PERFECT POWERS IN ARITHMETIC PROGRESSION

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Abstract

The conjecture of Masser-Oesterlé, popularly known as abc-conjecture have many consequences. We use an explicit version due to Baker to solve the equation

$$n(n+d)\cdots(n+(k-1)d) = by^l$$

in positive integer variables n, d, k, b, y, l such that b square free with the largest prime divisor of b at most k, $k \geq 2, l \geq 2$ and $\gcd(n,d) = 1$.

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1. Introduction

Let n, d, k, b, y be positive integers such that b is square free with $P(b) \leq k$, $k \geq 2, l \geq 2$ and $\gcd(n,d) = 1$. Here $P(m)$ denotes the largest prime divisor of m with the convention $P(1) = 1$. We consider the equation

$$n(n+d)\cdots(n+(k-1)d) = by^l \tag{1.1}$$

in variables n, d, k, b, y, l. If $k = 2$, we observe that (1.4) has infinitely many solutions. Therefore we always suppose that $k \geq 3$. It has been conjectured (see [Tij88], [SaSh05]) that

Conjecture 1.1. *Equation* (1.1) *implies that* $(k, \ell) \in \{(3,3), (4,2), (3,2)\}$.

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It is known that (1.1) has infinitely many solutions when $(k, \ell) \in \{(3,2),(3,3)(4,2)\}$. A weaker version of Conjecture 1.1 is the following conjecture due to Erdős.

Conjecture 1.2. *Equation* (1.1) *implies that k is bounded by an absolute constant.*

For an account of results on (1.1), we refer to Shorey [Sho02b] and [Sho06].

The well known conjecture of Masser-Oesterle states that

Conjecture 1.3. Oesterlé and Masser's abc-conjecture: *For any given $\epsilon > 0$ there exists a computable constant $\mathfrak{c}_\epsilon$ depending only on ϵ such that if*

$$a + b = c \tag{1.2}$$

where a, b and c are coprime positive integers, then

$$c \le \mathfrak{c}_\epsilon \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

It is known as abc-conjecture; the name derives from the usage of letters a, b, c in (1.2). For any positive integer $i > 1$, let $N = N(i) = \prod_{p|i} p$ be the radical of i, $P(i)$ be the greatest prime factor of i and $\omega(i)$ be the number of distinct prime factors of i and we put $N(1) = 1, P(1) = 1$ and $\omega(1) = 0$.

It has been shown in Elkies [Elk91] and Granville and Tucker [GrTu02, (13)] that abc-conjecture is equivalent to the following:

Conjecture 1.4. *Let $F(x,y) \in \mathbb{Z}[x,y]$ be a homogenous polynomial. Assume that F has pairwise non-proportional linear factors in its factorisation over $\mathbb{C}$. Given $\epsilon > 0$, there exists a computable constant κ_ϵ depending only on F and ϵ such that if m and n are coprime integers, then*

$$\prod_{p|F(m,n)} p \ge \kappa_\epsilon \left(\max\{|m|, |n|\} \right)^{\deg(F)-2-\epsilon}.$$

Shorey [Sho99] showed that abc-conjecture implies Conjecture 1.2 for $\ell \ge 4$ using $d \ge k^{c_1 \log\log k}$. Granville (unpublished) gave a proof of the preceding result without using the inequality $d \ge k^{c_1 \log\log k}$. Furthermore his proof is also valid for $\ell = 2, 3$.

Theorem 1.1. *The $abc-$conjecture implies Conjecture* (1.2)*.*

The proof was first published in the Master's Thesis of first author[Lai04]. We include the proof in this paper to have a published literature. This is given in Section 6. We would like to thank Professor A. Granville for allowing us to publish his proof.

An explicit version of Conjecture 1.2 due to Baker [Bak94] is the following:

Conjecture 1.5. Explicit abc-conjecture: *Let a, b and c be pairwise coprime positive integers satisfying* (1.2)*. Then*

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We observe that $N = N(abc) \geq 2$ whenever a, b, c satisfy (1.2). We shall refer to Conjecture 1.3 as *abc−conjecture* and Conjecture 1.5 as *explicit abc−conjecture*. Conjecture 1.5 implies the following explicit version of Conjecture 1.3 proved in [LaSh12].

Theorem 1.2. *Assume Conjecture 1.5. Let a, b and c be pairwise coprime positive integers satisfying (1.2) and $N = N(abc)$. Then we have*

$$c < N^{1+3/4} \tag{1.3}$$

Further for $0 < \epsilon \leq 3/4$, there exists an integer ω_ϵ depending only ϵ such that when $N = N(abc) \geq N_\epsilon = \prod_{p \leq p_{\omega_\epsilon}} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon}$$

where

$$\kappa_\epsilon = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \leq \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$$

with $\omega = \omega(N)$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ.

ϵ	$3/4$	$7/12$	$6/11$	$1/2$	$34/71$	$5/12$	$1/3$
ω_ϵ	14	49	72	127	175	548	6460
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

Thus $c < N^2$ which was conjectured in Granville and Tucker [GrTu02].

As a consequence of Theorem 1.2, we prove

Theorem 1.3. *Assume Conjecture 1.5. Then the equation*

$$n(n+d)\cdots(n+(k-1)d) = by^\ell \tag{1.4}$$

in integers $n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1, \ell > 1$ with gcd$(n, d) = 1$ and $P(b) \leq k$ implies $\ell \leq 7$. Further $k < e^{13006.2}$ when $\ell = 7$.

We observe that $e^{13006.2} < e^{e^{9.52}}$. Theorem 1.3 is a considerable improvement of Saradha [Sar12] where it is shown that (1.4) with $k \geq 8$ implies that $\ell \leq 29$ and further $k \leq 8, 32, 10^2, 10^7$ and $e^{e^{280}}$ according as $\ell = 29, \ell \in \{23, 19\}, \ell = 17, 13$ and $\ell \in \{11, 7\}$, respectively.

2. Notation and Preliminaries

For an integer $i > 0$, let p_i denote the $i-$th prime. We always write p for a prime number. For a real $x > 0$ and $d \in \mathbb{Z}, d \geq 1$, let

$$\pi_d(x) = \sum_{p \leq x, p \nmid d} 1, \;\; \pi(x) = \pi_1(x) = \sum_{p \leq x} 1, \;\; \Theta(x) = \prod_{p \leq x} p \;\text{ and }\; \theta(x) = \log(\Theta(x)).$$

We write $\log_2 i$ for $\log(\log i)$. Here we understand that $\log_2 1 = -\infty$.

Lemma 2.1. *We have*

(i) $\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$ for $x > 1$.

(ii) $p_i \geq i(\log i + \log_2 i - 1)$ for $i \geq 1$

(iii) $\theta(p_i) \geq i(\log i + \log_2 i - 1.076869)$ for $i \geq 1$

(iv) $\theta(x) < 1.000081x$ for $x > 0$

(v) $\sqrt{2\pi k}(\frac{k}{e})^k e^{\frac{1}{12k+1}} \leq k! \leq \sqrt{2\pi k}(\frac{k}{e})^k e^{\frac{1}{12k}}$.

The estimates (i) and (ii) are due to Dusart, see [Dus99b] and [Dus99a], respectively. The estimate (iii) is [Rob83, Theorem 6]. For estimate (iv), see [Dus99b]. The estimate (v) is [Rob55, Theorem 6].

3. Proof of Theorem 1.3

Let n, d, k, b, y be positive integers with $n \geq 1, d > 1, k \geq 4, b \geq 1, y \geq 1$, $\gcd(n, d) = 1$ and $P(b) \leq k$. We consider the Diophantine equation

$$n(n+d)\cdots(n+(k-1)d) = by^{\ell}. \tag{3.5}$$

Observe that $P(n(n+d)\cdots(n+(k-1)d)) > k$ by a result of Shorey and Tijdeman [ShTi90] and hence $P(y) > k$ and also $n + (k-1)d \geq (k+1)^{\ell}$. For every $0 \leq i < k$, we write

$$n + id = A_i X_i^{\ell} \text{ with } P(A_i) \leq k \text{ and } (X_i, \prod_{p \leq k} p) = 1.$$

Without loss of generality, we may assume that $k = 4$ or $k \geq 5$ is a prime which we assume throughout in this section. We observe that $(A_i, d) = 1$ for $0 \leq i < k$ and $(X_i, X_j) = 1$. Let

$$S_0 = \{A_0, A_1, \ldots, A_{k-1}\}.$$

For every prime $p \leq k$ and $p \nmid d$, let i_p be such that $\mathrm{ord}_p(A_i) = \mathrm{ord}_p(n+id) \leq \mathrm{ord}_p(n+i_p d)$ for $0 \leq i < k$. For a $S \subset S_0$, let

$$S' = S - \{A_{i_p} : p \leq k, p \nmid d\}.$$

Then $|S'| \geq |S| - \pi_d(k)$. By Sylvester-Erdős inequality(see [ErSe75, Lemma 2] for example), we obtain

$$\prod_{A_i \in S'} A_i | (k-1)! \prod_{p|d} p^{-\mathrm{ord}_p((k-1)!)}. \tag{3.6}$$

As a consequence, we have

Lemma 3.2. *Let* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \geq 1, \beta < 1$ *and* $e\beta < \alpha$. *Let*

$$S_1 := S_1(\alpha) := \{A_i \in S_0 : A_i \leq \alpha k\}.$$

For

$$k \geq \max\left\{\frac{\log(\frac{e\alpha}{\sqrt{\beta}}) + \frac{k\log(\alpha k)}{\log k}\left(1+\frac{1.2762}{\log k}\right) - \log(\alpha k)}{\log(e\alpha) + \beta\log\left(\frac{\beta}{e\alpha}\right)}, \exp\left(\frac{1+\frac{1.2762}{\log k}}{1-\beta}\right)\right\} \tag{3.7}$$

we have $|S_1| > \beta k$.

Proof. Let $S = S_0$, $s_1 = |S_1|$ and $s_2 = |S' - S_1|$. Then $s_2 \geq k - \pi(k) - s_1$. We get from (3.6) that

$$s_1! \prod_{i=1}^{k-\pi(k)-s_1} (\lfloor \alpha k + i \rfloor) \leq \prod_{A_i \in S'} A_i \leq (k-1)! \tag{3.8}$$

since elements of $S' - S_1$ are distinct and the product on the left side is taken to be 1 if $k - \pi(k) \leq s_1$.

Suppose $s_1 \leq \beta k$. If $k - \pi(k) \leq s_1$, then using Lemma 2.1 (i), we get $(1-\beta)\log k < 1 + 1.2762/\log k$ which is not possible by (3.7). Hence $k - \pi(k) > s_1$. By using Lemma 2.1 (v), we obtain

$$(\alpha k)^{k-\pi(k)} < \frac{(k-1)!}{s_1!}(\alpha k)^{s_1} < \begin{cases} \sqrt{2\pi(k-1)}\left(\frac{k-1}{e}\right)^{k-1} e^{\frac{1}{12(k-1)}} & \text{if } s_1 = 0 \\ \sqrt{\frac{k-1}{s_1}}\left(\frac{\alpha k e}{s_1}\right)^{s_1}\left(\frac{k-1}{e}\right)^{k-1} & \text{if } s_1 > 0. \end{cases}$$

We check that the expression for $s_1 = 0$ is less than that of $s_1 = 1$ since $\alpha \geq 1$. Observe that

$$\sqrt{\frac{k-1}{s_1}}\left(\frac{\alpha k e}{s_1}\right)^{s_1}$$

is an increasing function of s_1 since $s_1 \leq \beta k$ and $e\beta < \alpha$. This can be verified by taking $\log$ of the above expression and differentiating it with respect to s_1. Therefore

$$(\alpha k)^{k-\pi(k)} < \sqrt{\frac{k-1}{\beta k}}\left(\frac{e\alpha}{\beta}\right)^{\beta k}\left(\frac{k-1}{e}\right)^{k-1} < \sqrt{\frac{1}{\beta}}\left(\frac{e\alpha}{\beta}\right)^{\beta k}\left(\frac{k}{e}\right)^{k-1}$$

implying

$$(e\alpha)^k\left(\frac{\beta}{e\alpha}\right)^{\beta k} < \frac{e\alpha}{\sqrt{\beta}}(\alpha k)^{\pi(k)-1}.$$

Using Lemma 2.1 (i), we obtain

$$\log(e\alpha) + \beta\log\left(\frac{\beta}{e\alpha}\right) < \frac{1}{k}\log(\frac{e\alpha}{\sqrt{\beta}}) + \frac{\log(\alpha k)}{\log k}\left(1+\frac{1.2762}{\log k}\right) - \frac{\log(\alpha k)}{k}.$$

The right hand side of the above inequality is a decreasing function of k for k given by (3.7). This can be verified by observing that $\log(\alpha k)/\log k = 1 + \log\alpha/\log k$ and differentiating $(1.2762 + \log\alpha)/\log k - \log(\alpha k)/k$ with respect to k. This is a contradiction for k given by (3.7). $\square$

Corollary 3.1. *For $k > 113$, there exist $0 \le f < g < h < k$ with $h - f \le 8$ such that* $\max(A_f, A_g, A_h) \le 4k$.

Proof. By dividing $[0, k-1]$ into subintervals of the form $[9i, 9(i+1))$, it suffices to show $S_1(4) > 2(\lfloor k/9 \rfloor + 1)$ where S_1 is as defined in Lemma 3.2. Taking $\alpha = 4, \beta = 1/4$, we obtain from Lemma 3.2 that for $k \ge 700$, $|S_1(4)| > k/4 > 2(\lfloor k/9 \rfloor + 1)$. Thus we may suppose $k < 700$ and $|S_1(4)| \le 2(\lfloor k/9 \rfloor + 1)$. For each prime k with $113 < k < 700$, taking $\alpha = 4$ and $\beta k = 2(\lfloor k/9 \rfloor + 1)$ in Lemma 3.2, we get a contradiction from (3.8). Therefore $|S_1(4)| > 2(\lfloor k/9 \rfloor + 1)$ and the assertion follows. □

Given $0 \le f < g < h \le k-1$, we have

$$(h-f)A_g X_g^\ell = (h-g)A_f X_f^\ell + (g-f)A_h X_h^\ell. \tag{3.9}$$

Let $\lambda =$gcd$(h-f, h-g, g-f)$ and write $h-f = \lambda w, h-g = \lambda u, g-f = \lambda v$. Rewriting $h - f = h - g + g - f$ as

$$w = u + v \text{ with } \gcd(u, v) = 1,$$

(3.9) can be written as

$$w A_g X_g^\ell = u A_f X_f^\ell + v A_h X_h^\ell. \tag{3.10}$$

Let $G = \gcd(wA_g, uA_f, vA_h)$,

$$r = \frac{uA_f}{G}, s = \frac{vA_h}{G}, t = \frac{wA_g}{G} \tag{3.11}$$

and we rewrite (3.10) as

$$tX_g^\ell = rX_f^\ell + sX_h^\ell. \tag{3.12}$$

Note that $\gcd(rX_f^\ell, sX_h^\ell) = 1$.

From now on, we assume explicit $abc-$conjecture. Given $\epsilon > 0$, let $N(rstX_fX_gX_h) \ge N_\epsilon$ which we assume from now on till the expression (3.18). By Theorem 1.2, we obtain

$$tX_g^\ell < \kappa_\epsilon N(rstX_fX_gX_h)^{1+\epsilon} \tag{3.13}$$

i.e.,

$$X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}(X_fX_gX_h)^{1+\epsilon}}{t}. \tag{3.14}$$

Here $N_\epsilon = \kappa_\epsilon = 1$ if $\epsilon > 3/4$. For $\epsilon = 3/4$, by abuse of notation, we will be taking either $N_\epsilon = 1, \kappa_\epsilon = 1$ or $N_\epsilon = e^{37.1101}, \kappa_\epsilon \le 6/(5\sqrt{28\pi})$ if $N(rstX_fX_gX_h) \ge N_{3/4}$ and we will be using it without reference. We will be taking $\epsilon = 3/4$ for $\ell > 7$ and $\epsilon \in \{5/12, 1/3\}$ for $\ell = 7$. We have from (3.13) that

$$rst(X_fX_gX_h)^\ell < \kappa_\epsilon^3 N(rst)^{3(1+\epsilon)}(X_fX_gX_h)^{3(1+\epsilon)}.$$

Putting $X^3 = X_f X_g X_h$, we obtain

$$X^{\ell-3(1+\epsilon)} < \kappa_\epsilon N(rst)^{\frac{2}{3}+\epsilon} = \kappa_\epsilon N(\frac{uvwA_fA_gA_h}{G^3})^{\frac{2}{3}+\epsilon}. \tag{3.15}$$

Again from (3.12), we have

$$rs(X_fX_h)^\ell \le \left(\frac{rX_f^\ell + sX_h^\ell}{2}\right)^2 = \frac{t^2X_g^{2\ell}}{4}$$

implying

$$X_fX_hX_g \le \left(\frac{t^2}{4rs}\right)^{\frac{1}{\ell}} X_g^3 = \left(\frac{w^2A_g^2}{4uvA_fA_h}\right)^{\frac{1}{\ell}} X_g^3.$$

Therefore we have from (3.14) that

$$X_g^\ell < \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{t}\left(\frac{t^2}{4rs}\right)^{\frac{1+\epsilon}{\ell}} = \kappa_\epsilon \frac{N(rst)^{1+\epsilon}X_g^{3+3\epsilon}}{(4rst)^{\frac{1+\epsilon}{\ell}}t^{1-\frac{3(1+\epsilon)}{\ell}}} \tag{3.16}$$

i.e.,

$$X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon \frac{N(rst)^{(1+\epsilon)(1-\frac{1}{\ell})}}{4^{\frac{1+\epsilon}{\ell}}t^{1-\frac{3(1+\epsilon)}{\ell}}} \le \frac{N(rs)^{(1+\epsilon)(1-\frac{1}{\ell})}N(t)^{\epsilon+\frac{2(1+\epsilon)}{\ell}}}{4^{\frac{1+\epsilon}{\ell}}}. \tag{3.17}$$

We also have from (3.17) that

$$X_g^{\ell-3(1+\epsilon)} < \kappa_\epsilon \frac{N(\frac{uvA_fA_h}{G^2})^{(1+\epsilon)(1-\frac{1}{\ell})}N(\frac{wA_g}{G})^{\epsilon+\frac{2(1+\epsilon)}{\ell}}}{4^{\frac{1+\epsilon}{\ell}}}. \tag{3.18}$$

Lemma 3.3. *Let* $\ell \ge 11$. *Let* $S_0 = \{A_0, A_1, \ldots, A_{k-1}\} = \{B_0, B_1, \ldots, B_{k-1}\}$ *with* $B_0 \le B_1 \le \ldots \le B_{k-1}$. *Then*

$$B_0 \le B_1 < B_2 \ldots < B_{k-1}.$$

In particular $|S_0| \ge k-1$.

Proof. Suppose there exists $0 \le f < g < h < k$ with $\{f, g, h\} = \{i_1, i_2, i_3\}$ and

$$A_{i_1} = A_{i_2} = A \text{ and } A_{i_3} \le A.$$

By (3.10) and (3.11), we see that $\max(A_f, A_g, A_h) \le G$ and therefore $r \le u < k, s \le v < k$ and $t \le w < k$. Since $X_g > k$, we get from the first inequality of (3.17) with $\epsilon = 3/4, N_\epsilon = \kappa_\epsilon = 1$ that

$$k^{\ell-3(1+\epsilon)} < (rs)^{(1+\epsilon)(1-\frac{1}{\ell})}t^{\epsilon+\frac{2(1+\epsilon)}{\ell}} < k^{2+3\epsilon}$$

implying $\ell < 5 + 6\epsilon = 5 + 9/2$. This is a contradiction since $\ell \ge 11$. Therefore either A_i's are distinct or if $A_i = A_j = A$, then $A_m > A$ for $m \notin \{i, j\}$ implying the assertion. $\square$

As a consequence, we have

Corollary 3.2. *Let d be even and $\ell \geq 11$. Then $k \leq 14$.*

Proof. Let d be even and $\ell \geq 11$. Then we get from (3.6) with $S = S_0$ that

$$\prod_{A_i \in S'} A_i \leq (k-1)! 2^{-\mathrm{ord}_2((k-1)!)} = \prod_{2i+1 \leq k-1} (2i+1).$$

Observe that the right hand side of the above inequality is the product of all positive odd numbers less than k. On the other hand, since $\gcd(n, d) = 1$, we see that all A_i's are odd and $|S'| \geq |S_0| - \pi(k) \geq k - 1 - \pi(k)$ by Lemma 3.3. Hence

$$\prod_{A_i \in S'} A_i \geq \prod_{i=1}^{k-1-\pi(k)} (2i-1).$$

Observe here again that the right hand side of the above inequality is the product of first positive $k-1-\pi(k)$ odd numbers. Hence we get a contradiction if $2(k-1-\pi(k)) - 1 > k-1$. Assume $k \leq 2 + 2\pi(k)$. By Lemma 2.1 (i), we get $1 \leq 2/k + (2/\log k)(1 + 1.2762/\log k)$ which is not possible for $k \geq 30$. By using exact values of $\pi(k)$, we check that $k \leq 2 + 2\pi(k)$ is not possible for $15 \leq k < 30$. Hence the assertion. □

Lemma 3.4. *Let $\ell \geq 11$. Then $k < 400$.*

Proof. Assume that $k \geq 400$. By Corollary 3.2, we may suppose that d is odd. Further by Corollary 3.1, there exists $f < g < h$ with $h - f \leq 8$ and $\max(A_f, A_g, A_h) \leq 4k$. Since $n + (k-1)d > k^\ell$, we observe that $X_f > k, X_g > k, X_h > k$ implying $X > k$. First assume that $N = N(rstX_fX_gX_h) < e^{37.12}$. Then taking $\epsilon = 3/4$, $N_\epsilon = 1$ in (3.13), we get $400^{11} \leq k^{11} \leq tX_g^\ell < N^{1+3/4} \leq e^{37.12(1+3/4)}$ which is a contradiction. Hence we may suppose that $N \geq e^{37.12} \geq N_{3/4}$.

Note that we have $u + v = w \leq h - f \leq 8$. We observe that uvw is even. If $A_fA_gA_h$ is odd, then $h-f, g-f, h-g$ are all even implying $1 \leq u, v, w \leq 4$ or $N(uvw) \leq 6$ giving $N(uvwA_fA_gA_h) \leq 6A_fA_gA_h$. Again if $A_fA_gA_h$ is even, then $N(uvwA_fA_gA_h) \leq N((uvw)')A_fA_gA_h \leq 35A_fA_gA_h$ where $(uvw)'$ is the odd part of uvw and $N((uvw)') \leq 35$. Observe that $N((uvw)')$ is obtained when $w = 7, u = 2, v = 5$ or $w = 7, u = 5, v = 2$. Thus we always have $N(uvwA_fA_gA_h) \leq 35A_fA_gA_h \leq 35 \cdot (4k)^3$ since $\max(A_f, A_g, A_h) \leq 4k$. Therefore taking $\epsilon = 3/4$ in (3.15), we obtain using $\ell \geq 11$ and $X > k$ that

$$k^{11-3(1+\frac{3}{4})} < \frac{6}{5\sqrt{28\pi}} 35^{\frac{2}{3}+\frac{3}{4}} (4k)^{3(\frac{2}{3}+\frac{3}{4})}.$$

This is a contradiction since $k \geq 400$. Hence the assertion. □

4. Proof of Theorem 1.3 for $4 \leq k < 400$

We assume that $\ell \geq 11$. It follows from the result of Saradha and Shorey [SaSh05, Theorem 1] that $d > 10^{15}$. Hence we may suppose that $d > 10^{15}$ in this section.

Lemma 4.5. *Let* $r_k = \lfloor k+1-\pi(k) - (\sum_{i\le k}\log i)/(15\log 10)\rfloor$ *and*

$$I(k) = \{i \in [1,k] : P(n+id) > k\}.$$

Then $|I(k)| \ge r_k$.

Proof. Suppose not. Then $|I(k)| \le r_k - 1$. Let

$$I'(k) = \{i \in [1,k] : P(n+id) \le k\} = \{i \in [1,k] : n+id = A_i\}.$$

We have $A_i = n+id \ge (n+d)$ for $i \in I'(k)$. Let $S = \{A_i : i \in I'(k)\}$. Then $|S| \ge k+1-r_k$. From (3.6), we get

$$(k-1)! \ge \prod_{A_i \in S'} A_i \ge (n+d)^{|S'|} > d^{k+1-r_k-\pi(k)}.$$

Since $d > 10^{15}$, we get

$$k+1-\pi(k) - \frac{\sum_{i\le k}\log i}{15\log 10} < r_k = \lfloor k+1-\pi(k) - \frac{\sum_{i\le k}\log i}{15\log 10}\rfloor.$$

This is a contradiction. □

Here are some values of (k, r_k).

k	7	11	13	17	18	28	30	36
r_k	3	6	7	10	10	18	18	23

We give the strategy here. Let $I_k = [0, k-1] \cap \mathbb{Z}$ and a_0, b_0, z_0 be given. Let obtain a subset $I_0 \subseteq I_k$ with the following properties:

1. $|I_0| \ge z_0 \ge 3$.
2. $P(A_i) \le a_0$ for $i \in I_0$.
3. $I_0 \subseteq [j_0, j_0+b_0-1]$ for some j_0.
4. $X_0 = \max_{i\in I_0}\{X_i\} > k$ and let $i_0 \in I_0$ be such that $X_0 = X_{i_0}$.

For any $i, j \in I_0$, taking $\{f,g,h\} = \{i,j,i_0\}$, let $N = N(rstX_fX_gX_h)$. Observe that $X_0 \ge p_{\pi(k)+1}$ and further for any $f,g,h \in I_0$, we have $N(uvw) \le \prod_{p\le b_0-1} p$ and $N(A_fA_gA_h) \le \prod_{p\le a_0} p$. We will always take $\epsilon = 3/4, N_\epsilon = 1$ so that $\kappa_\epsilon = 1$ in (3.13) to (3.18).

Case I: Suppose there exists $i,j \in I_0$ such that $X_i = X_j = 1$. Taking $\{f,g,h\} = \{i,j,i_0\}$ and $\epsilon = 3/4$, we obtain from (3.14) and $\ell \ge 11$ that

$$p_{\pi(k)+1}^{\frac{37}{7}} \le X_0^{\frac{\ell}{1+\frac{3}{4}}-1} < N(uvwA_fA_gA_h) \le \prod_{p\le \max\{a_0, b_0-1\}} p. \tag{4.19}$$

Case II: There is at most one $i \in I_0$ such that $X_i = 1$. Then $|\{i \in I_0 : X_i > k\}| \ge z_0 - 1$. We take a_1, b_1, z_1 and find a subset $U_0 \subset I_0$ with the following properties:

1. $|U_0| \geq z_1 \geq 3$, $z_0/2 \leq z_1 \leq z_0$.
2. $P(A_i) \leq a_1$ for $i \in U_0$.
3. $U_0 \subseteq [i, i+b_1-1]$ for some i.

Let $X_1 = \max_{i\in U_0}\{X_i\} \geq p_{\pi(k)+z_1-1}$ and i_1 be such that $X_{i_1} = X_1$. Taking $\{f,g,h\} = \{i,j,i_1\}$ for some $i,j \in U_0$ and $\epsilon = 3/4$, we obtain from (3.17) and $\ell \geq 11$ that

$$p_{\pi(k)+z_1-1}^{\frac{23}{7}} \leq X_0^{\frac{\ell}{1+\frac{3}{4}}-3} < N(uvwA_fA_gA_h) \leq \prod_{p\leq\max\{a_1,b_1-1\}} p \tag{4.20}$$

since $\ell \geq 11$. One choice is $(U_0, a_1, b_1, z_1) = (I_0, a_0, b_0, z_0)$. We state the other choice.

Let $b' = \max(a_0, b_0-1)$. For each $b_0/2-1 < p \leq b'-1$, we remove those $i \in I_0$ such that $p|(n+id)$. There are at most $2(\pi(b'-1) - \pi(b_0/2-1))$ such i. Let I_0' be obtained from I_0 after deleting those i's. Then $|I_0'| \geq z_0 - 2(\pi(b'-1) - \pi(b_0/2-1))$. Let

$$U_1 = I_0' \cap [j_0, j_0 + \frac{b_0}{2} - 1] \text{ and } U_1 = I_0' \cap [j_0 + \frac{b_0}{2}, j_0 + b_0 - 1].$$

Let $U_0 \in \{U_1, U_2\}$ for which $|U_0| = \max(|U_1|, |U_2|)$ and choose one of them if $|U_1| = |U_2|$. Then $|U_0| \geq \lceil z_0/2 \rceil - \pi(b'-1) + \pi(b_0/2-1) = z_1$. Further $P(A_i) \leq b_0/2 - 1 = a_1$ and $b_1 = b_0/2$. Further $X_1 = \max_{i\in U_0}\{X_i\} \geq p_{\pi(k)+z_1-1}$. Our choice of z_0, a_0, b_0 will imply that $z_1 \geq 3$.

4.1. $k \in \{4, 5, 7, 11\}$

We take $I_0 = U_0 = I_k, a_i = b_i = z_i = k$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \leq \prod_{p\leq k} p$. And the assertion follows since both (4.19) and (4.20) are contradicted.

4.2. $k \in \{13, 17, 19, 23\}$

We take $I_0 = \{i \in [1, 11] : p \nmid (n+id) \text{ for } 13 \leq p \leq 23\}$. Then by $r_{11} = 6$ and Lemma 4.5 with $k = 11$, we see that $|I_0| \geq z_0 = 11 - 4 > 11 - r_{11} \geq 11 - |I(11)|$. Therefore there exist an $i \in I_0 \cap I_{11}$ and hence $X_i > 23$. We take $U_0 = I_0$, $a_i = b_i = 11, z_1 = z_0$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \leq \prod_{p\leq 11} p$. And the assertion follows since both (4.19) and (4.20) are contradicted.

4.3. $29 \leq k \leq 47$

We take $I_0 = \{i \in [1, 17] : p \nmid (n+id) \text{ for } 17 \leq p \leq k\}$. Then by $r_{17} = 10$ and Lemma 4.5 with $k = 17$, we have $|I_0| \geq z_0 = 17 - (\pi(k) - \pi(13)) = 23 - \pi(k) \geq 23 - \pi(47) = 8 > 17 - r_{17} \geq 17 - |I(17)|$ implying that there exists $i \in I_0$ with $X_i > k$. We take $a_i = 13, b_i = 17, z_i = 23 - \pi(k)$ for $i \in \{0, 1\}$ and hence $N(uvwA_fA_gA_h) \leq \prod_{p\leq 13} p$. And the assertion follows since both (4.19) and (4.20) are contradicted.

4.4. $k \geq 53$

Given m and q such that $mq < k$, we consider the q intervals

$$I_j = [(j-1)m+1, jm] \cap \mathbb{Z} \text{ for } 1 \leq j \leq q$$

and let $I' = \cup_{j=1}^{q} I_j$ and $I^{''} = \{i \in I' : m \leq P(A_i) \leq k\}$. There is at most one $i \in I'$ such that $mq - 1 < P(A_i) \leq k$ and for each $2 \leq j \leq q$, there are at most j number of $i \in I'$ such that $(mq-1)/j < P(A_i) \leq (mq-1)/(j-1)$. Therefore

$$\begin{aligned} |I^{''}| &\leq \pi(k) - \pi(mq-1) + \sum_{j=2}^{q} j \left(\pi(\frac{mq-1}{j-1}) - \pi(\frac{mq-1}{j}) \right) \\ &= \pi(k) + \sum_{j=1}^{q-1} \pi(\frac{mq-1}{j}) - q\pi(m-1) =: T(k,m,q). \end{aligned}$$

Hence there is at least one j such that $|I_j \cap I^{''}| \leq \lfloor T(k,m,q)/q \rfloor$. We will choose q such that $\lfloor T(k,m,q)/q \rfloor < r_m$. Let $I_0 = I_j \setminus I^{''}$ and let j_0 be such that $I_0 \subseteq [(j_0 - 1)m + 1, j_0 m]$. Then $p|(n+id)$ imply $p < m$ or $p > k$ whenever $i \in I_0$. Further $|I_0| \geq z_0 = m - \lfloor T(k,m,q)/q \rfloor$. Since $\lfloor T(k,m,q)/q \rfloor < r_m$, we get from Lemma 4.5 with $k = m$ and $n = (j_0 - 1)m$ that there is an $i \in I_0$ with $X_i > k$. Further $P(A_i) < m$ for all $i \in I_0$. Here are the choices of m and q.

$k \in$	$[53, 89)$	$[89, 179)$	$[179, 239)$	$[239, 367)$	$[367, 433)$
(m, q)	$(17, 3)$	$(28, 3)$	$(36, 5)$	$(36, 6)$	$(36, 10)$

We have $a_0 = m-1, b_0 = m$ and $z_0 = m - \lfloor T(k,m,q)/q \rfloor$ and we check that $z_0 \geq 3$. The Subsection 4.3($29 \leq k \leq 47$) is in fact obtained by considering $m = 17, q = 1$. Now we consider Cases I and II and try to get contradiction in both (4.19) and (4.20). For these choices of (m,q), we find that the Cases I are contradicted. Further taking $U_0 = I_0, a_1 = a_0 = m-1, b_1 = b_0 = m, z_1 = z_0$, we find that Case II is also contradicted for $53 \leq k < 89$. Thus the assertion follows in the case $53 \leq k < 89$. So, we consider $k \geq 89$ and try to contradict Cases II. Recall that we have $X_i > k$ for all but at most one $i \in I_0$. Write $I_0 = U_1 \cup U_2$ where $U_1 = I_0 \cap [(j_0-1)m+1, (j_0-1)m + m/2]$ and $U_2 = I_0 \cap [(j_0-1)m + m/2 + 1, j_0 m]$. Let $U_0' = U_1$ or $U_0' = U_2$ according as $|U_1| \geq z_0/2$ or $|U_2| \geq z_0/2$, respectively. Let $U_0 = \{i \in U_0' : p \nmid A_i \text{ for } m/2 \leq p < m\}$. Then $|U_0| \geq z_1 := z_0/2 - (\pi(m-1) - \pi(m/2)) = (m - \lfloor T(k,m,q)/q \rfloor)/2 - (\pi(m-1) - \pi(m/2)) \geq 3$. Further $p|(n+id)$ with $i \in U_0$ imply $p < m/2$ or $p > k$. Now we have Case II with $a_1 = m/2 - 1, b_1 = m/2$ and find that (4.20) is contradicted. Hence the assertion.

5. $\ell = 7$

Let $\ell = 7$. Assume that $k \geq \exp(13006.2)$. Taking $\alpha = 3$, $\beta = 1/15 + 2/9$ in Lemma 3.2, we get

$$|S_1(3)| = \{i \in [0, k-1] : A_i \leq 3k\}| > k \left(\frac{1}{15} + \frac{2}{9} \right).$$

For i's such that $A_i \in S_1(3)$, we have $X_i > k$ and we arrange these X_i's in increasing order as $X_{i_1} < X_{i_2} < \cdots <$. Then $X_{i_j} \ge p_{\pi(k)+j}$. Consider the set $J_0 = \{i : X_i \ge p_{\pi(k)+\lfloor k/15 \rfloor -2}\}$. We have

$$|J_0| > k\left(\frac{1}{15}+\frac{2}{9}\right) - \frac{k}{15} + 2 \ge 2\left(\lfloor \frac{k-1}{9} \rfloor + 1\right).$$

Hence there are $f, g, h \in J_0$, $f < g < h$ such that $h - f \le 8$. Also $A_i \le 3k$ and $X = (X_f X_g X_h)^{1/3} \ge p_{\pi(k)+\lfloor k/15 \rfloor -2}$.

First assume that $N = N(rstX_f X_g X_g) \ge \exp(63727) \ge N_{1/3}$. Observe that $uvw \le 70$ since $2 \le u + v = w \le 8$, obtained at $2 + 5 = 7$. Taking $\epsilon = 1/3$, we obtain from (3.15) and $\max(A_f, A_g, A_h) \le 3k$ that

$$p^3_{\pi(k)+\lfloor \frac{k}{15} \rfloor -2} < \frac{5}{6\sqrt{2\pi \cdot 6458}} N(uvwA_f A_g A_h) \le \frac{5 \cdot 70 \cdot (3k)^3}{6\sqrt{12920\pi}}.$$

Since $\pi(k) > 2$ we have $\pi(k) + \lfloor k/15 \rfloor - 2 > \frac{k}{15}$ and hence $p_{\pi(k)+\lfloor k/15 \rfloor -2} > (k/15)\log(k/15)$ by Lemma 2.1 (ii). Therefore

$$\left(\log \frac{k}{15}\right)^3 < \frac{350 \cdot (3 \cdot 15)^3}{6\sqrt{12920\pi}} \text{ or } k < 15 \cdot \exp\left(45 \cdot \left(\frac{350}{6\sqrt{12920\pi}}\right)^{\frac{1}{3}}\right)$$

which is a contradiction since $k \ge exp(13006.2)$.

Therefore we have $N = N(rstX_f X_g X_h) < exp(63727)$. We may also assume that $N > \exp(3895)$ otherwise taking $\epsilon = 3/4$ in (3.13), we get $k^7 < X_g^7 < N^{1+3/4} \le \exp(3895 \cdot 7/4)$ or $k < \exp(3895/4)$ which is a contradiction. Now we take $\epsilon = 5/12$ in (3.13) to get $k^7 < X_g^7 < N^{1+5/12} \le \exp(64266 \cdot 17/12)$ or $k < \exp(13006.2)$. Hence the assertion.

6. abc-conjecture implies Erdős conjecture: Proof of Theorem 1.1

Assume (3.5). We show that k is bounded by a computable absolute constant. Let $k \ge k_0$ where k_0 is a sufficiently large computable absolute constant. Let $\epsilon > 0$. Let $c_1, c_2, \cdots$ be positive computable constants depending only on ϵ. Let $I = \{i_p : p \le k \text{ and } p \nmid d\}$ where i_p be as defined in beginning of Section 3. Let $S = \{A_i : \lfloor k/2 \rfloor \le i < k \text{ or } i \in I\}$ and

$$\Phi = \prod_{\substack{i \ge \lfloor \frac{k}{2} \rfloor \\ i \notin I}} A_i.$$

By Sylvester-Erdős inequality(see [ErSe75, Lemma 2] for example), we get

$$\begin{aligned}\mathrm{ord}_p(\Phi) &\le \mathrm{ord}_p\left(\prod_{\substack{i\ge\lfloor\frac{k}{2}\rfloor\\ i\notin I}}(i-i_p)\right)\\ &\le \begin{cases}\mathrm{ord}_p\left((k-\lfloor\frac{k}{2}\rfloor-1-(i_p-\lfloor\frac{k}{2}\rfloor))!(i_p-\lfloor\frac{k}{2}\rfloor)!\right) & \text{if } i_p\ge\lfloor\frac{k}{2}\rfloor,\\ \mathrm{ord}_p\left(\binom{k-1-i_p}{k-\lfloor\frac{k}{2}\rfloor}(k-\lfloor\frac{k}{2}\rfloor)!\right) & \text{otherwise.}\end{cases}\end{aligned}$$

Since $\mathrm{ord}_p(r!s!)\le\mathrm{ord}_p((r+s)!)$ and $k-\lfloor k/2\rfloor=\lfloor(k+1)/2\rfloor$, we see that

$$p^{\mathrm{ord}_p(\Phi)}\le p^{\mathrm{ord}_p(\binom{k-1-i_p}{\lfloor\frac{k+1}{2}\rfloor})}p^{\mathrm{ord}_p(\lfloor\frac{k+1}{2}\rfloor!)}.$$

Using the fact that $p^{\mathrm{ord}_p(\binom{x}{k})}\le x$ for any $x\ge k$, we get

$$\Phi\le(k-1)^{\pi_d(k)}(\lfloor\frac{k+1}{2}\rfloor)!\le k^{\frac{k}{2}}e^{c_1k}$$

by using Lemma 2.1 $(i),(v)$.

Let D be a fixed positive integer and let

$$J=\left\{\frac{k-1}{2D}\le j\le\frac{k-1}{D}-1:\{Dj+1,Dj+2,\cdots,Dj+D\}\cap I=\phi\right\}.$$

We shall choose $D=20$. Let $j,j'\in J$ be such that $j\ne j'$. Then $Dj+i\ne Dj'+i'$ for $1\le i,i'\le D$ otherwise $D(j-j')=(i-i')$ and $|i'-i|<D$. Further we also see that $\lfloor k/2\rfloor\le Dj+i\le k-1$ for $1\le i\le D$ and consequently $|J|\ge(k-1)/2D-\pi(k)$. For each $j\in J$, let $\Phi_j=\prod_{i=1}^{D}A_{Dj+i}$. Then $\prod_{j\in J}\Phi_j$ divides Φ implying

$$\prod_{j\in J}\Phi_j\le\Phi\le k^{\frac{k}{2}}e^{c_1k}.$$

Thus there exists $j_0\in J$ such that

$$\Phi_{j_0}\le\left(k^{\frac{k}{2}}e^{c_1k}\right)^{\frac{1}{|J|}}\le\left(k^{\frac{k}{2}}e^{c_1k}\right)^{\frac{1}{\frac{k-1}{2D}-\pi(k)}}\le c_2^Dk^D.$$

Let

$$H:=\prod_{i=1}^{D}(n+(Dj_0+i)d).$$

Since $A_{Dj_0+i}X^{\ell}_{Dj_0+i} \leq n + (k-1)d$, we have $X_{Dj_0+i} \leq ((n+(k-1)d)/A_{Dj_0+i})^{1/\ell}$. Thus

$$\prod_{\substack{p|H \\ p>k}} p = \prod_{i=1}^{D} X_{Dj_0+i} \leq (n+(k-1)d)^{\frac{D}{\ell}} (\Phi_{j_0})^{-\frac{1}{\ell}}$$

Therefore

$$\begin{aligned}\prod_{p|H} p = \left(\prod_{\substack{p|H \\ p\leq k}} p\right)\left(\prod_{\substack{p|H \\ p>k}} p\right) &\leq \Phi_{j_0}(n+(k-1)d)^{\frac{D}{\ell}}(\Phi_{j_0})^{-\frac{1}{\ell}} \\ &\leq c_2^{D(1-\frac{1}{\ell})} k^{D(1-\frac{1}{\ell})} (n+(k-1)d)^{\frac{D}{\ell}}.\end{aligned}$$

On the other hand, we have $H = F(n + Dj_0 d, d)$ where

$$F(x,y) = \prod_{i=1}^{D}(x+iy)$$

is a binary form in x and y of degree D such that F has distinct linear factors. From Conjecture 1.4, we have

$$\prod_{p|H} p \geq c_3(n+Dj_0 d)^{D-2-\epsilon}.$$

Comparing the lower and upper bounds of $\prod_{p|H} p$ and using $n+Dj_0 d > (n+(k-1)d)/2$, we get

$$k > c_4(n+(k-1)d)^{1-\frac{2+\epsilon}{D(1-\frac{1}{\ell})}}.$$

We now use $n + (k-1)d > k^{\ell}$ to derive that

$$c_5 > k^{\ell(1-\frac{2+\epsilon}{D(1-\frac{1}{\ell})})-1}.$$

Taking $\epsilon = 1/2$ and putting $D = 20$, we get

$$c_6 > k^{\ell-1-\frac{\ell^2}{8(\ell-1)}} \geq k^{\frac{1}{2}}$$

since $\ell \geq 2$. This is a contradiction since $k \geq k_0$ and k_0 is sufficiently large. □

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References

[Bak94] A. Baker, *Experiments on the abc-conjecture*, Publ. Math. Debrecen **65**(2004), 253–260.

[Dus99a] P. Dusart, *The k^{th} prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$*, Math. Comp. **68** (1999), 411–415

[Dus99b] P. Dusart, *Inégalitiés explicites pour $\psi(X), \theta(X), \pi(X)$ et les nombres premiers*, C. R. Math. Acad. Sci. Soc. R. Can. **21(1)**(1999), 53-59.

[Elk91] N. Elkies, *ABC implies Mordell*, Int. Math. Res. Not. IMRN **7** (1991), 99–109.

[ErSe75] P. Erdős and J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292-301.

[GrTu02] A. Granville and T. J. Tucker, *It's as easy as abc*, Notices Amer. Math. Soc. **49**(2002), 1224-31.

[Lai04] S. Laishram, *Topics in Diophantine equations*, M.Sc. Thesis, TIFR/Mumbai University, 2004, online at http: //www.isid.ac.in/∼shanta/MScThesis.pdf.

[LaSh12] S. Laishram and T. N. Shorey, *Baker's Explicit abc-Conjecture and applications*, Acta Arith. **155** (2012), 419–429.

[Rob55] H. Robbins, *A remark on Stirling's formula*, Amer. Math. Monthly **62** (1955), 26-29.

[Rob83] G. Robin, *Estimation de la fonction de Tchebychef θ sur le k-ieme nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n*, Acta Arith. **42** (1983), 367–389.

[Sar12] N. Saradha, *Applications of Explicit abc-Conjecture on two Diophantine Equatiosn*, Acta Arith. **151** (2012), 401–419.

[SaSh05] N. Saradha and T. N. Shorey, *Contributions towards a conjecture of Erdos on perfect powers in arithmetic progressions*, Compos. Math. **141** (2005), 541-560.

[Sho06] T. N. Shorey, *Diophantine approximations, Diophantine equations, Transcendence and Applications*, Indian J. Pure Appl. Math. **37** (2006), 9–39.

[Sho02b] T.N. Shorey, *Powers in arithmetic progression* (II), Analytic Number Theory, RIMS Kokyuroku (2002), Kyoto University.

[Sho99] T. N. Shorey, *Exponential diophantine equations involving products of consecutive integers and related equations*, Number Theory ed. R.P. Bambah, V.C. Dumir and R.J. Hans-Gill, Hindustan Book Agency (1999), 463-495.

[ShTi90] T. N. Shorey and R. Tijdeman, *On the greatest prime factor of an arithmetical progression*, A tribute to Paul Erdős, ed. by A. Baker, B. Bollobás and A. Hajnal, Cambridge University Press (1990), 385–389.

[Tij88] R. Tijdeman, *Diophantine equations and diophantine approximations*, Number Theory and Applications, Kluwer (1988), 215–243.