Extensions of Schur's irreducibility results

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Abstract

We prove that the generalised Laguerre polynomials $L_n^{(\alpha)}(x)$ with $0 \leqslant \alpha \leqslant 50$ are irreducible except for finitely many pairs (n, α) and that these exceptions are necessary. In fact it follows from a more general statement.

1. Introduction

For $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$ with $n \geqslant 1$, we define the generalised Laguerre polynomials of degree n as

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{(n-j)!j!}.$$

There is an extensive literature on Laguerre polynomials. In particular, the irreducibility of these class of orthogonal polynomials has been well studied. The irreducibility of $L_n^{(-2n-1)}$ proved by Filaseta and Trifonov [FiTr02] is equivalent to the fact that all Bessel polynomials are irreducible. Also Laguerre polynomials provide examples of polynomials of degree n with associated Galois group A_n where A_n is the alternating group on n symbols and the irreducibility of $L_n^{(n)}$ proved by Filaseta, Kidd and Trifonov [FiKiTr] has been used to settle explicitly the *Inverse Galois problem* that for every $n > 1$ there exists an explicit polynomial of degree n with associated Galois group A_n. We prove

Theorem 1. *Let $0 \leqslant \alpha \leqslant 50$. Then $L_n^{(\alpha)}(x)$ is irreducible except when $n = 2, \alpha \in \{2, 7, 14, 23, 34, 47\}$ and $n = 4, \alpha \in \{5, 23\}$ where it has a linear factor.*

For the exceptions, we have

$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6)$; $L_2^{(7)}(x) = \frac{1}{2}(x-6)(x-12)$;
$L_2^{(14)}(x) = \frac{1}{2}(x-12)(x-20)$; $L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30)$;
$L_2^{(34)}(x) = \frac{1}{2}(x-30)(x-42)$; $L_2^{(47)}(x) = \frac{1}{2}(x-42)(x-56)$;
$L_4^{(5)}(x) = \frac{1}{24}(x-6)(x^3-30x^2+252x-504)$;
$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040)$.

Theorem 1 is an extension of a result of Filaseta, Finch and Leidy [FiFiLe] where they proved that $L_n^{(\alpha)}(x)$ is irreducible for all n and $0 \leqslant \alpha \leqslant 10$ except when $(n, \alpha) \in \{(2,2), (4,5), (2,7)\}$. Therefore we shall always assume that $\alpha > 10$ in the proof of Theorem 1. We also consider the problem of finding factors of Laguerre polynomials. We have

Theorem 2. *Let $1 \leqslant k \leqslant \frac{n}{2}$ and $0 \leqslant \alpha \leqslant 5k$. Then $L_n^{(\alpha)}(x)$ has no factor of degree k except when $k = 1, (n, \alpha) \in \{(2,2), (4,5)\}$.*

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The Laguerre polynomials are a special case of generalizations of following class of polynomials first considered by Schur. Let $n \geqslant 1, a \geqslant 0$ and $a_0, a_1, \ldots, a_n$ be integers. The *generalized Schur polynomials* are defined as

(1)
$$f(x) := f_{n,a}(x) := f_{n,a}(a_0, a_1, \cdots, a_n) = a_n \frac{x^n}{(n+a)!} + a_{n-1}\frac{x^n}{(n-1+a)!} + \ldots + a_1\frac{x}{(1+a)!} + a_0\frac{1}{a!}.$$

It is easy to see that by taking

$$a = \alpha \text{ and } a_j = (-1)^j \binom{n}{j} \text{ for } 0 \leqslant j \leqslant n,$$

we obtain $(n+\alpha)! f_\alpha(x) = n! L_n^{(\alpha)}(x)$.

Schur [Sch29] proved that $f(x)$ with $a = 0$ and $|a_0| = |a_n| = 1$ is irreducible. He also proved in [Sch73] that $f(x)$ with $a = 1$ and $|a_0| = |a_n| = 1$ is irreducible unless $n+1 = 2^r$ for some r where it may have a linear factor or $n = 8$ where it may have a quadratic factor. Also for $a = 2$ and many other values of a the polynomial $f(x)$ may have a linear factor. Clearly if $f(x)$ is reducible, then $f(x)$ has a factor of degree k with $1 \leqslant k \leqslant \frac{n}{2}$. Shorey and Tijdeman [ShTi] proved that $f(x)$ with $2 \leqslant k \leqslant \frac{n}{2}$, $0 \leqslant a \leqslant \frac{3}{2}k$ and $|a_0| = |a_n| = 1$ has no factor of degree k except when

$$(n,k,a) \in \{(6,2,3),(7,2,2),(7,2,3),(7,3,3),(8,2,1),(8,3,2), (12,3,4),(13,2,3),(22,2,3),(46,3,4),(78,2,3)\}. \tag{2}$$

Furthermore all the exceptions in (2) are necessary. They also showed that for $f(x)$ with $3 \leqslant k \leqslant \frac{n}{2}, |a_0| = |a_n| = 1$ and $0 \leqslant a \leqslant 10$ when $k = 3, 4$ or $0 \leqslant a \leqslant 30$ when $k \geqslant 5$ has no factor of degree k except when

$$(n,k,a) \in \{(7,3,3),(8,3,2),(12,3,4),(18,4,9),(18,4,10),(46,3,4), (56,4,10),(17,5,11),(19,5,9),(40,5,12)\}. \tag{3}$$

We extend the validity of their results as follows.

Theorem 3. *Let $2 \leqslant k \leqslant \frac{n}{2}$, $0 \leqslant a \leqslant 5k$ and $|a_0| = |a_n| = 1$. Then $f_{n,a}(x)$ has no factor of degree k except possibly when (n,k,a) is given by (2) or (3) or*

$$\begin{aligned} k = 2,\ (n,a) \in \{&(4,5),(6,4),(8,8),(12,4),(17,8),(21,4),(22,6),(23,5),\\ &(23,10),(24,9),(36,9),(43,6),(44,5),(46,9),(58,6),(59,5),\\ &(72,9),(73,8),(77,4),(91,9),(112,9),(233,10),(234,9)\};\\ k = 3,\ (n,a) \in \{&(14,12),(17,11),(53,12)\};\\ k = 4,\ (n,a) \in \{&(16,12),(17,11),(38,13),(39,18)\}. \end{aligned} \tag{4}$$

Theorem 4. *Let $2 \leqslant k \leqslant \frac{n}{2}$, $|a_0| = |a_n| = 1$ and $0 \leqslant a \leqslant 40$ if $k = 2$ and $0 \leqslant a \leqslant 50$ if $k \geqslant 3$. Then $f_{n,a}(x)$ has no factor of degree k except possibly when (n,k,a) is given by (2) or (3) or (4) or the cases $k = 2$ with*

$$n + a \leqslant 100 \text{ or } a \in \{13,14,19,33\}, n+a \in \{126,225,2401,4375\}$$

or

a	$n+a$	a	$n+a$	a	$n+a$
12	169, 729	15, 16	289	17	513
18	361, 513, 1216	19, 20	243	21	529
21, 22	121, 576	24	325, 625, 676	27	784
28	145	29	961	31	243
32	243, 289, 1089	33	136, 256, 289, 5832	36	1369
38	325, 625, 676	39	1025, 6561	40	288

It is likely to obtain factorizations in most of these cases but we have not carried out the computations. The following assertion follows from Theorem 4.

THEOREM 5. *The polynomial $f_{n,a}(x)$ with $a_0a_n = \pm 1, a_1 = a_2 = \ldots = a_{n-1} = 1$ and $a \leqslant 12$ is either irreducible or a product of a linear polynomial times a polynomial of degree $n-1$. factor.*

We shall use the results of [ShTi] stated above without reference in this paper. Thus we always suppose that $a > 3$ if $k = 2$, $a > 10$ if $k = 3, 4$ and $a > 30$ if $k \geqslant 5$ in Theorems 3 and 4. Further we observe that Theorem 4 with $k \geqslant 10$ follows from Theorem 3. Also Theorem 2 follows immediately from Theorem 1 for $k \leqslant 10$ and from Theorem 3 for $k > 10$. Thus it suffices to prove Theorems 1, 3, 4 with $k < 10$ and 5. The new ingredients in the proofs of our theorems are the following Irreducibility Lemma and sharper lower estimates for the greatest prime factor of $\Delta(m, k)$ where

$$\Delta(m, k) = m(m+1)\cdots(m+k-1). \tag{5}$$

LEMMA 1.1. *Let $a > 0, 1 \leqslant k \leqslant \frac{n}{2}$ and $u_0 = \frac{a}{k}$.*
(A) Assume that there is a prime $p \geqslant k+2$ with

$$p \Big| \prod_{i=1}^{k}(a+n-k+i), \quad p \nmid a_0a_n \tag{6}$$

and

$$p \nmid \prod_{i=1}^{k}(a+i). \tag{7}$$

Suppose

$$p \geqslant \min(2u_0, k+u_0) \tag{8}$$

or

$$p > 2k \text{ and } p^2 - p \geqslant a. \tag{9}$$

Then $f_{n,a}(x)$ has no factor of degree k.
(B) If there is a prime $p \geqslant k+2$ with

$$p \Big| \prod_{i=1}^{k}(n-k+i)(a+n-k+i) \tag{10}$$

and (7) and satisfying (8) or (9), then $L_n^{(a)}(x)$ has no factor of degree k.

We have stated Lemma 1.1 and some of the subsequent lemmas in a more general way than required for the proof of our theorems. We prove Lemma 1.1 in Section 2. In Section 3, we give a refinement of an argument of Erdős and Sylvester. In Sections $5-9$, we prove Theorems 1, 3, 4 and 5 by combining Lemma 1.1 with the refinement in Section 4, results on Grimm's conjecture (see Lemma 3.4) and estimates from prime number theory. Section 3 contains preliminaries required for the proof of our theorems. For any real $u > 0$, let $\lfloor u \rfloor$ and $\lceil u \rceil$ be the floor function of u and the ceiling function of u, respectively. Thus $\lfloor u \rfloor$ is the greatest integer less than or equal to u and $\lceil u \rceil$ is the least integer exceeding u.

2. Proof of Lemma 1.1

We will use the notations introduced in this section throughout the paper. We write

$$\Delta_j = \Delta(a+1, j) = (a+1)(a+2)\cdots(a+j).$$

We observe that $q|\Delta_k$ for all primes $k < q \leqslant \frac{a+k}{\lceil u_0 \rceil}$ since $a \leqslant k\lceil u_0 \rceil < q\lceil u_0 \rceil \leqslant a+k$. Suppose there is a prime p satisfying the condition of the lemma. Then $p > \frac{a+k}{\lceil u_0 \rceil}$ by (7). As in the proof of [ShTi, Lemma 4.2], it suffices to show that

$$\phi_j := \phi_j(p) := \frac{\mathrm{ord}_p(\Delta_j)}{j} < \frac{1}{k} \quad \text{for } 1 \leqslant j \leqslant n \tag{11}$$

for showing that $f_{n,a}(x)$ has no factor of degree k. Also as in the proof of [FiFiLe, Lemma 2.4], for showing $L_n^{(a)}(x)$ has no factor of degree k, it suffices to show

$$\phi'_j := \phi'_j(p) := \frac{\mathrm{ord}_p\left(\frac{\Delta_j}{\binom{n}{j}}\right)}{j} < \frac{1}{k} \quad \text{for } 1 \leqslant j \leqslant n. \tag{12}$$

Since $\phi'_j \leqslant \phi_j$, we show that (11) holds for all j.

Let j_0 be the minimum j such that $p|(a+j)$ and write $a+j_0 = pl_0$ for some l_0. Then $j_0 \leqslant p$ and $j_0 > k$ since $p \nmid \Delta_k$. Also we see that $l_0 \leqslant \lceil u_0 \rceil$ which we shall use in the proof without reference.

We may restrict to those j such that $a+j = pl$ for some l. Then $j - j_0 = p(l - l_0)$. Writing $l = l_0 + s$, we get $j = j_0 + ps$. Note that if $p|(a+j)$, then $a+j = p(l_0+r)$ for some r. Hence we have

$$\mathrm{ord}_p(\Delta_j) = \mathrm{ord}_p((pl_0)(p(l_0+1))\cdots(p(l_0+s))) = s+1+\mathrm{ord}_p(l_0(l_0+1)\cdots(l_0+s)). \tag{13}$$

Let r_0 be such that $\mathrm{ord}_p(l_0+r_0)$ is maximal. We consider two cases.

Case I: Assume that $l_0+s < p^2$. If $p \nmid (l_0+i)$ for $0 \leqslant i \leqslant s$, then $\phi_j = \frac{s+1}{j_0+ps} < \frac{s+1}{k+ks} = \frac{1}{k}$. Hence we may suppose that $p|(l_0+i)$ for some $0 \leqslant i \leqslant s$ and further $l_0+s = pl_1$ for some $1 \leqslant l_1 < p$. Assume $s = 0$. Then $p|l_0$ which together with $l_0 < p^2$ implies $\mathrm{ord}_p(\Delta_j)$ $=\mathrm{ord}_p(a+j_0) = 2$. Therefore $a+p \geqslant a+j_0 \geqslant p^2$ implying $a \geqslant p^2 - p$. If (8) holds, then $a \leqslant \max(k(p-k), \frac{pk}{2}) < p(p-1)$ which is not possible. Thus (9) holds and hence $p \geqslant 2k+1$ and $a = p^2 - p$ implying $j_0 = p$. Therefore $\phi_j = \frac{2}{j_0} = \frac{2}{p} < \frac{1}{k}$. Thus we have $s \neq 0$ and we obtain from (13) that $\mathrm{ord}_p(\Delta_j) = s+1+l_1$ implying $\phi_j \leqslant \frac{s+1+l_1}{j_0+ps}$. Hence $\phi_j < \frac{1}{k}$ if $(p-k)\frac{s}{l_1} \geqslant k$ since $\frac{j_0+sp}{k} > 1 + s\frac{p}{k}$.

Suppose p satisfies (9). Then we may assume that $s < l_1$. Since $l_1 < p$, we have $s < p$ implying $\mathrm{ord}_p(\Delta_j) \leqslant s+2$ giving $\phi_j < \frac{s+2}{k+ps} \leqslant \frac{1}{k}$ since $s > 0$.

Thus we assume that p satisfies (8). Since $p \geqslant k+2, s = pl_1 - l_0$ and $l_0 \leqslant \lceil u_0 \rceil$, we have $(p-k)\frac{s}{l_1} - k \geqslant 2(p - \frac{l_0}{l_1}) - k \geqslant 2p - k - 2\lceil u_0 \rceil$. Hence it suffices to show $2p - k \geqslant 2\lceil u_0 \rceil$. Since $p \geqslant \min(2u_0, k+u_0)$, we have

$$2p - k = p + p - k \geqslant \begin{cases} 2u_0 + 2 \geqslant 2\lceil u_0 \rceil & \text{if } p \geqslant 2u_0 \\ 2(k + \lceil u_0 \rceil) - k \geqslant 2\lceil u_0 \rceil & \text{if } p \geqslant k + u_0, \end{cases}$$

noting that $p \geqslant k+u_0$ implies $p \geqslant k + \lceil u_0 \rceil$.

Case II: Let $l_0 + s \geqslant p^2$. Then we get from (13) that

$$\mathrm{ord}_p(\Delta_j) \leqslant s + 1 + \mathrm{ord}_p(l_0 + r_0) + \mathrm{ord}_p(s!) \leqslant s + 1 + \frac{\log(l_0 + s)}{\log p} + \frac{s}{p-1}.$$

Since $\frac{j}{k} = \frac{j_0 + ps}{k} > 1 + \frac{p}{k}s$, it is enough to show that

$$\frac{p}{k} \geqslant 1 + \frac{1}{p-1} + \frac{\log(l_0 + s)}{s \log p}.$$

Observe that $\frac{\log(l_0+s)}{s\log p}$ is a decreasing function of s. Since $s \geqslant p^2 - l_0$, it suffices to show

$$\frac{p}{k} \geqslant 1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0}.$$

Suppose p satisfies (8). Then from $l_0 \leqslant \lceil u_0 \rceil \leqslant p$ and $p \geqslant k+2$, we have $p^2 - l_0 \geqslant (k+2)^2 - (k+2) \geqslant 2(k+1)$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leqslant 1 + \frac{1}{k+1} + \frac{2}{2(k+1)} < 1 + \frac{2}{k} \leqslant \frac{p}{k}.$$

Suppose p satisfies (9). Then from $l_0 \leqslant \lceil u_0 \rceil \leqslant a$ and $p > 2k$, we obtain $p^2 - l_0 \geqslant p^2 - a \geqslant p > 2k$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leqslant 1 + \frac{1}{2k} + \frac{2}{2k} < 1 + \frac{2}{k} \leqslant \frac{p}{k}.$$

Hence the assertion. □

Corollary 2.1. *Let k, p and $\mathfrak{A}_{k,p}$ be given by*

$$\begin{aligned}
&k = 1,\ p = 3,\ \mathfrak{A}_{1,3} = \{3r, 3r+1 : 0 \leqslant r \leqslant 16\} \setminus \{7, 16, 24, 25, 34, 43\}\\
&k = 1,\ p = 5,\ \mathfrak{A}_{1,5} = \{5r, 5r+1, 5r+2, 5r+3 : 0 \leqslant r \leqslant 9\} \cup \{50\} \setminus \{23, 48\}\\
&k = 1,\ p = 7,\ \mathfrak{A}_{1,7} = [0, 50] \cap \mathbb{Z} \setminus \{6, 13, 20, 27, 34, 41, 47, 48\}\\
&k = 2,\ p = 5,\ \mathfrak{A}_{2,5} = \{5r, 5r+1, 5r+2 : 0 \leqslant r \leqslant 8\} \cup \{45, 50\} \setminus \{21, 22\}\\
&k = 2,\ p = 7,\ \mathfrak{A}_{2,7} = [0, 50] \cap \mathbb{Z} \setminus (\{7r-1, 7r-2 : 1 \leqslant r \leqslant 7\} \cup \{45, 46\})\\
&k = 3,\ p = 5,\ \mathfrak{A}_{3,5} = \{0, 1, 5, 6, 10, 11, 15, 25, 26, 30, 31, 35, 36, 40, 50\}\\
&k = 3,\ p = 7,\ \mathfrak{A}_{3,7} = \{7r, 7r+1, 7r+2, 7r+3 : 0 \leqslant r \leqslant 5\} \cup \{42, 49, 50\}\\
&k = 4,\ p = 7,\ \mathfrak{A}_{4,7} = \{7r, 7r+1, 7r+2 : 0 \leqslant r \leqslant 4\} \cup \{35, 36, 49, 50\}\\
&k = 5,\ p = 7,\ \mathfrak{A}_{5,7} = \{0, 1, 7, 8, 14, 15, 21, 22, 28, 49, 50\}.
\end{aligned}$$

Suppose $n \geqslant 2k$ and p satisfies (6). Then $f_{n,a}(x)$ has no factor of degree k for $a \in \mathfrak{A}_{k,p}$. Further if p satisfies (10), then $L_n^{(a)}(x)$ has no factor of degree k for $a \in \mathfrak{A}_{k,p}$.

Proof. For k, p and $a \in \mathfrak{A}_{k,p}$ given in the statement of Corollary 2.1, we check that $p \nmid \Delta_k$ and $\frac{\mathrm{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for $j \leqslant 50$. As in the proof of Lemma 1.1, it suffices to check that $\frac{\mathrm{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for all $j \geqslant 1$. Since $\mathrm{ord}_p(s!) \leqslant \frac{s}{p-1}$, we have for $j > 50$ that

$$\frac{\mathrm{ord}_p(\Delta_j)}{j} = \frac{\mathrm{ord}_p((a+j)!) - \mathrm{ord}_p(a!)}{j} \leqslant \frac{\frac{a+j}{p-1} - \mathrm{ord}_p(a!)}{j} \leqslant \frac{1}{p-1} + \frac{\frac{a}{p-1} - \mathrm{ord}_p(a!)}{51} < \frac{1}{k}.$$

Thus $\frac{\mathrm{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for all $j \geqslant 1$. □

Corollary 2.2. *Let $a > 0$ and $1 \leqslant k \leqslant \frac{n}{2}$.*

(*i*) *If there is a prime $p > a + k$ satisfying (6), then $f_{n,a}(x)$ has no factor of degree k.*

(*ii*) *Let $p \geqslant k+2$ be a prime satisfying* (6) *and let*

$$\mathcal{A}_p = \bigcup_{i=1}^{r_p} ([ip-k, ip-1] \cap \mathbb{Z}_{>0}) \cup \{j > pr_p, j \in \mathbb{Z}\}$$

where

$$r_p = \lfloor \frac{k}{2} \rfloor \text{ if } p < 2k \text{ and } p-1 \text{ if } p \geqslant 2k.$$

Then $f_{n,a}(x)$ has no factor of degree k for $a \notin \mathcal{A}_p$.

(*iii*) *Let $P_1 > P_2 > \ldots > P_s \geqslant k+2$ be primes satisfying* (6)*. For a subset $\{Q_1, Q_2, \ldots, Q_g\} \subseteq \{P_1, P_2, \ldots, P_s\}$, let*

$$\mathcal{B}\{Q_1, \ldots, Q_g\} = \bigcap_{l=1}^{g} \mathcal{A}_{Q_l}.$$

Then $f_{n,a}(x)$ has no factor of degree k for $a \notin \mathcal{B}\{Q_1, \ldots, Q_g\}$.

In earlier results, Corollary 2.2 (i) has been used. This is possible only if there is a $p > k+a$ satisfying (6). But it is possible to apply Lemma 1.1 even when $p \leqslant k+a$ for all p satisfying (6). For example, take $n = 15, a = 13, k = 3$. Here $p < k+a$ for all p satisfying (6). However (6), (7) and (9) are satisfied with $p = 13$ and hence $f_{n,13}(x)$ has no factor of degree 3 by Lemma 1.1.

Proof. (i) is immediate from Lemma 1.1. Consider (ii). We may assume that $p \leqslant k+a$ by (i). Let $a \notin \mathcal{A}_p$. Then $a \leqslant pr_p$ implying $a \leqslant p^2 - p$ if $p \geqslant 2k$ and $2u_0 = \frac{2a}{k} \leqslant \frac{2pr_p}{k} \leqslant p$ if $p < 2k$ satisfying either (8) or (9). Since $a \notin \mathcal{A}_p$, there is some i for which $ip-1 < a < (i+1)p-k$ implying $ip < a+1 < a+k < (i+1)p$. Therefore $p \nmid \prod_{j=1}^{k}(a+j)$ which together with (6) and $p \geqslant k+2$ satisfy the conditions of Lemma 1.1. Now the assertion follows by Lemma 1.1. The assertion (iii) follows from (ii). □

3. Preliminaries for Theorems 3-5

For a positive integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime factors and the greatest prime factor of ν, respectively, and we put $\omega(1) = 0, P(1) = 1$. For positive integers ν, we write

$$\pi(\nu) = \sum_{p \leqslant \nu} 1,$$
$$\theta(\nu) = \sum_{p \leqslant \nu} \log p.$$

Let p_i denote the $i-th$ prime.

We begin with some results on primes.

Lemma 3.1. *Let $k \in \mathbb{Z}$ and $\nu \in \mathbb{R}$. We have*

(*i*) $\pi(\nu) \geqslant \frac{\nu}{\log \nu - 1}$ *for* $\nu \geqslant 5393$ *and* $\pi(\nu) \leqslant \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$ *for* $\nu > 1$.

(*ii*) $\pi(\nu_1 + \nu_2) \leqslant \pi(\nu_1) + \pi(\nu_2)$ *for* $2 \leqslant \nu_1 < \nu_2 \leqslant \frac{7}{5}\nu_1(\log \nu_1)(\log\log \nu_1)$.

(*iii*) $\nu(1 - \frac{3.965}{\log^2 \nu}) \leqslant \theta(\nu) < 1.00008\nu$ *for* $\nu > 1$.

(*iv*) $p_k \geqslant k \log k$ for $k \geqslant 1$.

(*v*) $\mathrm{ord}_p((k-1)!) \geqslant \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ *for* $k \geqslant 2$.

(*vi*) $\sqrt{2\pi k}\ e^{-k}k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\ e^{-k}k^k e^{\frac{1}{12k}}$.

The estimates $(i), (ii)$ and (iii) are due to Dusart ([Dus99] and [Dus02], respectively). The estimate (iv) is due to Rosser [Ros38] and estimate (vi) is due to Robbins [Rob55, Theorem 6]. For a proof of (v), see [LaSh04, Lemma 2(i)]. $\square$

We derive from Lemma 3.1 the following results.

Corollary 3.2. *Let $10^{10} < m \leqslant 123k$. Then there are primes p, q with $m \leqslant p < m+k$ and $\frac{m}{2} \leqslant q < \frac{m+k}{2}$.*

Proof. Let $10^{10} < m \leqslant 123k$. We observe that the assertion holds if

$$\theta(\frac{m+k-1}{s}) - \theta(\frac{m-1}{s}) = \sum_{\frac{m-1}{s} < p \leqslant \frac{m+k-1}{s}} \log p > 0$$

for $s = 1, 2$. Now from Lemma 3.1 and since $m > 10^{10}$, it suffices to show

$$\theta(\frac{m+k-1}{s}) - \theta(\frac{m-1}{s}) > \frac{m+k-1}{s}\left(1 - \frac{3.965}{\log^2(5\cdot 10^9)}\right) - 1.00008\frac{m-1}{s} > 0$$

or

$$k(1 - \frac{3.965}{\log^2(5\cdot 10^9)}) > (m-1)(\frac{8}{10^5} + \frac{3.965}{\log^2(5\cdot 10^9)}).$$

This is true since $m \leqslant 123k$ and

$$\frac{1 - \frac{3.965}{\log^2(5\cdot 10^9)}}{\frac{8}{10^5} + \frac{3.965}{\log^2(5\cdot 10^9)}} > 123.$$

$\square$

Corollary 3.3. *We have*

$$\pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{6k}{5}) \leqslant \begin{cases} k-2 & \text{for } k \geqslant 61 \\ \pi(4k) & \text{for } k \geqslant 8000. \end{cases} \tag{14}$$

Proof. Let $k \geqslant 30000$. We have from $\frac{\log y}{\log x} = 1 + \frac{\log y/x}{\log x}$ and Lemma 3.1 (i) that

$$\begin{aligned} &(\log 4k)\left(\pi(4k) - \pi(\frac{6k}{5}) - \pi(k) - \pi(\frac{k}{2}) - \pi(\frac{k}{3}) - \pi(\frac{k}{4})\right) \\ \geqslant &\frac{4k}{\log 4k - 1} + \\ &k\left(4 - \frac{6}{5}\left(1 + \frac{\log\frac{10}{3}}{\log\frac{6k}{5}}\right)\left(1 + \frac{1.2762}{\log\frac{6k}{5}}\right) - \sum_{j=1}^{4}\frac{1}{j}\left(1 + \frac{\log 4j}{\log\frac{k}{j}}\right)\left(1 + \frac{1.2762}{\log\frac{k}{j}}\right)\right). \end{aligned}$$

The right hand side of the above inequality is an increasing function of k and it is positive at $k = 30000$. Therefore the left hand side of (14) is at most $\pi(4k)$ for $k \geqslant 30000$. By using exact values, we find that it is valid for $k \geqslant 8000$.

Also $\pi(4k) \leqslant \frac{4k}{\log 4k}\left(1 + \frac{1.2762}{\log 4k}\right) \leqslant k-2$ is true for $k \geqslant 8000$. Therefore the left hand side of (14) is at most $k-2$ for $k \geqslant 8000$. Finally we check using exact values of the $\pi-$function that the left hand side of (14) is at most $k-2$ for $61 \leqslant k < 8000$. $\square$

The following result is on Grimm's Conjecture, [LaSh06b, Theorem 1]. Grimm's Conjecture states that *given integers $n \geqslant 1$ and $k \geqslant 1$ such that whenever $n+1, \cdots, n+k$ are all composite numbers, we can find distinct primes P_i with $P_i | (n+i)$ for $1 \leqslant i \leqslant k$.* This is a difficult conjecture having several interesting consequences. For example, this conjecture implies $p_{i+1} - p_i < p_i^{0.46}$ for sufficiently large i, a result better than that given by Riemann hypothesis. This follows by taking $n = p_i$ in [LaMu00, Theorem 1(i)]. We refer to [RST75] and [LaMu00] for a survey and results on Grimm's Conjecture.

Lemma 3.4. *Let $m \leqslant 1.9 \cdot 10^{10}$ and $l \geqslant 1$ be such that $m+1, m+2, \cdots, m+l$ are all composite numbers. Then there are distinct primes P_i such that $P_i|(m+i)$ for each $1 \leqslant i \leqslant l$.*

The following result follows from [SaSh03, Lemma 3].

Lemma 3.5. *Let $m+k-1 < k^{\frac{3}{2}}$. Let $|\{i : P(m+i) \leqslant k\}| = \mu$. Then*

$$\binom{m+k-1}{k} \leqslant (2.83)^{k+\sqrt{m+k-1}}(m+k-1)^{k-\mu}.$$

4. An upper bound for m when $\omega(\Delta(m,k)) \leqslant t$

Let m, k and t be positive integers such that

$$\omega(\Delta(m,k)) \leqslant t. \tag{15}$$

For every prime p dividing $\Delta(m,k)$, we delete a term $m+i_p$ in $\Delta(m,k)$ such that $\mathrm{ord}_p(m+i_p)$ is maximal. Then we have a set T of terms in $\Delta(m,k)$ with

$$|T| = k - t := t_0.$$

We arrange the elements of T as $m+i_1 < m+i_2 < \cdots < m+i_{t_0}$. Let

$$\mathfrak{P} := \prod_{\nu=1}^{t_0}(m+i_\nu) \geqslant m^{t_0}. \tag{16}$$

Now we obtain an upper bound for $\mathfrak{P}$. For a prime p, let r be the highest power of p such that $p^r \leqslant k-1$ and let i_0 be such that $\mathrm{ord}_p(m+i_0d)$ is maximal. Let $w_l = |\{m+i : p^l|(m+i), m+i \in T\}|$ for $1 \leqslant l \leqslant r$. By an argument that was first given by Sylvester and Erdős(see []), we have $w_l \leqslant [\frac{i_0}{p^l}] + [\frac{k-1-i_o}{p^l}] \leqslant [\frac{k-1}{p^l}]$. Let $h_p > 0$ be such that $[\frac{k-1}{p^{h_p+1}}] \leqslant t_0 < [\frac{k-1}{p^{h_p}}]$. Then there are at most $t_0 - w_{h_p+1}$ terms in T exactly divisible by p^l with $l \leqslant h_p$. Hence

$$\begin{aligned}\mathrm{ord}_p(\mathfrak{P}) &\leqslant rw_r + \sum_{u=h_p+1}^{r-1} u(w_u - w_{u+1}) + h_p(t_0 - w_{h_p+1})\\ &= w_r + w_{r-1} + \cdots + w_{h_p+1} + h_pt_0\\ &\leqslant \sum_{u=1}^{r}\lfloor\frac{k-1}{p^u}\rfloor + h_pt_0 - \sum_{u=1}^{h_p}\lfloor\frac{k-1}{p^u}\rfloor = \mathrm{ord}_p((k-1)!) + h_pt_0 - \sum_{u=1}^{h_p}\lfloor\frac{k-1}{p^u}\rfloor.\end{aligned}$$

It is also easy to see that $\mathrm{ord}_p(\mathfrak{P}) \leqslant \mathrm{ord}_p((k-1)!)$. Let $L_0(p) = \min(0, h_pt_0 - \sum_{u=1}^{h_p}\lfloor\frac{k-1}{p^u}\rfloor)$. For any $l \geqslant 1$, we have from (16) that

$$m \leqslant (\mathfrak{P})^{\frac{1}{t_0}} \leqslant \left((k-1)!\prod_{p\leqslant p_l} p^{L_0(p)}\right)^{\frac{1}{t_0}} =: L(k,l). \tag{17}$$

Observe that

$$m^{t_0} \leqslant (L(k,l))^{t_0} \leqslant (k-1)!. \tag{18}$$

5. Prelude to the proof of Theorems 3-5

Let $k \geqslant 2$, $n \geqslant 2k$, $a \geqslant 0, m = n + a - k + 1$ and $|a_0 a_n| = 1$. Then $m > k + a$. We consider the polynomials $f_{n,a}(x)$ with $3 < a \leqslant 40$ when $k = 2$; $10 < a \leqslant 50$ when $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \leqslant \max(50, 5k)$ when $k \geqslant 5$. Let $P_1 > P_2 > \ldots > P_s \geqslant k+2$ be primes dividing $\Delta(m, k)$. We write $P_{m,k} = \{P_1, P_2, \ldots, P_s\}$. We use Corollaries 2.1 and 2.2 to apply the following procedure which we refer to as *Procedure* $\mathcal{R}$.

Procedure $\mathcal{R}$: Let k be fixed. For all a with $3 < a \leqslant 40$ if $k = 2$; $10 < a \leqslant 50$ if $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \leqslant \max(50, 5k)$ if $k \geqslant 5$, it suffices to consider only (m, k, a) with $P_1 \leqslant k + a$ by Corollary 2.2 (i). We restrict to such triples (m, k, a) with $P_1 \leqslant k+a$. By Corollary 2.2 (iii), we have $a \in \mathfrak{B}_0(m, k) := \mathfrak{B}\{P_1, P_2, \ldots, P_s\}$. Therefore we further restrict to (m, k, a) with $a \in \mathfrak{B}_0(m, k)$. Further for $k \in \{2, 3, 4, 5\}$ and $p = 5 \in P_{m,k}$ if $k = 2$; $p = 5 \in P_{m,k}$ or $p = 7 \in P_{m,k}$ if $k = 3$ and $p = 7 \in P_{m,k}$ if $k \in \{4, 5\}$, we restrict to those (m, k, a) with $a \notin \mathfrak{A}_{k,p}$ by using Corollary 2.1 and recalling $n = m + k - 1 - a$. Every (m, k, a) gives rise to the triplet (n, k, a).

We try to exclude the triplets (n, k, a) given by *Procedure* $\mathcal{R}$ to prove our theorems.

Let

$$\omega_0(a) = \begin{cases} \pi(a+k) & \text{if } a \leqslant k+1 \\ \sum_{j=1}^{2} \left(\pi(\frac{a+k}{j}) - \pi(\max(k+1, \frac{a}{j}))\right) + \pi(k+1) & \text{if } k+1 < a \leqslant 2k+2 \\ \sum_{j=1}^{3} \left(\pi(\frac{a+k}{j}) - \pi(\max(k+1, \frac{a}{j}))\right) + \pi(k+1) & \text{if } 2k+2 < a \leqslant 3k+3 \\ \sum_{j=1}^{4} \left(\pi(\frac{a+k}{j}) - \pi(\max(k+1, \frac{a}{j}))\right) + \pi(k+1) & \text{if } 3k+3 < a \leqslant 4k+4 \\ \sum_{j=1}^{5} \left(\pi(\frac{a+k}{j}) - \pi(\max(k+1, \frac{a}{j}))\right) + \pi(k+1) & \text{if } 4k+4 < a \leqslant 5k \end{cases}$$

and ω_1 be the maximum of $\omega_0(a)$ for $1.5k < a \leqslant 5k$. Then $\omega(\Delta(a+1, k)) \leqslant \omega_1$.

Let $k \geqslant 10$. Assume that $\omega(\Delta(m, k)) > \omega_1$. Then there is a prime $p \geqslant k+2$ with $p|\Delta(m, k)$ such that $p \nmid \Delta(a+1, k)$ and $p \nmid a_0 a_n$. Further $p \geqslant 13 > 2u_0$ since $u_0 \leqslant 5$. Hence $f(x)$ has no factor of degree k by Lemma 1.1. Therefore we may suppose that

$$\omega(\Delta(m, k)) \leqslant \omega_1 \text{ for } k \geqslant 10. \tag{19}$$

Let $k \geqslant 100$. Let $(i-1)(k+1) < a \leqslant i(k+1)$ with $1 \leqslant i \leqslant 5$. For $1 \leqslant j < i$, we have $\frac{a}{j} > \frac{k}{j} \geqslant \frac{100}{4}$ implying $\frac{\frac{a}{j}}{\frac{k}{j}} = \frac{a}{k} \leqslant 5 \leqslant \frac{7}{5}\log(25)\log\log(25) \leqslant \frac{7}{5}\log(\frac{k}{j})\log\log(\frac{k}{j})$. Hence $\pi(\frac{a+k}{j}) - \pi(\frac{a}{j}) \leqslant \pi(\frac{k}{j})$ for $1 \leqslant j < i$ by Lemma 3.1 (ii). Therefore

$$\omega_0(a) \leqslant \begin{cases} \pi(k+k+1) & \text{if } a \leqslant k+1 \\ \pi(k) + \pi(\frac{k}{2}+k+1) & \text{if } k+1 < a \leqslant 2k+2 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}+k+1) & \text{if } 2k+2 < a \leqslant 3k+3 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}+k+1) & \text{if } 3k+3 < a \leqslant 4k+4 \\ \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{k}{5}+k) & \text{if } 4k+4 < a \leqslant 5k \end{cases}$$

which, again by Lemma 3.1 (ii), implies

$$\omega_1 \leqslant \pi(k) + \pi(\frac{k}{2}) + \pi(\frac{k}{3}) + \pi(\frac{k}{4}) + \pi(\frac{6k}{5}) =: \omega_2 \text{ for } k \geqslant 100. \tag{20}$$

Let $N_1(p) = \{N : P(N(N-1)) \leqslant p\}$ and $N_2(p) = \{N : P(N(N-2)) \leqslant p, N \text{ odd}\}$. Then N_1 and N_2 are given by [Leh64, Table IA] for $p \leqslant 41$ and [Leh64, Table IIA] for $p \leqslant 31$, respectively and we shall use them without reference. For given k, N and j with $1 \leqslant j < k$, we put

$$M_j(N,k) = \prod_{i=0}^{k-1}(N-j+i).$$

Let

$$\mathcal{N}_j(k) := \{N \in N_1(41) : P(M_j(N,k)) \leqslant 59\}.$$

By observing that

$$M_1(N,k+1) = M_1(N,k)(N-1+k),\ M_k(N,k+1) = (N-k)M_{k-1}(N,k)$$

and

$$M_j(N,k+1) = M_j(N,k)(N-j+k) = (N-j)M_{j-1}(N,k) \quad \text{for } 1 < j < k,$$

we can compute $\mathcal{N}_j(k)$ recursively as follows. Recall that $P(N(N-1)) \leqslant 41$ for $N \in N_1(41)$. Hence we have

$$\mathcal{N}_1(3) = \{N \in N_1(41) : P(N+1) \leqslant 59\},\ \mathcal{N}_2(3) = \{N \in N_1(41) : P(N-2) \leqslant 59\}.$$

For $k \geqslant 3$ and $1 \leqslant j \leqslant k$, we obtain $\mathcal{N}_j(k+1)$ recursively by

$$\mathcal{N}_1(k+1) = \{N \in \mathcal{N}_1(k) : P(N-1+k) \leqslant 59\},\ \mathcal{N}_k(k+1) = \{N \in \mathcal{N}_{k-1}(k) : P(N-k) \leqslant 59\}$$

and

$$\mathcal{N}_j(k+1) = \{N \in \mathcal{N}_j(k) : P(N-j+k) \leqslant 59\} \cup \{N \in \mathcal{N}_{j-1}(k) : P(N-j) \leqslant 59\} \text{ for } 1 < j < k.$$

6. Proof of Theorems 3 and 4 for $k < 10$

Let $k = 2$. Then $a \leqslant 40$. By Corollary 2.2 (i), we first restrict to those m for which $P(m(m+1)) \leqslant 41$. They are given by $m = N - 1$ with $N \in N_1(41)$. By *Procedure* $\mathcal{R}$, we obtain the tuples $(n, 2, a)$ given in the following table.

a	$n+a$	a	$n+a$	a	$n+a$
$4,5$	9	4	10	$5,6$	$28,49,64$
$4,8,9$	$16,25,81$	9	$33,45,55,100,121,243$	10	$33,243$
12	$27,28,49,64,91,169,729$	13	$21,25,28,36,50,64$	14	25
$13,14$ $19,33$	$81,126,225,2401,4375$	$15,16$	289	17	513
18	$25,76,81,96,361,513,1216$	19	$25,28,36,49,50,64,243$	20	$28,33,49,64,243$
21	$25,33,45,55,529$	$21,22$	$46,81,100,121,576$	23	81
24	$40,81,65,325,625,676$	26	$49,64$	27	$49,64,784$
28	$81,145$	29	$81,125,961$	31	243
32	$243,289,1089$	33	$49,50,51,64,85,$ $136,256,289,5832$	34	$49,50,64,81$
36	1369	38	$65,81,325,625,676$	39	$81,82,1025,6561$
40	$49,64,82,288$				

Let $3 \leqslant k \leqslant 9$. Then $10 < a \leqslant 50$ if $k = 3, 4$ and $30 < a \leqslant 50$ if $5 \leqslant k \leqslant 9$. Thus we may assume that $P(\Delta(m,k)) \leqslant 59$ by Corollary 2.2 (i).

Let $m \leqslant 10000$. We need to consider $[k, 59] \cup \mathcal{M}(k)$ where $\mathcal{M}(k) = \{60 \leqslant m \leqslant 10000 : P(\Delta(m,k)) \leqslant 59\}$. We compute $\mathcal{M}(3)$ and further from the identity $\Delta(m,k+1) = (m+k)\Delta(m,k)$, we obtain $\mathcal{M}(k+1) = \{m \in \mathcal{M}(k) : P(m+k) \leqslant 59\}$ for $k \geqslant 3$ recursively. In fact we get

$$\mathcal{M}(6) = \{90, 91, 116, 184, 185, 285, 340\}, \quad \mathcal{M}(7) = \{90, 184\}$$

and $\mathcal{M}(8) = \mathcal{M}(9) = \emptyset$. We now apply Procedure $\mathcal{R}$ on $m \in [k, 59] \cup \mathcal{M}(k)$. We get

a	$n+a$	a	$n+a$
11	28	12	26, 27, 28, 65
19, 20	56, 100	20	46, 162
21	46	32	51, 56, 100, 121
33	51	38, 39	82
41, 43	56, 100	43, 44, 45	162

or $a \in \{12, 13, 18, 19, 20, 27, 32, 33, 34, 39, 41, 43, 44\}$, $n+a = 50$ if $k = 3$ and

a	$n+a$	a	$n+a$	a	$n+a$	a	$n+a$
11, 12	27, 28	13, 31, 32, 33	51	18	57	10	66

if $k = 4$.

Thus $m > 10000$. Suppose that $m + j = N \in N_1(41)$ for some $1 \leqslant j < k$. Then $\Delta(m, k) = M_j(N, k)$ which implies $N \in \mathcal{N}_j(k)$ since $P(\Delta(m, k)) \leqslant 59$. Let $\mathcal{N}'_j(k) = \{m \in \mathcal{N}_j(k) : m > 10000\}$. We find that

$$\mathcal{N}'_1(3) = \{13311, 13455, 17576, 17577, 19551, 29601, 32799, 212381\}$$
$$\mathcal{N}'_2(3) = \{10881, 11662, 13312, 13456, 13690, 16170, 17577, 23375, 27456, 31213, 134850, 212382, 1205646\}$$
$$\mathcal{N}'_1(4) = \{17576\}, \ \mathcal{N}'_2(4) = \{17577\}, \ \mathcal{N}'_3(4) = \{10881\}$$

and $\mathcal{N}'_j(k) = \emptyset$ for $k \geqslant 5$ and $1 \leqslant j < k$. We now take $m = N - j$ with $N \in \mathcal{N}_j(k)$ for $1 \leqslant j < k$ and apply Procedure $\mathcal{R}$ to find that there are no triplets (n, k, a).

Thus we may suppose that $m + j \notin N_1(41)$ for all $1 \leqslant j < k$. Then $P((m+i)(m+i+1)) > 41$ for each $0 \leqslant i < k-1$. By Corollary 2.2 (*iii*), we may suppose that $P(\Delta(m, k)) \leqslant 53$ for $k \leqslant 8$ and $P(\Delta(m, k)) \leqslant 59$ for $k = 9$. Taking $V(m, k) = \{P((m+2i)(m+2i+1)) : 0 \leqslant i < \frac{k}{2}\}$, we have $V(m, k) \subseteq \{43, 47, 53\}$ for $4 \leqslant k \leqslant 7$ and $V(m, k) = \{43, 47, 53, 59\}$ if $k = 8, 9$. Then $k \neq 8$ and computing $\{a \leqslant 50 : a \in \mathfrak{B}\{Q_1, Q_2\}$ for $(Q_1, Q_2) \in \{(47, 43), (53, 43, (53, 53)\}\}$ if $k = 4, 5$; $(Q_1, Q_2) = (53, 43)$ if $k = 6, 7, 9$, we find that the set is empty except when $k = 5, (Q_1, Q_2) = (43, 47)$ where it is $\{42\}$. Thus we may assume that $k = 5$ and $a = 42$. Further $P(\Delta(m, k)) = 47$ and $43|\Delta(m, k)$. If $p|\Delta(m, k)$ with $13 \leqslant p \leqslant 41$, then $42 \notin \mathfrak{B}\{47, p\}$ by Corollary 2.2 (*iii*). Thus we may further suppose that $p|\Delta(m, k)$ with $p \leqslant 11$ or $p \in \{43, 47\}$. Also $P(m) \leqslant 41$ otherwise each of $P(m), P((m+1)(m+2)), P((m+3)(m+4))$ is > 41 which is not possible. Again we get $P(m+2) \leqslant 41$ since otherwise each of $P(m(m+1)), P(m+2), P((m+3)(m+4))$ is > 41. Therefore $P(m(m+2)) \leqslant 41$ implying $P(m(m+2)) \leqslant 11$. If m is odd, then $m = N - 2$ for $N \in N_2(11)$ and we check that there is a prime $p > 11, p \notin \{43, 47\}$ with $p|\Delta(m, k)$ which is a contradiction. Thus m is even and we have $P(\frac{m}{2}(\frac{m}{2}+1)) \leqslant 11$ implying $m = 2N - 2$ with $N \in N_1(11)$. This is again not possible as above.

Let $k = 3$. Then $P(\Delta(m, k)) \leqslant 53$ by Corollary 2.2 (*i*). Recall that $P_1 > P_2 > \cdots \geqslant k+2$ are all the primes dividing $\Delta(m, k)$. We observe that $P_1 > 41$ since $m + j \notin N_1(41)$ for $1 \leqslant j < k$. Further $P((m+1)(m+2)) > 41$ if $P(m) > 41$ and $P(m(m+1)) > 41$ if $P(m+2) > 41$ which are excluded by Corollary 2.2 (*iii*) as above. Thus we may suppose that $P_1 = P(m+1) > 41$ and $P(m(m+2)) \leqslant 41$. If m is even, then $m = 2N - 2$ for $N \in N_1(41)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \ldots\}$. Thus m is odd. If $P(m(m+2)) \leqslant 31$, then $m = N - 2$ with $N \in N_2(31)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \ldots\}$ which is excluded. Thus

$P_2 = P(m(m+2)) \in \{37, 41\}$ which together with $41 < P_1 \leqslant 53$ implies $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2\}$ except when $P_1 = 43, P_2 = 41$ where $a = 40 \in \mathfrak{B}\{P_1, P_2\}$. Thus $a = 40, P(m+1) = 43$ and $P(m(m+2)) = 41$. Further by Corollary 2.2 (iii), we may assume $p \in \{2, 3, 7, 41, 43\}$ for $p|\Delta(m,3)$ and $2 \cdot 43|(m+1)$. By looking at the possible prime factorisations of $m, m+1, m+2$ and taking $(m+2) - m$ or $m - (m+2)$, we have the following possibilities.

$$\begin{array}{ll} m+1 = 2^r \cdot 7^y \cdot 43^t, & 3^x - 41^z = \pm 2; \\ m+1 = 2^r \cdot 3^x \cdot 43^t, & 7^y - 41^z = \pm 2; \\ m+1 = 2^r \cdot 43^t, & 3^x - 41^z = \pm 2; \\ m+1 = 2^r \cdot 43^t, & 3^x \cdot 7^y - 41^z = \pm 2; \\ m+1 = 2^r \cdot 43^t, & 3^x - 7^y \cdot 41^z = \pm 2; \\ m+1 = 2^r \cdot 43^t, & 7^y - 3^x \cdot 41^z = \pm 2; \end{array}$$

where r, x, y, z, t are positive integers. The second and fourth equations are excluded by taking remainders modulo 7. Calculating modulo 8 for the remaining possibilities, we get the following four simultaneuos equations.

$C1:$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 7^y \cdot 43^t = 1,$	$2^r \cdot 7^y \cdot 43^t - 41^z = 1,\ x$ odd
$C2:$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 41^z = 1,\ x$ odd
$C3:$	$3^x - 7^y \cdot 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y \cdot 41^z = 1$
$C4:$	$3^x \cdot 41^z - 7^y = 2,$	$3^x \cdot 41^z - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y = 1$

If $4|2^r$ in $C2$, we get a contradiction by taking remainders modulo 4 since x is odd, thus $2^r = 2$. Calculating modulo 7 in all the possibilities, we find that $C1$ is excluded since x is odd. Further $6|(x-1)$ in $C2$; $6|(x-2)$, $3|r$ in $C3$ and $3|r$ in $C4$. Note that $x \geqslant 2$. Taking remainders modulo 9 again, we find that $3|(z+1)$ in $C2$; $3|t$ in $C3$ and $3|t, 3|(y-1)$ in $C4$. Thus we have $(-41^{\frac{z+1}{3}})^3 + 3 \cdot 41(3^{\frac{x-1}{3}})^3 = 2 \cdot 41$ in $C2$, $(-2^{\frac{r}{3}} \cdot 43^{\frac{t}{3}})^3 + 9(3^{\frac{x-2}{3}})^3 = 1$ in $C3$ and $(2^{\frac{r}{3}} \cdot 43^{\frac{t}{3}})^3 + 7(-7^{\frac{y-1}{3}})^3 = 1$ in $C4$. We solve the Thue equations $X^3 + 123Y^3 = 82, X^3 + 9Y^3 = 1$ and $X^3 + 7Y^3 = 1$ with X, Y integers in **PariGp** to find that it is not possible.

We recall that Theorem 4 follows from Theorem 3 when $k \geqslant 10$. Therefore we prove Theorem 3 with $k \geqslant 10$ in Sections 7, 8 and this will complete the proofs of Theorems 3 and 4.

7. Proof of Theorem 3 for $k \geqslant 10$

We may suppose by Corollary 2.2 (i) that $P(\Delta(m,k)) \leqslant a + k \leqslant 6k$. Let $k \leqslant 17$. We may suppose that $\max(30, 1.5k) < a \leqslant 5k$. First assume that $m + j \notin N_1(41)$ for any $1 \leqslant j < k$. Let

$$\mathfrak{L}_i(k,a) := \{p : \max(41, \frac{a}{i}) < p \leqslant \frac{a+k}{i}\} \text{ for } 1 \leqslant i \leqslant 5$$

and $\ell(k) := \max\limits_{1.5k < a \leqslant 5k} |\cup_{i=1}^5 \mathfrak{L}_i(k,a)|$. There are at most $\ell(k)$ primes > 41 dividing $\Delta(a+1,k)$ and we delete numbers in $\{m, m+1, \cdots, m+k-1\}$ divisible by those primes. We are left with at least $k - \ell(k)$ numbers. We observe that the prime factors of each of these numbers are at most 41 otherwise the assertion follows by Lemma 1.1. We call U the largest such number. From [Leh64, Tables IA], we may assume that each of these numbers is at least at a distance 2 from the preceding one. Thus $m + k - 1 \geqslant U \geqslant m + 2(k - \ell(k) - 1)$. Hence we have a contradiction if $k - 2\ell(k) - 1 > 0$. This is the case since $\ell(k) = 2, 3, 4, 5$ when $k = 10, k \in \{11, 12\}$, $k \in \{13, 14\}$, $k \in \{15, 16, 17\}$, respectively. Therefore we suppose that $m + j_0 = N \in N_1(41)$ for some $1 \leqslant j_0 \leqslant k-1$. Then $\Delta(m,k) = M_{j_0}(N,k)$. We check that $P(M_j(N,7)) > 102$ for $1 \leqslant j < 7$ when $N > 10000$ and $N \in N_1(41)$. Thus $m < N \leqslant 10000$. For each $m < 10000$, we check that $P(\Delta(m,10)) > 102$ for $m \geqslant 118$. Therefore $P(\Delta(m,k)) > 6k$ when $m \geqslant 118$. Further we find that $p_{i+1} - p_i \leqslant 10$ for $p_i < 118$. Hence for $m < 118$, $P(\Delta(m,k)) \geqslant m$ since $k \geqslant 10$. Therefore we have $P(\Delta(m,k)) \geqslant \min(m, 6k+1) > k + a$

for all m. Now the assertion follows by Corollary 2.2 (i).

Thus $k \geqslant 18$. First we check that $\omega_1 < k$ for $k \leqslant 100$ which together with (20) and Corollary 3.3 implies $\omega_1 < k$ for all k. Suppose $m \leqslant 10^{10}$. If at least one of $m, m+1, \ldots, m+k-1$ is a prime, then $P(\Delta(m,k)) \geqslant m > k+a$ and therefore the assertion follows from Corollary 2.2 (i). Hence we may suppose that each of $m, m+1, \ldots, m+k-1$ is composite. By Lemma 3.4, we obtain $\omega(\Delta(m,k)) \geqslant k > \omega_1$ which contradicts (19). Therefore we have $m > 10^{10}$ which implies $k > 500$ by (19) and (17) with $t_0 = \omega_1$.

By (19) and (20), we have $\omega(\Delta(m,k)) \leqslant \omega_2$. We obtain from (18), Lemma 3.1 (vi) and $k > 500$ that

$$m^{k-\omega_2} < (k-1)! = \frac{k!}{k} \leqslant \frac{\sqrt{2\pi k}}{k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}} < \left(\frac{k}{e}\right)^k. \tag{21}$$

Since $m \geqslant 10^{10}$, we get

$$\log k - 1 > \frac{(k-\omega_2)\log m}{k} \geqslant 10(\log 10)(1 - \frac{\omega_2}{k}).$$

By using estimates of $\pi(\nu)$ from Lemma 3.1 (i), we obtain

$$k > e^{\left(1+10(\log 10)\left(1-\frac{\frac{6}{5}}{\log\frac{6k}{5}}\left(1+\frac{1.2762}{\log\frac{6k}{5}}\right)-\sum_{j=1}^{4}\frac{1}{j\log\frac{k}{j}}\left(1+\frac{1.2762}{\log\frac{k}{j}}\right)\right)\right)} =: J(k)$$

Since $J(k)$ is an increasing function of k and $k > 500$, we have $k > J(500) \geqslant 4581$. Further $k > J(4581) \geqslant 578802$ and hence $k > J(578802) > 4.5 \times 10^7$. Let $m \leqslant 123k$. Then, by Corollary 3.2, there is a prime $P_1 \geqslant m$ such that $P_1 | \Delta(m,k)$. Since $m > a+k$, the assertion follows by Corollary 2.2 (i). Therefore we may suppose that $m > 123k$.

Assume that $m+k-1 \geqslant k^{\frac{3}{2}}$. Then $m > \frac{k^{\frac{3}{2}}}{e}$ and we get from (21) and Corollary 3.3 that

$$k^k > (k^{\frac{3}{2}})^{k-\pi(4k)}$$

which together with estimates of $\pi(\nu)$ from Lemma 3.1 implies

$$0 > \frac{k-3\pi(4k)}{k} \geqslant 1 - \frac{12}{\log 4k}\left(1+\frac{1.2762}{\log 4k}\right).$$

The right hand expression is an increasing function of k and the inequality does not hold at $k = 10^6$. Therefore $m+k-1 < k^{\frac{3}{2}}$. By Lemma 3.5, we get

$$\binom{m+k-1}{k} \leqslant (2.83)^{k+k^{\frac{3}{4}}} k^{\frac{3}{2}(\pi(4k)-\pi(k))}$$

since $|\{i : P(m+i) \leqslant k\}| \geqslant k - (\pi(4k) - \pi(k))$ by (15) and Corollary 3.3. On the other hand, we have $m > 123k$ implying

$$\begin{aligned}\binom{m+k-1}{k} \geqslant \binom{124k}{k} = \frac{(124k)!}{k!(123k)!} &> \frac{\sqrt{2\pi(124k)}(\frac{124k}{e})^{124k}}{\sqrt{2\pi k}(\frac{k}{e})^k \mathrm{e}^{\frac{1}{12k}}\sqrt{2\pi(123k)}(\frac{123k}{e})^{123k}\mathrm{e}^{\frac{1}{12\cdot 123k}}} \\ &> \frac{0.4}{\sqrt{k}} e^{-\frac{1}{8k}}(335.7)^k\end{aligned}$$

using estimates of $\nu!$ from Lemma 3.1. Comparing the upper and lower bounds, we obtain

$$0 > \log(0.4) - \frac{1}{8k} - 0.5\log k + k\log(\frac{335.7}{2.83}) - k^{\frac{3}{4}}\log(2.83) - \frac{3}{2}(\pi(4k)-\pi(k))\log k.$$

By using estimates of $\pi(\nu)$ from Lemma 3.1 again, we obtain

$$\begin{aligned}\frac{(\pi(4k)-\pi(k))\log k}{k} &\leqslant \frac{4\log k}{\log 4k}(1+\frac{1.2762}{\log 4k}) - \frac{\log k}{\log k - 1}\\ &\leqslant 4\left(1-\frac{\log 4}{\log 4k}\right)\left(1+\frac{1.2762}{\log 4k}\right)-1\\ &\leqslant 4\left(1-\frac{\log 4 - 1.2762}{\log 4k}\right)-1 \leqslant 3.\end{aligned}$$

Therefore we have

$$0 > \frac{\log(0.4)-\frac{1}{8k}-0.5\log k}{k} + \log(\frac{335.7}{2.83}) - k^{-\frac{1}{4}}\log(2.83) - 4.5.$$

The right hand side of the above inequality is an increasing function of k and the inequality is not valid at $k = 10^6$. This is a contradiction. □

8. Proof of Theorem 5

By Theorem 4, we restrict to those triplets (n, a, k) given in the statement of Theorem 4 with $a \leqslant 12$. We now factorize $f_{n,a}(x)$ with $a_0a_n = \pm 1, a_1 = a_2 = \ldots = a_{n-1} = 1$ to find that these $f_{n,a}(x)$ are irreducible. Hence the assertion follows. □

9. Proof of Theorem 1

For the proof of Theorem 1, we put $\alpha = a$ throughout this section. As remarked in Section 1 after the statement of Theorem 1, we may assume that $10 < a \leqslant 40$. For $n \leqslant 18$ and $n \in \{24, 25, 27, 30, 32, 36, 45, 48, 54, 60, 64, 72, 75, 80, 90, 112, 120\}$, we find that $L_n^{(a)}(x)$ is irreducible except for (n, a) listed in Theorem 1. Thus we assume $n > 18$, $n \notin \{24, 25, 27, 30, 32, 36, 45, 48, 54, 60, 64, 72, 75, 80, 90, 112, 120\}$. Assume that $L_n^{(\alpha)}(x)$ is reducible. Then $L_n^{(\alpha)}(x)$ has a factor of degree k with $1 \leqslant k \leqslant \frac{n}{2}$. First we prove the following lemma.

Lemma 9.1. *Let $k \geqslant 2$. Then $L_n^{(a)}(x)$ has no factor of degree k.*

Proof. Let $k \geqslant 2$ and $a \leqslant 40$ if $k = 2$. We may restrict to those (n, k, a) given in the list of exceptions in Theorem 4. For each of these triplets (n, k, a), we first check if there is a prime $p \geqslant k + 2$ with (10) such that either (8) or (9) is satisfied and they can be excluded by Lemma 1.1. We are now left with triples (n, k, a) given by $k = 2, (n, a) \in \{(100, 21), (40, 24), (256, 33), (42, 40)\}$. For these (n, a), we check that $L_n^{(a)}(x)$ is irreducible.

Let $k = 2$ and $40 < a \leqslant 50$. Suppose $n \notin N_1(23)$ and $n+a \notin N_1(23)$. Then $P_1 = P(n(n-1)) > 23$ and $P_2 = P((n + a)(n + a - 1)) > 23$. Further either $P_1 \nmid (a + 1)(a + 2)$ or $P_2 \nmid (a + 1)(a + 2)$ and then the assertion follows by Lemma 1.1. Therefore we may assume that either $n = N \in N_1(23)$ or $n+a = N \in N_1(23)$. Further we may also suppose that $P(n(n-1)(n+a)(n+a-1)) \leqslant P((a+1)(a+2))$ since otherwise the assertion follows by Lemma 1.1. For $N \in N_1(23)$ and $N > 10000$, we check that $P((N - a)(N - a - 1)) > P((a + 1)(a + 2))$ and $P((N + a)(N + a - 1)) > P((a + 1)(a + 2))$ except when $(a, N) \in \{(45, 10648), (46, 12168)\}$ where $P(N(N - 1)) \in \{13, 23\}$, respectively. Observe that $N(N-1)|n(n-1)(n+a)(n+a-1)$. By taking $p = P(N(N-1))$, the assertion follows from Lemma 1.1. We now consider $n \leqslant 10000$. Let a be given. By Lemma 1.1, we first restrict to those n for which $P(n(n - 1)(n + a)(n + a - 1)) \leqslant P((a + 1)(a + 2))$. Further we check that there is a prime $p|n(n-1)(n+a)(n+a-1), p > 7$ and $p \nmid (a+1)(a+2)$. Lemma 1.1 implies the assertion now. □

By Lemma 9.1, we only need to consider $k = 1$. If there is a prime $p|n(n + a), p \nmid (a + 1)$ with either $p \geqslant 11$ or $p = 7, a \neq 47$ or $p = 5, a \notin \{23, 48\}$ or $p = 3, a \notin \{16, 24, 25, 34, 43\} =: S_1$, then the assertion follows by Lemma 1.1 and Corollary 2.1. Let $P_a = \{2\} \cup \{p : p|(a+1)\}$ if $a \notin S_1 \cup \{23, 47, 48\}$, $P_a = \{2, 3\} \cup \{p : p|(a+1)\}$ if $a \in S_1$, $P_a = \{2, 3, 5\}$ if $a = 23$, $P_a = \{2, 3, 7\}$ if $a = 47$ and $P_a = \{2, 5, 7\}$ if $a = 48$. Thus for a given a, we may assume that $p|n(n+a)$ implies $p \in P_a$.

Let a be given. Let $p|n$ with $p > 2$. Then $p \in P_a$. As in the proof of Lemma 1.1, if we have $\phi'_j < 1$ for all $1 \leqslant j \leqslant n$, then $L_n^{(\alpha)}(x)$ does not have a linear factor and we are done. Let $1 \leqslant j \leqslant 50$. We compute ϕ_j to find that $\phi_j < 1$ for $j > 1$ except when $(p, a) \in T_1 := \{(3, 16), (3, 17), (3, 34), (3, 35), (3, 43), (3, 44), (5, 23), (5, 24), (5, 48), (5, 49), (7, 47), (7, 48)\}$ where $\phi_j < 1$ for $j > 2$ and except when $23 \leqslant a \leqslant 26, p = 3$ where $\phi_j < 1$ for $j > 4$. Let $j > 50$. By using $\mathrm{ord}_p(s!) \leqslant \frac{s}{p-1}$, we find that

$$\phi_j = \frac{\mathrm{ord}_p((a+j)!) - \mathrm{ord}_p(a!)}{j} \leqslant \frac{\frac{a+j}{p-1} - \mathrm{ord}_p(a!)}{j} \leqslant \frac{1}{p-1} + \frac{\frac{a}{p-1} - \mathrm{ord}_p(a!)}{51} < 1.$$

It suffices to show that $\phi'_1 < 1$ except when $(p, a) \in T_1$ for which we need to show $\phi'_j < 1$, $1 \leqslant j \leqslant 2$ and except when $23 \leqslant a \leqslant 26, p = 3$ for which we need to show $\phi'_j < 1$ for $1 \leqslant j \leqslant 4$. Let $\phi'_0 = \max\{\phi'_i\}$ for $1 \leqslant i \leqslant 4$. It suffices to show $\phi'_0 < 1$ is always valid. This is the case except when $a \in \{24, 49\}, p = 5$; $a \in \{17, 24, 25, 26, 35, 44\}, p = 3$ and $a = 48, p = 7$. Further $\mathrm{ord}_5(n) \leqslant 1$ when $a \in \{24, 49\}$, $\mathrm{ord}_7(n) \leqslant 1$ when $a = 48$, $\mathrm{ord}_3(n) \leqslant 1$ when $a \in \{17, 24, 25, 35, 44\}$ and $\mathrm{ord}_3(n) \leqslant 2$ when $a = 26$ otherwise $\phi'_0 < 1$. Let $a \in \{17, 26, 35\}$ and $\mathrm{ord}_3(n) = 1$ or $\mathrm{ord}_3(n) = 2$. Then from $n(n+a) = 2^\alpha 3^{\beta_3}$ and $gcd(n, n+a) \leqslant 2$, we obtain $n \in \{3, 6, 9, 18\}$ which is not possible. Let $a = 49$ and $\mathrm{ord}_5(n) = 1$. Then from $n(n+a) = 2^\alpha 5^{\beta_5}$ and $\gcd(n, n+a) = 1$, we obtain $n = 5$ which is again not possible. Here $\gcd(a, b)$ stands for greatest common divisor of a and b.

Therefore n is a power of 2 except when $a = 24$ where $\mathrm{ord}_3(n) \leqslant 1$ or $\mathrm{ord}_5(n) \leqslant 1$; $a = 25$ where $\mathrm{ord}_3(n) \leqslant 1$; $a = 44$ where $\mathrm{ord}_3(n) \leqslant 1$ and $a = 48$ where $\mathrm{ord}_7(n) \leqslant 1$. From the definition of P_a, we observe that $n(n+a)$ has at most two odd prime factors except when $a = 34$ where it has at most three odd prime factors. Hence we always have $n, n+a$ of the form

$$\begin{array}{llll} & n = 2^{\alpha+\delta}, \ \frac{n+a}{2^\delta} = & p^{\beta_p} & \text{if } P_a = \{2, p\} \\ & n = 2^{\alpha+\delta}, \ \frac{n+a}{2^\delta} \in & \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}\} & \text{if } P_a = \{2, p_1, p_2\} \\ (22) & n = 2^{\alpha+\delta}, \ \frac{n+a}{2^\delta} \in & \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_3^{\beta_{p_3}}, & \\ & & p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}\} & \text{if } P_a = \{2, p_1, p_2, p_3\}. \end{array}$$

where $2^\delta || a$ and in addition $n, n+a$ is of the form

$$\begin{array}{llll} & n = 15 \cdot 2^{\alpha+3}, & n+a = 8 \cdot 3^{\beta_3+1} \quad \text{or} & \\ & n = 3 \cdot 2^{\alpha+3}, & n+a \in \{8 \cdot 3^{\beta_3+1}, 8 \cdot 3^{\beta_3+1} 5^{\beta_5}\} & \text{if } a = 24 \\ (23) & n = 3 \cdot 2^{\alpha}, & n+a = 13^{\beta_{13}} & \text{if } a = 25 \\ & n = 3 \cdot 2^{\alpha+2}, & n+a = 4 \cdot 5^{\beta_5} & \text{if } a = 44 \\ & n = 7 \cdot 2^{\alpha+4}, & n+a = 16 \cdot 5^{\beta_5} & \text{if } a = 48. \end{array}$$

Here all the exponents of odd prime powers appearing in (22) and (23) are positive. For $n < 512$ and n of the form given by (22) or (23) which are given by $n \in \{96, 128, 192, 224, 240, 256, 384, 448, 480\}$, we check that there is a prime $p|(n+a), p \notin P_a$ except when $(n, a) \in \{(256, 14), (128, 16), (256, 16), (96, 24), (192, 24), (256, 32), (256, 33), (128, 34)\}$. We find that for each of these (n, a), the polynomial $L_n^{(a)}(x)$ is irreducible. Therefore we have $n \geqslant 512$.

From the equality $\frac{n+a}{2^\delta} - \frac{n}{2^\delta} = \frac{a}{2^\delta}$, we obtain an equation of the form

$$p^{\beta_p} - 2^\alpha = \frac{a}{2^\delta} \quad \text{or} \quad p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} - 2^\alpha = \frac{a}{2^\delta}$$

or further $3^{\beta_3}5^{\beta_5}7^{\beta_7} - 2^\alpha = 17$ (only when $a = 34$) or $3^{\beta_3} - 5 \cdot 2^\alpha = 1$ (only when $a = 24$) or $13^{\beta_{13}} - 3 \cdot 2^\alpha = 25$ (only when $a = 25$) or $5^{\beta_5} - 3 \cdot 2^\alpha = 11$ (only when $a = 44$) or $5^{\beta_5} - 7 \cdot 2^\alpha = 3$ (only when $a = 48$). In each of the equations thus obtained, we note that $8|2^\alpha$ since $n \geqslant 512$. Out of all the equations, we need to consider only those which are valid under remainders modulo 8 and hence we restrict to those. Here we use $p^{\beta_p} \equiv 1$ or p modulo 8 according as β_p is even or odd, respectively. They are now expressed as the Thue equation

$$X^3 + AY^3 = B$$

and we solve them in **PariGp**. For instance, let $a = 32$. Then we obtain equations of the form $3^{\beta_3} - 2^\alpha = 1$, $11^{\beta_{11}} - 2^\alpha = 1$, $3^{\beta_3}11^{\beta_{11}} - 2^\alpha = 1$. By taking remainders modulo 8, we find that $\beta_3, \beta_{11}, \beta_3 + \beta_{11}$ are even for the first, second and third equation, respectively. This implies $3^{\frac{\beta_3}{2}} - 1 = 2, 3^{\frac{\beta_3}{2}} + 1 = 2^{\alpha-1}$ giving $3^{\beta_3} = 9, 2^\alpha = 8$ for the first equation and $11^{\frac{\beta_{11}}{2}} - 1 = 2, 11^{\frac{\beta_{11}}{2}} + 1 = 2^{\alpha-1}$ giving a contradiction for the second equation. Observe that $2^\alpha > 8$ since $n \geqslant 512$. Thus we are left with $3^{\beta_3}11^{\beta_{11}} - 2^\alpha = 1$. For some $0 \leqslant r, s, t \leqslant 2$, we have $\alpha + r, \beta_3 - s, \beta_{11} - t$ all are multiples of 3 and from $-2^{\alpha+r} + 2^r3^s11^t3^{\beta_3-s}11^{\beta_{11}-t} = 2^r$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 2^r, A = 2^r3^s11^t, 0 \leqslant r, s, t \leqslant 2$ and with X a power of 2 and $33|AY$. There are 27 possibilities of pairs (A, B). If $A = 1$, then $B = 1$ and we factorise $X^3 + Y^3$ to get a contradiction. Thus the case $A = 1$ is excluded. For all other values of (A, B) than those given by $t = 2$, we check in **PariGp** that none of the solutions (X, Y) of Thue equations thus obtained satisfy the condition X a power of 2 and $33|AY$ except when $A = 66, B = 2$ where $X = -4$ and $Y = 1$ from which we obtain $n = 1024$. When $t = 2$, from $3^{\beta_3-s+3}11^{\beta_{11}-2+3} - 2^{3-r}3^{3-s} \cdot 11 \cdot 2^{\alpha+r-3} = 3^{3-s} \cdot 11$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 3^{3-s} \cdot 11, A = 2^{3-r}3^{3-s} \cdot 11, 0 \leqslant r, s \leqslant 2$ and $33|X$ and Y a power of 2. We check again in **PariGp** that none of the solutions (X, Y) of these Thue equations thus satisfy the condition $33|X$ and Y a power of 2. Hence we need to consider $n = 1024$ when $a = 32$. For another example, let $a = 48$. We obtain equations of the form $5^{\beta_5} - 2^\alpha = 3$, $7^{\beta_7} - 2^\alpha = 3$, $5^{\beta_5} - 7 \cdot 2^\alpha = 3$ and $5^{\beta_5}7^{\beta_7} - 2^\alpha = 3$. The first three equations are excluded modulo 8 and for the last equation, we find that β_5, β_7 are both odd. Taking remainders modulo 7 imply $3|(\alpha - 2)$ or $3|(\alpha + 1)$ and hence from the equation $-2^{\alpha+1} + 2 \cdot 5^{\beta_5}7^{\beta_7} = 6$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 6, A = 2 \cdot 5^s7^t, 0 \leqslant s, t \leqslant 2$ and X a power of 2 and $70|AY$. When $t = 2$, from $5^{\beta_5-s+3}7^{\beta_7+1} - 4 \cdot 5^{3-s} \cdot 7 \cdot 2^{\alpha-2} = 3 \cdot 5^{3-s} \cdot 7$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 21 \cdot 5^{3-s}, A = 28 \cdot 5^{3-s}, 0 \leqslant s \leqslant 2$ and $35|X$ and Y a power of 2. We check in **PariGp** that all the solutions (X, Y) of these Thue equations are excluded except when $(A, B) = (70, 6)$ where $X = -4, Y = -1$ and we obtain $n = 512$. Hence we need to consider $n = 512$ when $a = 48$. Similarly, all other a's are excluded except when $a \in \{20, 24\}$ where we obtain $(n, a) \in \{(4096, 20), (1920, 24)\}$.

Thus we now exclude the cases $(n, a) \in \{(4096, 20), (1920, 24), (1024, 32), (512, 48)\}$. We take $p = 2$ and show that $\phi'_j < 1$ for all $1 \leqslant j \leqslant n$. This is shown by checking $\text{ord}_2(\Delta_j) - \text{ord}_2(\binom{n}{j}) < j$ for j such that $\text{ord}_2(\Delta_j) \geqslant j$ for these pairs (n, a). Hence they are all excluded. □

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References

Dus99 P. Dusart, *Inégalitiés explicites pour $\psi(X), \theta(X), \pi(X)$ et les nombres premiers*, C. R. Math. Rep. Acad. Sci. Canada **21(1)**(1999), 53-59.

Dus02 P. Dusart, *Sur la conjecture* $\pi(x+y) \leqslant \pi(x)+\pi(y)$, Acta Arith. **102** (2002), 295-308.

ErSe75 P. Erdős and J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292-301.

FiFiLe M. Filaseta, C. Finch and J. R. Leidy, *T. N. Shorey's influence in the theory of irreducible polynomials*, Diophantine Equations, ed. by N. Saradha, Narosa Publ., New Delhi etc., (2008), 77-102.

FiKiTr M. Filaseta, T. Kidd and O. Trifonov, *Laguerre polynomials with Galois group* A_m *for each* m, submitted for publication.

FiTr02 M. Filaseta, O. Trifonov, *The irreducibility of Bessel polynomials*, J. Reine Angew. Math. **550** (2002), 125-140.

LaMu00 S. Laishram and R. Murty, *Grimm's Conjecture and Smooth Numbers*, Michigan Math. Journal, in press.

LaSh04 S. Laishram and T. N. Shorey, *Number of prime divisors in a product of terms of an arithmetic progression*, Indag. Math. **15(4)** (2004), 505-521.

LaSh06b S. Laishram and T. N. Shorey, *Grimm's Conjecture on consecutive integers*, Int. Jour. Number Theory **2** (2006), 207-211.

Leh64 D. H. Lehmer, *On a problem of Stőrmer*, Illinois J. of Math. **8** (1964), 57-79.

RST75 K. Ramachandra, T. N. Shorey and R. Tijdeman, *On Grimm's problem relating to factorisation of a block of consecutive integers*, J. reine angew. Math. **273** (1975), 109-124.

Rob55 H. Robbins, *A remark on Stirling's formula*, Amer. Math. Monthly **62** (1955), 26-29.

Ros38 B. Rosser, *The* $n-th$ *prime is greater than* $n \log n$, Proc. London Math. Soc. **45** (1938), 21-44.

SaSh03 N. Saradha and T. N. Shorey, *Almost squares and factorisations in consecutive integers*, Compositio Math. **138** (2003), 113-124.

Sch73 I. Schur, *Affektlose Gleichungen in der Theorie der Laguerreschen und Hermitschen Polynome*, J. Reine Angew. Math. **165** (1931), 52-58.

Sch29 I. Schur, *Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen, II*, Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl. **14** (1929), 370-391.

ShTi T. N. Shorey and R. Tijdeman, *Generalizations of some irreducibility results by Schur*, Acta Arith. **145** (2010), 341-371.

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