Some irreducibility results for truncated binomial expansions

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Abstract. For positive integers $n > k$, let $P_{n,k}(x) = \sum_{j=0}^{k} \binom{n}{j} x^j$ be the polynomial obtained by truncating the binomial expansion of $(1+x)^n$ at the k^{th} stage. These polynomials arose in the investigation of Schubert calculus in Grassmannians. In this paper, the authors prove the irreducibility of $P_{n,k}(x)$ over the field of rational numbers when $2 \leqslant 2k \leqslant n < (k+1)^3$.

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1. Introduction

For positive integers k and n with $k \leqslant n-1$, let $P_{n,k}(x)$ denote the polynomial $\sum_{j=0}^{k} \binom{n}{j} x^j$, where $\binom{n}{j} = \frac{n!}{j!\ (n-j)!}$. In 2007, Filaseta, Kumchev and Pasechnik considered the problem of irreducibility of $P_{n,k}(x)$ over the field $\mathbb{Q}$ of rational numbers. This problem arose during the 2004 MSRI program on "topological aspects of real algebraic geometry" in the work of Inna Scherbak [6]. These polynomials have also arisen in the context of work of Iossif V. Ostrovskii [3]. In the case $k = 2$, $P_{n,k}(x)$ has negative discriminant and hence is irreducible over $\mathbb{Q}$. In fact it is already known that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for all $n \leqslant 100$, $k+2 \leqslant n$ (cf. [2, p.455]). In [2], Filaseta et al. pointed out that when $k = n-1$, then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ if and only if n is a prime number. They also proved that for any fixed integer $k \geqslant 3$, there exists an integer n_0 depending on k such that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for every $n \geqslant n_0$. So there are indications that $P_{n,k}(x)$ is irreducible for every n, k with $3 \leqslant k \leqslant n-2$.

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In this paper, we prove the irreducibility of $P_{n,k}(x)$ for all n, k such that $2 \leqslant 2k \leqslant n < (k+1)^3$. We consider the irreducibility of the polynomial $P_{n,k}(x-1) = \sum_{j=0}^{k} c_j x^j$, where $c_j = \sum_{i=j}^{k} \binom{n}{i}\binom{i}{j}(-1)^{i-j}$. As in [2], on using the identity

$$\sum_{j=0}^{a}(-1)^j \binom{b}{j} = (-1)^a \binom{b-1}{a}, \quad a < b \ \text{ non-negative integers,}$$

a simple calculation shows that

$$c_j = (-1)^{k-j}\binom{n}{j}\binom{n-j-1}{k-j} = \frac{(-1)^{k-j} \quad n(n-1)\cdots(n-k)}{j!(k-j)!} \; \frac{1}{(n-j)}. \tag{1}$$

In fact we shall prove the irreducibility of $P_{n,k}(x)$ using Newton polygons with respect to primes exceeding k dividing $\binom{n}{k}$ and some results of Erdős, Selfridge, Saradha, Shorey and Laishram regarding such primes (cf. [7], [5]). The same method gives the irreducibility of polynomial

$$F_{n,k}(x) = \sum_{j=0}^{k} a_j c_j x^j, \tag{2}$$

where $a_0, a_1, \ldots, a_k$ are non-zero integers and each a_i has all of its prime factors $\leqslant k$. We prove

Theorem 1.1. *Let k and n be positive integers such that $2k \leqslant n < (k+1)^3$. Then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$.*

Theorem 1.1 is derived from the following more general result.

Theorem 1.2. *Let k and n be positive integers such that $8 \leqslant 2k \leqslant n < (k+1)^3$ and $F_{n,k}(x)$ be as in (2). Then $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$ except possibly when (n,k) belongs to the set $\{(9,4),(18,4),(27,4),(10,5),(12,5),(27,5)\}$.*

In the course of the proof of the above theorem, we prove the following result which is of independent interest as well.

Theorem 1.3. *Let k, n be integers such that $n \geqslant k+2 \geqslant 4$. Suppose there exists a prime $p > k, p|(n-l)$ with $1 \leqslant l \leqslant k-1$ and $ord_p(n-l) = e$ such that $gcd(e,l) \leqslant 2$ and $gcd(e, k-l) \leqslant 2$. If $l_1 < k/2$ is a positive integer such that $l \notin \{l_1, 2l_1, k-l_1, k-2l_1\}$, then $F_{n,k}(x)$ cannot have a factor of degree l_1 over $\mathbb{Q}$.*

2. Notation and Preliminary Results

For any non-zero integer a, let $v_p(a)= \mathrm{ord}_p(a)$ denote the p-adic valuation of a, i.e., the highest power of p dividing a and denote $v_p(0)$ by ∞. Let $g(x) = \sum_{j=0}^{k} a_j x^j$ be a polynomial over $\mathbb{Q}$ with $a_0 a_k \neq 0$. To each term $a_i x^i$, we associate a point $(n-i, v_p(a_i))$ ignoring however the point $(n-i, \infty)$ if $a_i = 0$ and form the set

$$S = \{(0, v_p(a_k)), \ldots, (n-j, v_p(a_j)), \ldots, (k, v_p(a_0))\}.$$

The Newton polygon of $g(x)$ with respect to p is the polygonal path formed by the lower edges along the convex hull of points of S. Slopes of the edges are increasing when calculated from left to right.

We begin with the following well known results, see [1] for Theorem 2.A and [4, 5.1.F] for Theorem 2.B.

Theorem 2.A. *Let p be a prime and $g(x), h(x)$ belong to $\mathbb{Q}(x)$ with $g(0)h(0) \neq 0$ and $u \neq 0$ be the leading coefficient of $g(x)h(x)$. Then the edges of the Newton polygon of $g(x)h(x)$ with respect to p can be formed by constructing a polygonal path beginning at $(0, v_p(u))$ and using the translates of the edges in the Newton polygon of $g(x)$ and $h(x)$ with respect to p taking exactly one translate for each edge. The edges are translated in such a way as to form a polygonal path with slopes of edges increasing.*

Theorem 2.B. *Let $(x_0, y_0), (x_1, y_1), \ldots, (x_r, y_r)$ denote the successive vertices of the Newton polygon of a polynomial $g(x)$ with respect to a prime p. Let $\tilde{v}_p$ denote the unique extension of v_p to the algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers. Then $g(x)$ factors over $\mathbb{Q}_p$ as $g_1(x)g_2(x)\cdots g_r(x)$ where the degree of $g_i(x)$ is $x_i - x_{i-1}$, $i = 1, 2, \ldots, r$ and all the roots of $g_i(x)$ in the algebraic closure of $\mathbb{Q}_p$ have $\tilde{v}_p$ valuation $\frac{y_i - y_{i-1}}{x_i - x_{i-1}}$. In particular all the roots of an irreducible factor of $g(x)$ over $\mathbb{Q}_p$ will have the same $\tilde{v}_p$ valuation.*

For an integer $\nu > 1$, let $P(\nu)$ denote the greatest prime divisor of ν and let $\pi(\nu)$ denote the number of primes not exceeding ν. As in [5], $\delta(k)$ will denote the integer defined for $k \geqslant 3$ by

$$\delta(k) = \begin{cases} 2, & \text{if } 3 \leqslant k \leqslant 6; \\ 1, & \text{if } 7 \leqslant k \leqslant 16; \\ 0, & \text{otherwise.} \end{cases}$$

For numbers n, k and h, $[n, k, h]$ will stand for the set of all pairs $(n, k), (n+1, k), \ldots,$ $(n+h-1, k)$. In particular $[n, k, 1] = \{(n, k)\}$.

We shall denote by S the union of the sets

$[6,3,1],[8,3,3],[18,3,1],[9,4,1],[10,5,4],[16,5,1],[18,5,3],[27,5,2],[12,6,2],[20,6,1],$ $[14,7,3],[18,7,1],[20,7,2],[30,7,1],[16,8,1],[21,8,1],[26,13,3],[30,13,1],[32,13,2],$ $[36,13,1],[28,14,1],[33,14,1],[36,17,1]$

and by T the union of the sets

$[38,19,3],[42,19,1],[40,20,1],[94,47,3],[100,47,1],[96,48,1],[144,71,2],[145,72,1],$ $[146,73,3],[156,73,1],[148,74,1],[162,79,1],[166,83,1],[172,83,1],[190,83,1],$ $[192,83,1],[178,89,1],[190,89,1],[192,89,1],[210,103,2],[212,103,2][216,103,2],$ $[213,104,1],[217,104,1],[214,107,12],[216,108,10],[218,109,9],[220,110,7]$ $[222,111,5],[224,112,3],[226,113,7],[250,113,1],[252,113,2],[228,114,5],[253,114,1],$ $[230,115,3],[232,116,1],[346,173,1],[378,181,1],[380,181,2],[381,182,1],[392,193,2],$ $[393,194,1],[396,197,1],[398,199,3],[400,200,1],[552,271,5],[553,272,1],[555,272,2],$ $[556,273,1],[554,277,3],[558,277,5],[556,278,1],[559,278,4],[560,279,3],[561,280,1],$ $[562,281,7],[564,282,5],[566,283,5],[576,283,1],[568,284,3],[570,285,1],[586,293,1].$

With the above notations, we shall use the following theorem due to Laishram and Shorey [5, Theorem 3].

Theorem 2.C. *Let $n \geqslant 2k \geqslant 6$ and $f_1 < f_2 < \cdots < f_\mu$ be integers in $[0,k)$. Assume that the greatest prime factor of $(n-f_1)\dots(n-f_\mu) \leqslant k$. Then $\mu \leqslant k - \left[\frac{3}{4}\pi(k)\right] + 1 - \delta(k)$ unless $(n,k) \in S \cup T$.*

The following corollary is an immediate consequence of Theorem 2.C.

Corollary 2.D. *Let n and k be positive integers with $n \geqslant 2k \geqslant 38$. Then there are at least five distinct terms of the product $n(n-1)\cdots(n-k+1)$ each divisible by a prime exceeding k except when $(n,k) \in T$.*

For the proof of Theorem 1.3, we need the following propositions.

Proposition 2.1. *Let $k \geqslant 6$ and $n > k^2$. Then there exist two distinct terms $n+r$ and $n+s$ of the product $n(n+1)\cdots(n+k-1)$ which are divisible by primes $> k$ exactly to an odd power.*

Proof. Suppose the proposition is false for some n and k with $k \geqslant 6$ and $n > k^2$. Let $\Delta(n,k) = n(n+1)\cdots(n+k-1)$. Thus either $\mathrm{ord}_p(\Delta(n,k))$ is even for all primes $p > k$ or there is exactly one term $n+i$ and a prime $p > k$ such that $\mathrm{ord}_p(\Delta(n,k))$ is odd. The first possibility is excluded since for any positive integer b with $P(b) \leqslant k$, the equation

$$n(n+1)\cdots(n+k-1) = by^2$$

has no solution in positive integers n,k,y when $n > k^2 \geqslant 4^2$ by [7, Theorem A]. We now consider the case when there is exactly a term $n+i$ and a prime $p > k$ such that $\mathrm{ord}_p(\Delta(n,k))$ is odd. Suppose first that $0 < i < k-1$. Removing the term $n+i$ from $\Delta(n,k)$, we see that $n(n+1)\cdots(n+i-1)(n+i+1)\cdots(n+k-1) = b_1y_1^2$ where $P(b_1) \leqslant k$ which is impossible by virtue of [7, Theorem 2[1]].

It remains to consider the case when $i = 0$ or $k-1$. Let Δ' denote the product $(n+1)\cdots(n+k-1)$ or $n(n+1)\cdots(n+k-2)$ according as $i = 0$ or $k-1$. Then Δ' is a product of $k-1$ consecutive integers such that

$$\Delta' = b_2y_2^2 \tag{3}$$

with $P(b_2) \leqslant k$. This is impossible when $P(b_2) \leqslant k-1$ by [7, Theorem A]. It only remains to deal with the situation when $P(b_2) = k$. Then k will be a prime dividing only one term of the product Δ', say k divides $n+j, j \neq i$. We remove the term $n+j$ of the product Δ' and it is clear from (3) that

$$\frac{\Delta'}{n+j} = b_3y_3^2\ , \quad P(b_3) \leqslant k-2. \tag{4}$$

It is immediate from (4) and [7, Theorem 2] that $n+j$ is the first or last term of the product Δ' as $k-1 \geqslant 5$. Thus we see that $\frac{\Delta'}{n+j}$ is the product of $k-2$ consecutive integers. This is impossible by [7, Theorem A].

Proposition 2.2. *Let n, k be positive integers with $n \geqslant k+2 \geqslant 4$ and $F_{n,k}(x)$ be given by (2). Suppose there exists a prime $p > k$ such that $p^e || (n-l)$ for some l, $1 \leqslant l \leqslant k-1$. Let d =gcd(e,l) and d' =gcd$(e,k-l)$. Then the following hold.*
(i) The edges of the Newton polygon of $F_{n,k}(x)$ with respect to p have slopes $\frac{-e}{k-l}$, $\frac{e}{l}$.
(ii) $F_{n,k}(x)$ has at least two distinct irreducible factors over $\mathbb{Q}_p$; one of them has degree a multiple of $\frac{l}{d}$ and other has degree a multiple of $\frac{k-l}{d'}$.
(iii) If $d = d' = 1$, then $F_{n,k}(x)$ factors over $\mathbb{Q}_p$ as a product of two distinct irreducible polynomials of degrees l and $k-l$.

Proof. We consider the Newton polygon of $F_{n,k}(x)$ with respect to the prime p. In view of (1), the vertices of the Newton polygon are $(0,e), (k-l,0), (k,e)$. Thus the Newton polygon has two edges, one from $(0,e)$ to $(k-l,0)$ and other from $(k-l,0)$ to (k,e) with respective slopes $\frac{-e}{k-l}$ and $\frac{e}{l}$ proving (i).

Note that equations of the two edges are given by:

$$y - e = \frac{-e}{k-l}\ x \quad \text{and} \quad y = \frac{e}{l}\ (x-k+l).$$

[1]It states that for $n > k^2 \geqslant 5^2$ the equation $n(n+1)\cdots(n+i-1)(n+i+1)\cdots(n+k-1) = by^2$ has no solution in positive integers n, k, b, y with $P(b) \leqslant k$ and $0 < i < k-1$.

On the first edge, the x-coordinates of the lattice points occur at multiples of $\frac{k-l}{d'}$, i.e., when $x = \frac{k-l}{d'}.M$ where $0 \leqslant M \leqslant d'$; on the second edge the x-coordinates of lattice points are given by $k - l + \frac{l}{d}.N$ where $0 \leqslant N \leqslant d$. By Theorem 2.B, all the roots of an irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ have the same slope. Since the slopes of the two edges as shown in (i) are different, we see that any irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ must lie on the first edge or on the second edge. Hence assertion (ii) now follows from Theorem 2.A. Assertion (iii) is an immediate consequence of (ii).

The last assertion quickly yields the following result.

Corollary 2.3. *If for a pair (n, k), $n \geqslant k+2$, there exist terms $n - l_1$, $n - l_2$, $1 \leqslant l_1 < l_2 < k$, divisible respectively by primes p_1, p_2 exceeding k exactly to the first power such that $l_1 + l_2 \neq k$, then $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$.*

Proposition 2.4. *Let n, k and $F_{n,k}(x)$ be as in Proposition 2.2. Let p be a prime $> k$ and $e > 0$ be such that $p^e || n$. Then every irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ has degree a multiple of $\frac{k}{D}$, where $D =$gcd(e, k).*

Proof. The vertices of the Newton polygon of $F_{n,k}(x)$ with respect to p are $(0, e), (k, 0)$. Thus the Newton polygon has only one edge whose equation is given by $y - e = \frac{-e}{k}\, x$. The x-coordinates of the lattice points on this edge occur at multiples of k/D. So arguing as in Proposition 2.2, any irreducible factor of $F_{n,k}(x)$ must have degree a multiple of k/D.

3. Proof of Theorem 1.3

For a polynomial $f(x)$, *deg* $f(x)$ will stand for the degree of $f(x)$. We denote $F_{n,k}(x)$ by $F(x)$ and $\gcd(e, l)$, $\gcd(e, k - l)$ by d, d' respectively. Suppose to the contrary that $F(x)$ has a factor of degree l_1 over $\mathbb{Q}$. We write $F(x) = a_k c_k g_1(x) h_1(x)$, where $g_1(x), h_1(x)$ are monic polynomials over $\mathbb{Q}$ with *deg* $h_1(x) = l_1 < \frac{k}{2}$. Observe that $F(x)$ cannot factor as a product of irreducible polynomials of degree l and $k - l$ over $\mathbb{Q}_p$ because one of the factors will be a constant multiple of g_1 and the other will be a constant multiple of h_1, which is impossible as $l \neq l_1, k - l_1$. Therefore in view of Proposition 2.2 (iii), we may suppose that either $d > 1$ or $d' > 1$. The proof is split into three cases.

Case I. $d = 2$ and $d' = 1$. In view of Proposition 2.2 (ii), $F(x)$ factors over $\mathbb{Q}_p$ either as a product of two irreducible polynomials of degree l, $k - l$ or three irreducible polynomials of degree $k - l$, $l/2$, $l/2$. The first possibility does not arise as pointed out in the above paragraph. Thus the factorization of $F(x)$ into monic irreducible polynomials over $\mathbb{Q}_p$ is of the type

$$F(x) = a_k c_k G(x) H_1(x) H_2(x)$$

where $G(x)$ is of degree $k - l$ and $H_1(x), H_2(x)$ are of degree $\frac{l}{2}$ each. Since $F(x) = a_k c_k g_1(x) h_1(x)$ and $deg\ g_1(x) > \frac{k}{2}$, we have either

$$g_1(x) = G(x) \ \text{ or } \ g_1(x) = H_1(x)H_2(x) \ \text{ or } \ g_1(x) = G(x)H_i(x), \ i = 1 \text{ or } 2.$$

On comparing degrees, we see that

$$\text{either} \quad k - l_1 = k - l \quad \text{or} \quad k - l_1 = l \quad \text{or} \quad k - l_1 = k - \frac{l}{2}\ ,$$

which implies that $l \in \{l_1,\ k - l_1,\ 2l_1\}$ contrary to the hypothesis of the theorem.

Case II. $d = 1$ and $d' = 2$. By Proposition 2.2 (ii) and the observation at the beginning of the proof, the factorization of $F(x)$ over $\mathbb{Q}_p$ is

$$F(x) = a_k c_k G_1(x) G_2(x) H(x)$$

where $G_1(x), G_2(x), H(x)$ are monic irreducible polynomials of degree $\frac{k-l}{2}$, $\frac{k-l}{2}$ and l respectively. Hence

$$\text{either} \quad g_1(x) = G_j(x) \quad \text{or} \quad g_1(x) = H(x) \quad \text{or} \quad g_1(x) = G_j(x)H(x), \quad j \in \{1, 2\}.$$

Comparing the degrees, we see that

$$\text{either} \quad k - l_1 = \frac{k-l}{2} \ \text{ or } \ k - l_1 = l \ \text{ or } \ k - l_1 = \frac{k-l}{2} + l. \tag{5}$$

The first possibility can hold only when $l_1 = \frac{k}{2}$ and $l = 0$ which is impossible by the hypothesis. The second and third equalities in (5) imply that $l = k - l_1$ or $k - 2l_1$ again contradicting the hypothesis.

Case III. $d = 2$ and $d' = 2$. Arguing as above, we see that over $\mathbb{Q}_p$, $F(x)/a_k c_k$ factors as $G(x)H_1(x)H_2(x)$ or $G_1(x)G_2(x)H(x)$ or $G_1(x)G_2(x)H_1(x)H_2(x)$ where $G(x), H(x), G_1(x), G_2(x), H_1(x), H_2(x)$ are monic irreducible polynomials with $deg\ G(x) = k - l, deg\ H(x) = l,\ deg\ H_1(x) = deg\ H_2(x) = l/2$ and $deg\ G_1(x) = \ deg\ G_2(x) = \frac{k-l}{2}$. The first two possibilities can occur only when $l \in \{l_1, 2l_1, k - l_1, k - 2l_1\}$ as shown in Cases I and II. So it remains to consider the case when

$$F(x) = a_k c_k G_1(x) G_2(x) H_1(x) H_2(x).$$

Since $F(x) = a_k c_k g_1(x) h_1(x)$, we see that

$$g_1 \in \{H_1H_2, G_1G_2, H_1H_2G_1, H_1H_2G_2, H_2G_1G_2, H_1G_2, H_2G_1, H_1G_1, H_2G_2, G_1, G_2, H_1, H_2\}.$$

Now on comparing the degrees, it is immediate that

$$k - l_1 \in \{l,\ k - l,\ l + \frac{k-l}{2},\ k - l + \frac{l}{2},\ \frac{l}{2} + \frac{k-l}{2},\ \frac{k-l}{2},\ \frac{l}{2}\}.$$

Note that $k-l_1$ cannot be equal to any of the first five elements of the above set since $l \notin \{l_1, 2l_1, k-l_1, k-2l_1\}$ and $l_1 < k/2$. The possibilities $k-l_1 = \frac{k-l}{2}$ and $k-l_1 = l/2$ cannot occur as $k-l_1 > \frac{k}{2}$ while $\frac{k-l}{2}$ and $\frac{l}{2}$ are less than $\frac{k}{2}$. This completes the proof of the theorem.

4. Proof of Theorem 1.2

With S and T as in Theorem 2.C, we first prove

Lemma 4.1. *For $(n,k) \in S \cup T$, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$ except possibly when (n,k) belongs to the subset S' of S given by*
$S' = \{(6,3), [8,3,3], (18,3), (9,4), (10,5), (12,5), (12,6), (16,8), (18,5), [27,5,2], (30,7)\}$.
Proof. Observe that if n is divisible by a prime $p > k$ with $\mathrm{ord}_p(n) = 1$, then $x^k F_{n,k}(1/x)$ is an Eisenstein polynomial with respect to p and so $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. Further if two distinct terms $n-l_1, n-l_2$ of the product $n(n-1)\cdots(n-k+1)$ are divisible by primes p_1 and p_2 exceeding k such that $\mathrm{ord}_{p_i}(n-l_i) = 1$ and $l_1 + l_2 \neq k$, then in view of the above observation and Corollary 2.3, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. For each (n,k) belonging to $T \cup (S \setminus S')$, n not a prime, Table 1 at the end of this section indicates two primes p_1 and p_2 satisfying the above property which completes the proof of the lemma.

Proof of Theorem 1.2. Suppose that $F_{n,k}(x)$ is reducible over $\mathbb{Q}$. Consider first the case when $F_{n,k}(x)$ factors over $\mathbb{Q}$ into two irreducible polynomials of degree $\frac{k}{2}$ each. To prove the irreducibility of $F_{n,k}(x)$ in this case, it is enough to show that there exists $l' \neq k/2$, $0 \leqslant l' \leqslant k-1$ such that $n-l'$ is divisible by a prime $p' > k$ exactly with the first power. If $l' = 0$, then as pointed out in the opening lines of the proof of Lemma 4.1, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. If $l' \geqslant 1$ then by Proposition 2.2 (*iii*), $F_{n,k}(x)$ has two irreducible factors of degree l' and $k-l'$ over $\mathbb{Q}_{p'}$ which is impossible as $l' \neq k/2$ thereby proving the irreducibility of $F_{n,k}(x)$ over $\mathbb{Q}$. The existence of a term $n-l' \neq n-(k/2)$, $0 \leqslant l' \leqslant k-1$, which is divisible by some prime $p' > k$ with $\mathrm{ord}_{p'}(n-l') = 1$ is guaranteed for $k \geqslant 6$ by Theorem 2.C when $2k \leqslant n < (k+1)^2$ unless $(n,k) \in S \cup T$, and by Proposition 2.1 when $(k+1)^2 \leqslant n < (k+1)^3$. For $k = 4$, Table 2 at the end of this section indicates such a term $n-l'$ when $8 \leqslant n < 5^3, n \neq 9$ and n is not divisible by any prime $p > 4$ with $\mathrm{ord}_p(n) = 1$. Thus in view of Lemma 4.1 and the fact that k is even in this case, the irreducibility of $F_{n,k}(x)$ needs to be proved when $(n,k) \in \{(9,4), (12,6), (16,8)\}$. In view of Proposition 2.2 (*iii*), over $\mathbb{Q}_{11}$ each of the polynomials $F_{12,6}(x)$ and $F_{16,8}(x)$ has an irreducible factor of degree 5 which is impossible in the present case. So the theorem is proved in this case.

Consider now the possibility when $F_{n,k}(x)$ has a factor of degree $l_1 < k/2$ over $\mathbb{Q}$. We

make some claims.
Claim 1: $P(n) \leqslant k$.
Suppose not. Let $p_0 = P(n) > k$ and $\mathrm{ord}_p(n) = e_0$. Then $e_0 \leqslant 2$ since $n < (k+1)^3$. So by Proposition 2.4, every irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_{p_0}$ has degree a multiple of k or $\frac{k}{2}$ according as $e_0 = 1$ or 2 respectively. This is not possible.
Claim 2: There are at most four distinct terms in the product $n(n-1)\cdots(n-k+1)$ which are divisible by a prime $> k$.
Assume the contrary. Then there is a term $n-l$ with $0 \leqslant l < k$ and a prime $p > k$ with $p|(n-l)$ such that $l \notin \{l_1,\ 2l_1,\ k-l_1,\ k-2l_1\}$. Note that $l > 0$ in view of Claim 1. Further e $=\mathrm{ord}_p(n-l) \leqslant 2$ implying that $F_{n,k}(x)$ cannot have a factor of degree l_1 over $\mathbb{Q}$ by Theorem 1.3, which contradicts the assumption of the present case.
Claim 3: There are at most two distinct terms in the product $n(n-1)\cdots(n-k+1)$ which are divisible by a prime $> \sqrt{n}$.
Suppose not. Let $1 \leqslant l_1' < l_2' < l_3'$ be such that there exist prime $p_i > \sqrt{n}$ and e_i $=\mathrm{ord}_{p_i}(n-l_i')$. Then $e_i = 1$ for each $i \in \{1,2,3\}$. By using Proposition 2.2 (*iii*) for each $i \in \{1,2,3\}$, $F_{n,k}(x)$ factors over $\mathbb{Q}_{p_i}$ as a product of two non-associate irreducible polynomials of degree l_i' and $k-l_i'$. It is clear that over $\mathbb{Q}$, $F_{n,k}(x)$ has a factorization of the type $F_{n,k}(x) = a_k c_k G_i(x) H_i(x)$ where $G_i(x), H_i(x)$ are monic irreducible polynomials belonging to $\mathbb{Q}[x]$ with degrees $k-l_i'$, l_i' respectively. Clearly at least one of $l_2' \neq k-l_1'$ or $l_3' \neq k-l_1'$ must hold and hence we have two different factorizations of $F_{n,k}(x)$ into irreducible factors over $\mathbb{Q}$ leading to a contradiction.

From Claim 2, Corollary 2.D and Lemma 4.1, it follows that $k \leqslant 18$. In view of Claim 1, we may first restrict to those n for which $P(n) \leqslant k$. Further by Claims 2 and 3, those n can be excluded for which $n(n-1)\cdots(n-k+1)$ has either five terms divisible by a prime $> k$ or three terms divisible by a prime $> \sqrt{n}$. We use *Sage* mathematics software for the above computations. Then we are left with the following pairs (n,k) given by

$$k = 4,\ n \in \{9,12,18,27,32,48,64,72,81,108\};$$
$$k = 5,\ n \in \{10,12,20,27,50,64,100,128,200\};$$
$$k = 6,\ n \in \{50\}.$$

Out of these pairs, we can exclude those pairs (n,k) for which the hypothesis of Corollary 2.3 is satisfied. It can be easily checked that we are finally left with only the pairs given in the statement of Theorem 1.2. This completes the proof of the theorem.

Table 1.

$(n,k) \in [n,k,h] \to Primes$		$(n,k) \in [n,k,h] \to Primes$		$(n,k) \in [n,k,h] \to Primes$	
$[16,5,1]$	$11,13$	$[162,79,1]$	$131,139$	$[346,173,1]$	$293,307$
$[20,5,1]$	$17,19$	$[166,83,1]$	$131,139$	$[378,181,1]$	$293,307$
$[20,6,1]$	$17,19$	$[172,83,1]$	$137,139$	$[380,181,2]$	$293,307$
$[14,7,3]$	$13,11$	$[190,83,1]$	$131,139$	$[381,182,1]$	$293,307$
$[18,7,1]$	$13,17$	$[192,83,1]$	$131,139$	$[392,193,2]$	$293,307$
$[20,7,1]$	$17,19$	$[178,89,1]$	$131,139$	$[393,194,1]$	$293,307$
$[21,7,1]$	$17,19$	$[190,89,1]$	$131,139$	$[396,197,1]$	$293,307$
$[21,8,1]$	$17,19$	$[192,89,1]$	$139,149$	$[398,199,3]$	$293,307$
$[26,13,3]$	$17,19$	$[210,103,1]$	$139,149$	$[400,200,1]$	$283,307$
$[30,13,1]$	$19,23$	$[212,103,2]$	$139,149$	$[552,271,5]$	$421,431$
$[32,13,2]$	$23,29$	$[216,103,2]$	$139,149$	$[553,272,1]$	$421,431$
$[36,13,1]$	$29,31$	$[213,104,1]$	$139,149$	$[555,272,2]$	$421,431$
$[28,14,1]$	$17,19$	$[217,104,1]$	$139,149$	$[556,273,1]$	$421,431$
$[33,14,1]$	$29,31$	$[214,107,12]$	$139,149$	$[554,277,3]$	$421,431$
$[36,17,1]$	$29,31$	$[216,108,10]$	$139,149$	$[558,277,5]$	$421,431$
$[38,19,3]$	$23,29$	$[218,109,9]$	$139,149$	$[556,278,1]$	$421,431$
$[42,19,1]$	$37,41$	$[220,110,7]$	$139,149$	$[559,278,4]$	$421,431$
$[40,20,1]$	$31,37$	$[222,111,5]$	$139,149$	$[560,279,3]$	$421,431$
$[94,47,3]$	$89,83$	$[224,112,3]$	$139,149$	$[561,280,1]$	$421,431$
$[100,47,1]$	$83,89$	$[226,113,7]$	$139,149$	$[562,281,7]$	$409,431$
$[96,48,1]$	$79,83$	$[250,113,1]$	$139,149$	$[564,282,5]$	$409,431$
$[144,71,2]$	$101,103$	$[252,113,2]$	$139,149$	$[566,283,5]$	$421,431$
$[145,72,1]$	$101,103$	$[228,114,5]$	$139,149$	$[576,283,1]$	$421,431$
$[146,73,3]$	$101,103$	$[253,114,1]$	$139,149$	$[568,284,3]$	$419,431$
$[156,73,1]$	$109,113$	$[230,115,3]$	$139,149$	$[570,285,1]$	$421,431$
$[148,74,1]$	$107,113$	$[232,116,1]$	$139,149$	$[586,293,1]$	$421,431$

Table 2.

$n \quad \to$	$n-l', p'$	$n \quad \to$	$n-l', p'$	$n \quad \to$	$n-l', p'$
8	$7,7$	36	$35,7$	81	$78,13$
12	$11,11$	48	$47,47$	96	$95,19$
16	$13,13$	49	$46,23$	98	$95,19$
18	$17,17$	50	$47,47$	100	$99,11$
24	$23,23$	54	$51,17$	108	$107,107$
25	$22,11$	64	$63,7$	121	$118,59$
27	$26,13$	72	$69,23$		
32	$29,29$	75	$74,37$		

5. Proof of Theorem 1.1

In view of Theorem 1.2., we need to prove the irreducibility of $P_{n,k}(x)$ only when $1 \leqslant k \leqslant 3$, $2k \leqslant n < (k+1)^3$ or (n,k) belongs to $\{(9,4),(18,4),(27,4),(10,5),(12,5),(27,5)\}$. Using Maple, we have verified the irreducibility of $P_{n,k}(x)$ for these values of (n,k).

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References.

[1] M. G. Dumas, Sur quelques cas d'irréducibilité des polynomes à coefficients rationnels, *J. Math. Pures Appl.* 2 (1906) 191-258.

[2] M. Filaseta, A. Kumchev and D. Pasechnik, On the irreducibility of a truncated binomial expansion, *Rocky Mountain J. Math.* 37 (2007), 455464.

[3] I. V. Ostrovskii, On a problem of A. Eremenko, *Comput. Methods Funct. Theory* 4 (2004), No. 2, 275-282.

[4] P. Ribenboim, *The Theory of Classical Valuations*, Springer-Verlag New York, 1999.

[5] Shanta Laishram and T. N. Shorey, Number of prime divisors in a product of consecutive integers, *Acta Arithmetica* 113.4 (2004), 327-341.

[6] I. Scherbak, Intersections of Schubert varieties and highest weight vectors in tensor products sl_{N+1}- representations, ArXiv e-print math.RT/0409329, Sept. 2004.

[7] N. Saradha and T. N. Shorey, Almost squares and factorisations in consecutive integers, *Compositio Math.* 138 (2003), 113-124.

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