2}}\right]$$

for $k \ge 1000$. This does not hold for $k \ge 34500$. Thus $k < 34500$. Further $k \le 5266$ from (34) by taking $n_0 = k+1$ and hence for $n > k$. Similarly we derive for $d = 4$ that (34) does not hold except for the values of k stated in Lemma 9.

Let $d = 3$ and 5. We continue as in $d = 2$ case to derive that (34) does not hold for $k \ge 5775$ if $d = 3$ and $k \ge 2164$ if $d = 5$. Let $d = 3$. We may assume that $k < 5775$. Taking $n_e = 2\lceil\frac{k+1}{2}\rceil, n_o = 2\lceil\frac{k}{2}\rceil + 1, d_e = d_o = 3$, we see that (41) and (42) does not hold for $k \ge 3235$ except at $k = 3501, 3510, 3522$. Therefore $k \le 3226$ or $k = 3501, 3510, 3522$ since $2k-1$ is a prime. For $d = 5$, the assertion follows similarly from (41) and (42) with $n_e = 2\lceil\frac{k+1}{2}\rceil, n_o = 2\lceil\frac{k}{2}\rceil + 1, d_e = d_o = 5$.

Let $d = 7$ and $n > 2k$. We take $\alpha_0 = \frac{2k+1}{7}$. Then $\alpha_0 \ge \pi(2k) - 1$ for $k \ge 1526$. As in the case $n > k$ and $d = 2$, we see from (35) that $k < 1750$. Let $n_e = 2k+2, n_o = 2k+1, d_e = d_o = 7$. Then we see that (41) and (42) do not hold. $\square$

3.1 $d = 2, 4$

Let $d = 2$. From Lemmas 8 and 9, we may suppose that $k \le 5266$ and $n > k$. Let $n < 1.5k$. For the values of n and k given by $k < n < 1.5k$ and $k \le 5266$, we check that (26) is valid except at $k = 9, 10, 12$. By Lemma 5, we may restrict to $k = 9, 10, 12$. Now we check that (9) holds in each of the above possibilities. Let $n \ge 1.5k$. We check that (34) with $n_0 = \lceil 1.5k\rceil$ does not hold except at $k = 16, 24, 54, 55, 57, 100, 142$. Thus we may assume that $k = 16, 24, 54, 55, 57, 100, 142$. Let $n_0 = 2k+1$. Then (34) does not hold for these values of n_0 and k. Therefore, we may suppose that $n \le 2k$. Then we check that (9) holds.

Let $d = 4$. From Lemma 8 and 9, we may suppose that $n > k$ and $k = 12, 16, 22, 24, 31, 37, 40\ 42, 52, 54, 55, 57, 100, 142$. For these values of k, we see that (34) with $n_0 = \lceil 1.5k\rceil$ does not hold and hence for $n \ge 1.5k$. Thus $k < n < 1.5k$. Then we check that (9) is valid.

Thus it remains to consider only the cases $3 \le d \le 53$ with d prime by Lemmas 6 and 7. From now onwards, we always assume that d is prime.

3.2 $d = 3, 5, 7$

Let $d = 3$. By Lemma 8 and 9, we may assume that $k \le 3226$ or $k = 3501, 3510, 3522$ and $n > k$. Let $n \ge 1.5k$. Taking $n_e = 2\lceil \frac{1.5k}{2} \rceil, n_o = 2\lceil \frac{1.5k-1}{2} \rceil + 1, d_e = d_o = 3$, we see that (41) and (42) do not hold for $k \ge 9$ except at $k = 54, 55, 57$. For these values of k, taking $n_e = 2k+2, n_o = 2k+1, d_e = d_o = 3$, we see that (41) and (42) do not hold. Thus $n \le 2k$. For these values of n and k, we check that (9) holds. Let $k < n < 1.5k$. We observe that $nr' \equiv 1 (\text{mod } 3)$ or $2 (\text{mod } 3)$. We check that (26) holds for $9 \le k \le 3226$ or $k = 3501, 3510, 3522$. Now the assertion follows from Lemma 5. The proof for the case $d = 5$ and $k \ne 10$ is similar. Let $d = 5$ and $k = 10$. Putting $n_e = 2k+6, n_o = 2k+1, d_e = d_o = 5$, we see that (41) and (42) does not hold. Thus $n \le 2k+4$. For $k < n < 2k+4$, we check that (9) is valid. If $d = 7$, then we derive from Lemmas 8 and 9 that (9) holds.

3.3 $d \ge 11$

Let $d = 11$. From (39), we see that $k \le 11500$. By (36), we see that $k \le 5589$. Putting $n_e = 2, n_o = 1, d_e = d_o = 11$, we see from (41) and (42) that either $k \le 2977$ or $k = 3181, 3184, 3187, 3190, 3195, 3199$. Now we check (9) for $1 \le n < 11$. Then $n > k$ by Lemma 4. Taking $n = k+1$, we see that (34) does not hold for $k > 252$. Thus we may assume that $k \le 252$. Taking $n_e = 2\lceil \frac{k+1}{2} \rceil, n_o = 2\lceil \frac{k}{2} \rceil + 1, d_e = d_o = 11$ in (41) and (42), we see that $k = 9, 10, 12, 16, 21, 22, 24, 27, 31, 37, 40, 42, 45, 52, 54, 55, 57, 70, 91, 99, 100, 121, 142$. Let $n_e = 2\lceil \frac{1.5k}{2} \rceil, n_o = 2\lceil \frac{1.5k-1}{2} \rceil, d_e = d_o = 11$. Then (41) and (42) imply that $k = 10, 22, 37, 42, 54, 55, 57$. For these values of k, we apply (41) and (42) with $n_e = 2k+2, n_o = 2k+1, d_e = d_o = 11$. Then $k = 10, 22$. For these values of k, we check that (41) and (42) are not valid at $n_e = 4k, n_o = 4k+1, d_e = d_o = 11$. Now, we check that (9) holds at $k < n < \lceil 1.5k \rceil$ with $k = 9, 10, 16, 22, 24, 31, 37, 40, 42, 45, 52, 54, 55, 57, 70, 99, 100, 142$, at $1.5k \le n \le 2k$ with $k = 10, 22, 42, 54, 55, 57$ and at $2k < n \le 4k$ with $k = 10, 22, 55$.

The proof for the case $d = 13$ is similar to that of $d = 11$. For $d = 17, 19, 23$, we apply (44) given in the proof of Lemma 7 to estimate k and continue as in the case $d = 11$. For $d > 23$, $k \in \{10, 12, 16, 22, 24, 31, 37, 40, 42, 54, 55, 57\}$ by (44). Firstly we check that (9) holds for $1 \le n \le \min(d, k)$ and coprime to d. Thus $n > k$. Taking $n_e = 2\lceil \frac{k+1}{2} \rceil, n_o = 2\lceil \frac{k}{2} \rceil + 1, d_e = d_o = 53$ in (41) and (42), we see that $k \in \{10, 12, 16, 24, 37, 55, 57\}$. For these values of k, taking $n_e = 2\lceil \frac{3k+1}{2} \rceil, n_o = 2\lceil \frac{3k}{2} \rceil + 1, d_e = d_o = 53$, we see that (41) and (42) does not hold. Thus $n < k \le 3k$. We check that (9) holds at the above possibilities of n, d and k.

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School of Mathematics
Tata Institute of Fundamental Research
Mumbai 400 005 India
shanta@math.tifr.res.in
shorey@math.tifr.res.in