IRREDUCIBILITY OF GENERALIZED LAGUERRE POLYNOMIALS $L_n^{(\frac{1}{2}+u)}(x)$ WITH INTEGER u

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ABSTRACT. Generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ are classical orthogonal polynomial sequences that plays an important role in various branches of analysis and mathematical physics. Schur (1929) was the first to study the algebraic properties of these polynomials by proving that $L_n^{(\alpha)}(x)$ where $\alpha \in \{0, 1, -n-1\}$ are irreducible. For $\alpha = u + \frac{1}{2}$ with integer u satisfying $1 \le u \le 45$, we prove that $L_n^{(\alpha)}(x)$ and $L_n^{(\alpha)}(x^2)$ of degrees n and $2n$, respectively, are irreducible except when $(u, n) = (10, 3)$ where we give a factorization. The cases $u = -1, 0$ are due to Schur. Further we consider more general polynomials $G_\alpha(x)$ and $G_\alpha(x^2)$ of degrees n and $2n$, respectively, and prove that they are either irreducible or have a factor of degree in $\{1, n-1\}$, $\{1, 2, 2n-2, 2n-1\}$, respectively, except for an explicitly given finite set of pairs (u, n). We also show that these exceptional pairs other than one for $G_\alpha(x)$ and six for $G_\alpha(x^2)$ are necessary. Further for a general $u > 0$ we give an upper bound for the degree of factor of $G_\alpha(x)$ and $G_\alpha(x^2)$ in terms of u.

1. INTRODUCTION

For positive integer n and real number α, the generalized Laguerre polynomials are given by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\dots(j+1+\alpha)}{(n-j)!j!}(-x)^j$$

and $L_n^{(0)}(x)$ is called the Laguerre polynomial. If $\alpha = -k$ with k a positive integer, then

$$L_n^{(-k)}(x) = (-x)^k \frac{L_{n-k}^{(k)}(x)}{n(n-1)\cdots(n-k+1)}, \tag{1}$$

see [Sz75, formula 5.2.1]. We shall restrict ourselves to the case that α is a rational number. The generalized Laguerre polynomial satisfies second order linear differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0, \ y = L_n^{(\alpha)}(x)$$

and the difference equation

$$L_n^{(\alpha)}(x) - L_n^{(\alpha-1)}(x) = L_{n-1}^{(\alpha)}(x).$$

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They have been extensively studied in various branches of mathematics and mathematical physics. Schur [Sch31], [Sch73] was the first to establish interesting and important algebraic properties of these polynomials. His work, along with the work of Gow [Gow89] and Filaseta, Kidd, Trifonov [FilKiTr] on the irreducibilty of $L_n^{(\alpha)}(x)$ for $n \equiv 2 \pmod 4$ with $n > 2$, settled the inverse Galois problem explicitly that for every positive integer $n > 1$, there exists an explicit Laguerre polynomial of degree n whose Galois group is the alternating group A_n. We consider whether $L_n^{(\alpha)}(x)$ is irreducible over $\mathbb{Q}$. By irreducibility of a polynomial, we shall always mean its irreducibility over $\mathbb{Q}$. Schur [Sch31], [Sch73] proved that $L_n^{(0)}(x)$ and $L_n^{(1)}(x)$ are irreducible for all n. Filaseta and Lam [FilLa02] proved that for a fixed rational number α which is not a negative integer, $L_n^{(\alpha)}(x)$ is irreducible for all but finitely many n. We observe from (1) that $L_n^{(-k)}(x)$ is reducible whenever k is a positive integer. The irreducibility of generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ for an arbitrary given rational number α with denominator greater than 4 is not known. We give an account of the known results on irreducibilty of $L_n^{(\alpha)}(x)$ and its generalizations when the denominator of α is at most 4. Let α be a rational number with denominator equal to $d \geq 1$ written in its reduced form. Then α can be uniquely written as

$$\alpha = \alpha(u) = u + \frac{a}{d} \tag{2}$$

where $u, a \in \mathbb{Z}$ with $a = 0$ if $d = 1$ and $1 \leq a < d$, $\gcd(a, d) = 1$ if $d > 1$. Thus $\alpha = u$ if $d = 1$.

Let $d = 1$. Laishram and Shorey [LaSh11] showed that for integers α with $0 \leq \alpha \leq 50$, $L_n^{(\alpha)}(x)$ is irreducible for all n except for $n = 2, \alpha \in \{2, 7, 14, 23, 34, 47\}$ and $n = 4, \alpha \in \{5, 23\}$ where it has a linear factor. The above result with $0 \leq \alpha \leq 10$ was already proved by Filaseta, Finch and Leidy [FilFiLe08]. Further it has been established that $L_n^{(-1-n-r)}(x)$ with $0 \leq r \leq 22$ is irreducible. The case $r = 0$ was proved by Schur [Sch73] and we observe that this is the truncated exponential series. A new proof of this case was given by Coleman [Col87] depending on Newton polygons. This initiated a new method and it was considerably refined by Filaseta [Fil94]. The case $r = 2$ was proved by Sell [Sell04], $r = 1$ and $3 \leq r \leq 8$ by Hajir [Haj95], [Haj09] and $9 \leq r \leq 22$ by Nair and Shorey [NaSh15]. The case $r = n$ was established by Filaseta and Trifonov [FilTr02] and it confirms immediately a conjecture of Grosswald that the Bessel polynomials are irreducible. Hajir [Haj09] conjectured that $L_n^{(-1-n-r)}(x)$ with $r \geq 0$ is irreducible for all n. He proved that for a given r there exists an explicit number $B(r)$ such that for $n > B(r)$, $L_n^{(-1-n-r)}(x)$ is irreducible and the value of $B(r)$ is considerably improved in [NaSh15].

Let $d = 2$. Then $\alpha = u + \frac{1}{2}$ and $L_n^{(\alpha)}(x)$ with $u \in \{-1, 0\}$ are connected with Hermite polynomials given by

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2) \text{ and } H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{(\frac{1}{2})}(x^2).$$

Schur [Sch31], [Sch73] proved that $L_n^{(-\frac{1}{2})}(x^2)$ and $L_n^{(\frac{1}{2})}(x^2)$ are irreducible implying the irreducibility of $H_{2n}(x)$ and $H_{2n+1}(x)/x$. We prove

Theorem 1. *Let* $1 \le u \le 45$ *and* $\alpha = u + \frac{1}{2}$. *The polynomials* $L_n^{(\alpha)}(x^2)$ *are irreducible except when* $(u, n) = (10, 3)$. *In such a case* $L_3^{(\frac{21}{2})}(x^2) = \frac{-1}{48}(2x^2 - 15)(4x^4 - 132x^2 + 1035)$.

Since the irreducibility of $L_n^{(\alpha)}(x^2)$ implies the irreducibility of $L_n^{(\alpha)}(x)$, we derive the following result for $L_n^{(\alpha)}(x)$.

Corollary 1.1. *Let* $1 \le u \le 45$ *and* $\alpha = u + \frac{1}{2}$. *Then* $L_n^{(\alpha)}(x)$ *are irreducible except when* $(u, n) = (10, 3)$. *In such a case* $L_3^{(\frac{21}{2})}(x) = -\frac{1}{48}(2x - 15)(4x^2 - 132x + 1035)$.

Let $d \in \{3, 4\}$. Laishram and Shorey [LaSh15] proved that $L_n^{(\alpha)}(x)$ is irreducible whenever $\alpha \in \{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}\}$.

Now we consider the irreducibility of some extension of generalized Laguerre polynomials. For integers $a_0, a_1, \ldots, a_n$ and α given by (2), let

$$\begin{aligned} G_\alpha(x) = G_n(x; \alpha) &= \sum_{j=0}^{n} a_j (n + \alpha)(n - 1 + \alpha) \cdots (j + 1 + \alpha) d^{n-j} x^j \\ &= \sum_{j=0}^{n} a_j x^j \Big(\prod_{i=j+1}^{n} (a + (u + i)d) \Big). \end{aligned}$$

We observe that

$$G_\alpha(x) = d^n n! L_n^{(\alpha)}\Big(\frac{x}{d}\Big) \text{ if } a_j = (-1)^j \binom{n}{j} \tag{3}$$

and therefore the irreducibility of $G_\alpha(x)$ implies the irreducibility of $L_n^{(\alpha)}(x)$. We have

$$G_\alpha(x^d) = G_n(x^d; \alpha) = \sum_{j=0}^{dn} b_j x^j \quad \text{where} \quad b_j = \begin{cases} a_l \displaystyle\prod_{i=l+1}^{n} (a + (u + i)d) & \text{if } j = dl \\ 0 & \text{otherwise.} \end{cases}$$

We observe that the irreducibility of $G_\alpha(x^d)$ implies the irreducibility of $G_\alpha(x)$. The result of Filaseta and Lam [FilLa02] already stated for $L_n^{(\alpha)}(x)$ is also valid for $G_\alpha(x)$. If $|a_0 a_n| = 1$, Schur [Sch31] proved that $G_\alpha(x)$ is irreducible if $\alpha = 0$ and also if $\alpha = 1$ unless $n + 1$ is a power of 2 where it may have a linear factor or $n = 8$ where it may have a quadratic factor. Further Laishram and Shorey [LaSh11] showed that for $k \ge 2$, $G_\alpha(x)$ with $|a_0 a_n| = 1$ has no factor of degree k when α is an integer satisfying $0 \le \alpha \le 40$ if $k = 2$ and $0 \le \alpha \le 50$ if $k \ge 3$ except for an explicitly given finite set of triples (n, k, α) and we refer to [LaSh11] for a complete list of exceptions. In fact it has no factor of degree ≥ 5 unless $(n, k, \alpha) \in \{(17, 5, 11), (19, 5, 9), (40, 5, 12)\}$. Shorey and Tijdeman [ShTi10] proved the above assertion when $0 \le \alpha \le 30$ if $k \ge 5$ and $0 \le \alpha \le 10$ if $3 \le k \le 4$. Further Laishram and Shorey [LaSh12] proved that $G_\alpha(x)$ with $\alpha \in \{\frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\}$ and $|a_0 a_n| = 1$ is either irreducible or linear polynomial times an irreducible polynomial of degree $n - 1$.

For an integer $|\nu| > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and we put $P(\pm 1) = 1$. Let A be the set of all integers a with $P(a) \leq 2$. Thus

$$A = \{\pm 2^t : t \geq 0, t \in \mathbb{Z}\}.$$

Let $S = \{(1, 121), (8, 59), (8, 114), (9, 4), (9, 113), (9, 163), (9, 554), (15, 23), (15, 107),$ $(16, 106), (20, 102), (21, 101), (26, 155), (26, 287), (30, 92), (36, 86), (43, 1158), (44, 716)\}$. Observe that if a polynomial of degree m has a factor of degree $k < m$, then it has a factor of degree $m - k$. *Therefore given a polynomial of degree m, we always consider a factor of degree k where $1 \leq k \leq \frac{m}{2}$.* We prove

Theorem 2. *Let $1 \leq u \leq 45$ and $\alpha = u + \frac{1}{2}$. Let $a_0, a_n \in A$. Then $G_\alpha(x^2)$ has no factor of degree ≥ 3 except when $(u, n) \in \{(1, 12), (6, 7), (9, 113), (10, 3), (21, 101)\}$ or $(u, n) \in S$ or $(u, n) = (44, 79)$ where it may have a factor of degree 3 or 4 or 6, respectively.*

Since irreducibility of $G_\alpha(x^2)$ implies the irreducibility of $G_\alpha(x)$, the following result for $G_\alpha(x)$ is a straightforward consequence of Theorem 2.

Corollary 1.2. *Let $1 \leq u \leq 45$ and $\alpha = u + \frac{1}{2}$. Let $a_0, a_n \in A$. Then $G_\alpha(x)$ has no factor of degree ≥ 2 except when $(u, n) \in S$ or $(u, n) = (44, 79)$ where it may have a factor of degree 2 or 3, respectively.*

The exceptions $(u, n) \in S$ are necessary as we see that in these cases $G_\alpha(x)$ has a quadratic factor with suitable choice of a_j's. See Section 9 for details. We are not able to find a suitable choice of a_j's in the case $(u, n) = (44, 79)$ for which $G_\alpha(x)$ may have a factor of degree 3 and $(u, n) \in \{(1, 12), (6, 7), (9, 113), (10, 3), (21, 101)\}$ for which $G_\alpha(x^2)$ may have a factor of degree 3. Finch and Saradha [FiSa10] showed that for $1 \leq u \leq 13$, the polynomials $G_\alpha(x)$ with $a_0, a_n \in A$ have no factor of degree ≥ 2 except for $(u, n) \in \{(1, 121), (8, 59), (8, 114), (9, 4), (9, 113), (9, 163), (9, 554)\}$ where it may have either a linear factor or a quadratic factor. We remark that this result is not assumed in the proof of Corollary 1.2.

We follow the method of Coleman- Filaseta based on Newton polygons. For applying this method, we restrict a_0, a_n to $|a_0 a_n| = 1$ in the above results on $G_\alpha(x)$ or $G_\alpha(x^2)$ and more generally to $a_0, a_n \in A$ in Theorem 2. This is because we need that the Newton polygon of $G_\alpha(x)$ lies above the Newton polygon of $G_\alpha(x)$ with a_j's equal to 1 [See Corollary 4.2] and we consider Newton polygons with respect to small primes including 3 in the proof of Theorem 2. The arguments of the proof of Theorem 2 are valid for larger values of u but the bound $u \leq 45$ is close to the optimal in the sense that otherwise tables of Najman [Naj10] on $S_M = \{n \geq 1 : n \text{ odd}, P(n(n + 2)) \leq M\}$ with $M \leq 100$ will not suffice and extending these tables S_M with $M > 100$ is computationally very difficult. Before we turn to state our next result, we give two remarks on Theorem 2.

Remarks (i) For the exceptions given in Theorem 2, we check that $G_\alpha(x)$ with $|a_0 a_n| = 1$ and $a_j = 1$ for $1 \leq j < n$ have no factor of degree ≥ 2 and hence these polynomials have no factor of degree ≥ 2. Thus for $1 \leq u \leq 45$, the polynomials given by $\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1+2(u+i))x^j$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$.

(ii) We check the assertion of Theorem 1 when $(u,n)=(10,3)$. Further we check that $G_\alpha(x^2)$ with $a_j=(-1)^j\binom{n}{j}$ is irreducible for the exceptions other than $(u,n)=(10,3)$ given in Theorem 2. Therefore we derive from Theorem 2 and (3) that it suffices to prove in Theorem 1 that $L_n^{(\alpha)}(x^2)$ has no factor in degree $\{1,2\}$ when $(u,n)\neq(10,3)$.

In the next result, we bound the degree of any factor of $G_\alpha(x^2)$ in terms of u where $\alpha=u+\frac{1}{2}$ and u is any positive integer.

Theorem 3. *Let $\alpha=u+\frac{1}{2}$ with $u\geq 1$ and $(u,n)\notin\{(1,12),(1,121)\}$. Assume that $G_\alpha(x^2)$ with $a_0,a_n\in A$ has a factor of degree l with $3\leq l\leq n$. Then*

(i) $u>\frac{1.35l}{2}-1.2$, *if $l=n$ and n odd*

(ii) $u>\frac{1.35l}{2}-0.5$, *otherwise.*

Thus $u>\frac{1.35l}{2}-1.2$ always in Theorem 3. Further it admits the following consequence for $G_\alpha(x)$.

Corollary 1.3. *Let $\alpha=u+\frac{1}{2}$ with $u\geq 1$ and $(u,n)\neq(1,121)$. Assume that $G_\alpha(x)$ with $a_0,a_n\in A$ has a factor of degree $l\geq 2$. Then $u>1.35l-0.5$.*

As already pointed out after the statement of Corollary 1.2, the assumption $(u,n)\neq(1,121)$ is necessary. A weaker estimate $u>l$ in Corollary 1.3 is proved in [LaSh12, Theorem 2]. The proofs of our theorems depend on the following estimate on the greatest prime factor of product of consecutive odd integers and it is of independent interest. For positive integers $m,\ d$ and $k\geq 2$ with $\gcd(m,d)=1$, we write

$$\Delta(m,d,k)=m(m+d)\cdots(m+d(k-1)).$$

We prove

Theorem 4. *Let $k\geq 2,\ m>2k$ and m is odd. Then*

$$P(\Delta(m,2,k))>\begin{cases}3.5k & \text{if } m\leq 2.5k\\ 4k & \text{if } m>2.5k\\ 4.7k & \text{if } m>3.5k\\ 5k & \text{if } m>3.5k \text{ unless } 76\leq k\leq 149 \text{ or } 152\leq k\leq 155\\ 6k & \text{if } m>4.5k \text{ and } k\leq 38\end{cases}$$

except for $(m,k)\in T$, where $T=\{(5,2),(7,2),(25,2),(33,2),(75,2),(243,2),(11,3),(117,3),(9,4),(15,4),(19,4),(21,4),(115,4),(13,5),(19,5),(17,6),(15,7),(21,8),(37,8),(19,9),(41,9),(87,19),(89,19),(81,23)\}$.

We observe that the exceptions in Theorem 4 are necessary. The first result in the direction of Theorem 4 is due to Sylvester [Syl1892] who proved that a product of k consecutive positive integers each exceeding k is divisible by a prime greater than k. This result has been extended to

$$P(\Delta(m,d,k)) > k \text{ for } m \geq d+k$$

by Sylvester [Syl1892] and

$$P(\Delta(m,d,k)) > k \text{ for } d > 1, (m,d,k) \neq (2,7,3)$$

by Shorey and Tijdeman [ShTi10]. Sharper estimates have been obtained and we refer to [ShTi15] for an account of results on the greatest prime of a product of consecutive terms in arithmetic progressions. We give two earlier results on Theorem 4. For $d=2$ and m odd, it follows from [LaSh06a, Corollary 1] that

$$P(\Delta(m,2,k)) > 2k$$

unless $(m,k) = (1,2)$. Further, Laishram and Shorey [LaSh12] proved that

$$P(\Delta(m,2,k)) > 4k \text{ for } m > 2.5k$$

unless $(m,k) \in \{(5,2),(7,2),(25,2),(243,2),(9,4),(13,5),(17,6),(15,7),(21,8),(19,9)\}$ and the exceptions are necessary. The proof of Theorem 4 depends on the combinatorial ideas of Erdős, elementary prime number theory [Lemmas 2.1,2.2], computational result on Grimm's problem [LaSh09], the tables of Lehmer [Leh64] and Najman [Naj10]. Sharper results can be obtained for sufficiently large k but they are not relevant for our purpose. For $\epsilon > 0$, it is shown in [ShTi15] that *abc* conjecture implies

$$P(\Delta(m,2,k)) > (\frac{1}{2} - \epsilon)k \log m \quad \text{for } k \geq 2,\ m \geq m_0$$

where m_0 is a number depending only on ϵ.

The techniques in this paper are also valid for more general polynomials $G_\alpha(x)$ and $G_\alpha(x^2)$ where the denominator d of α is ≥ 5, but we need sharper lower bounds for the greatest prime factor of products of terms in arithmetic progression with common difference d. For example assuming

$$P(\Delta(m,d,k)) > (k+u)d \quad \text{for} \quad m > kd,\ (m,d) = 1,$$

it follows immediately from [ShTi07, Lemma 10.1], that $G_\alpha(x)$ with $a_0, a_n \in A$ has no factor of degree $k \geq 2$. In particular $G_{\frac{1}{d}}(x)$ has no factor of degree $k \geq 2$ whenever

$$P(\Delta(m,d,k)) > kd \quad \text{for} \quad m > kd,\ (m,d) = 1.$$

As already remarked, the preceeding estimate with $d = 2$ is proved in [LaSh06a, Corollary 1] and in [LaSh12], [LaSh15] for $d = 3, 4$ respectively, except for $(m,d,k) \in \{(125,3,2),(21,4,2),(45,4,2)\}$.

We give preliminaries for the proofs of this paper in Section 2 and we prove Theorem 4 in Section 3. We introduce Newton polygons and state further results based on Newton polygons required in the proof of Theorem 2 in Section 4. Further we prove irreducibility criterion for $G_\alpha(x^2)$ for the proof of Theorem 2 in Section 5. We continue with the proofs of lemmas required for Theorem 2 in Section 6. Further we prove Theorem 2, Corollary 1.2 in Section 6. Finally we prove Theorem 1, Corollary 1.1 in Section 7 and Theorem 3, Corollary 1.3 in Section 8. We give in Section 9 the factorizations of exceptional cases of Theorem 2. All the calculations other than finding the solutions of Thue equations have been carried out by using MATHEMATICA. In particular, the irreducibility of polynomials has been checked by using Factor command in MATHEMATICA. The Thue equations have been solved in integers by SAGE.

2. Preliminaries

We always write p for a prime number. For an integer $\nu > 1$, we denote by $\omega(\nu)$ the number of distinct prime divisors of ν and we put $\omega(1) = 0$. For a positive real ν, we write

$$\pi(\nu) = \sum_{p \leq \nu} 1 \ \ , \ \ \theta(\nu) = \sum_{p \leq \nu} \log p.$$

For a prime p and a non zero integer r, we denote $\mathrm{ord}_p(r)$ to be the maximal power of p dividing r. Further we write $\nu_p(r)$ for $\mathrm{ord}_p(r)$ and $\nu_p(r) = \nu(r)$ if it is clear from the context. We define $\nu(0) = +\infty$. Further $\lfloor \nu \rfloor$ will denote the greatest integer less than or equal to ν and $\lceil \nu \rceil$ the least integer greater than or equal to ν. For sets A and B, we denote by $A \setminus B$ the set of all elements in A which are not in B. We recall some well-known estimates from prime number theory.

Lemma 2.1. *For $\nu > 1$, we have*

(i) $\pi(\nu) \leq \dfrac{\nu}{\log \nu}\left(1 + \dfrac{1.2762}{\log \nu}\right)$

(ii) $\nu(1 - \dfrac{3.965}{\log^2 \nu}) \leq \theta(\nu) < 1.00008\nu$

(iii) $p_\nu \geq \nu(\log \nu + \log\log \nu - 1)$

(iv) $\sqrt{2\pi\nu}(\frac{\nu}{e})^\nu e^{\frac{1}{12\nu+1}} \leq \nu! \leq \sqrt{2\pi\nu}(\frac{\nu}{e})^\nu e^{\frac{1}{12\nu}}$

(v) $\mathrm{ord}_p((\nu-1)!) \geq \frac{\nu-p}{p-1} - \frac{\log(\nu-1)}{\log p}$.

The estimates $(i), (ii), (iii)$ and (iv) are due to Dusart [Dus98, p.14], [Dus99] and [Rob55, Theorem 6], respectively. We give a proof for (v).
Let $p^h \leq \nu - 1 < \nu \leq p^{h+1}$ so that $h = \left\lfloor \frac{\log(\nu-1)}{\log p} \right\rfloor$. We have

$$\mathrm{ord}_p((\nu-1)!) = \left\lfloor \frac{\nu-1}{p} \right\rfloor + \cdots + \left\lfloor \frac{\nu-1}{p^h} \right\rfloor$$

and

$$\left\lfloor \frac{\nu-1}{p^i} \right\rfloor \geq \frac{\nu-1}{p^i} - 1 + \frac{1}{p^i} = \frac{\nu}{p^i} - 1 \text{ for } 1 \leq i \leq h.$$

Therefore

$$\begin{aligned}\mathrm{ord}_p((\nu-1)!) &\geq \frac{\nu}{p}\left(\frac{1-\frac{1}{p^h}}{1-\frac{1}{p}}\right)-h\\ &= \frac{\nu}{p-1}-\frac{\nu}{p^h(p-1)}-h\\ &\geq \frac{\nu-p}{p-1}-\frac{\log(\nu-1)}{\log p}.\end{aligned}$$

□

Lemma 2.2. *Let p be a prime. For any integer $l \geq 1$, write l in base p as $l = l_t p^t + l_{t-1}p^{t-1} + \cdots + l_1 p + l_0$ where $0 \leq l_i \leq p-1$ for $0 \leq i \leq t$ and $l_t > 0$. Then*

$$\nu_p(l!) = \frac{l-\sigma_p(l)}{p-1}$$

where $\sigma_p(l) = l_t + l_{t-1} + \cdots + l_1 + l_0$.

This is due to Legendre. For a proof, see [Hasse, Ch.17, p263].

In the sequel, we will assume that m is an odd positive integer. Let

$$M_0 = 1.9 \times 10^{10}. \tag{4}$$

The next result is [LaSh12, Corollary 2.3].

Lemma 2.3. *Let $M_0 < m \leq 131 \times 2k$. Then $P(\Delta(m,2,k)) \geq m$.*

Lemma 2.4. *Let $k \geq 2$ and $2k < m < 6k$. Then*

$$P(\Delta(m,2,k)) > \begin{cases} 3.5k & \text{if } m \leq 2.5k\\ 4k & \text{if } 2.5k < m \leq 3.5k\\ 5k & \text{if } 3.5k < m \leq 4.5k\\ 6k & \text{if } 4.5k < m < 6k\end{cases} \tag{5}$$

unless $(m,k) \in \{(5,2),(7,2),(11,3),(9,4),(15,4),(19,4),(21,4),(13,5),(19,5),(17,6),(15,7),(21,8),(37,8),(19,9),(41,9),(87,19),(89,19),(81,23)\}$.

Proof. We observe that the exceptions in Lemma 2.4 are necessary. The first two estimates are contained in [LaSh12, Theorem 3] and we prove the remaining two. Let $m > 3.5k$. For $2 \leq k \leq 20$ and $m < 6k$, the assertion follows by computing $P(\Delta(m,2,k))$. Thus $k > 20$. We observe that $\{m, m+2, \ldots, m+2(k-1)\}$ contains all primes between $5k$ and $5.4k$ if $3.5k < m \leq 4.5k$ since $m \leq 4.5k < 5k < 5.4k < 5.5k-2 < m+2(k-1)$. Similarly the above set contais all primes between $6k$ and $6.4k$ if $4.5k < m < 6k$. Therefore (5) holds if $\theta(5.4k) > \theta(5k)$ and $\theta(6.4k) > \theta(6k)$.

Let $(r,s) = (5,5.4)$ or $(6,6.4)$. Then from Lemma 2.1 (ii), we see that $\theta(sk) > \theta(rk)$ if

$$sk\left(1-\frac{3.965}{\log^2(sk)}\right) > 1.00008 \times rk$$

that is, if

$$k > \frac{1}{s} \exp\left(\sqrt{\frac{3.965s}{s - 1.00008r}}\right).$$

This is true for $k \geq 452$. For $21 \leq k \leq 451$, we check that there is always a prime in the intervals $(5k, 5.4k)$ and $(6k, 6.4k)$ except for $k = 23$ in the first interval. For $k = 23$, the assertion follows by computing $P(\Delta(m, 2, 23))$ for each $46 < m < 138$. □

The following result concerns Grimm's conjecture [LaSh06b, Theorem 1].

Lemma 2.5. *Let $x \leq M_0$ where M_0 is given by* (4) *and l be such that $x+1, x+2, \ldots, x+l$ are all composite integers. Then there exist distinct primes p_i such that $p_i \mid (x+i)$ for each $1 \leq i \leq l$.*

As a consequence, we have

Lemma 2.6. *Let $6k < m \leq M_0$. Then $P(\Delta(m, 2, k)) > 5k$ for $k \geq 66$.*

Proof. We may assume that $m + 2i$ is composite for all $0 \leq i < k$, otherwise the assertion follows since $m > 6k$. Since m is odd, $m + 2i + 1$ with $0 \leq i < k$ are even and then $m, m+1, m+2, \ldots, m+2k-1$ are all composite. Then by Lemma 2.5, there are distinct primes P_j with $P_j|(m+j-1)$ for $1 \leq j \leq 2k$. Therefore $\omega(\Delta(m, 2, k)) \geq k$ implying $P(\Delta(m, 2, k)) \geq p_{k+1}$. By Lemma 2.1 (*iii*), we have $p_{k+1} \geq k \log k$ which is $> 5k$ for $k \geq 149$. For $66 \leq k < 149$, we check that $p_{k+1} > 5k$. Hence the assertion follows. □

Let

$$S_M = \{n \geq 1 : n \text{ odd}, P(n(n+2)) \leq M\} \text{ and } S'_M = \{n \geq 1 : n \text{ odd}, P(n(n+4)) \leq M\}.$$

The sets S_M and S'_M for $M \leq 31$ are given by tables in Lehmer [Leh64, Tables IIA] and for $M = 100$ by tables in Najman [Naj10]. These tables are written with entry $n+1$ and $n+2$ when $n \in S_M$ and $n \in S'_M$, respectively. These tables, in particular the tables of Najman, turn out to be very useful in finding lower bounds for the greatest prime factor of a product of terms in arithmetic progression and their applications to divisibility properties of generalized Laguerre polynomials. It is convenient to use the tables of Najman for values larger than 10^6 and carry out the computations directly for values $\leq 10^6$. We prove

Lemma 2.7. *Let $m > 6k$ and $2 \leq k \leq 38$. Then $P(\Delta(m, 2, k)) > 6k$ unless $(m, k) \in \{(25, 2), (33, 2), (75, 2), (243, 2), (117, 3), (115, 4)\}$.*

Proof. Let $k \geq 2$, $m > 6k$. Assume that $P(\Delta(m, 2, k)) < 6k$. Then $\omega(\Delta(m, 2, k)) \leq \pi(6k) - 1$ since $2 \nmid \Delta(m, 2, k)$.

Let $k = 2$. Then $P(m(m+2)) \leq 11$ which implies $m \in S_{11}$. Since $m \geq 13$, we have $m \in \{25, 33, 75, 243\}$.

Let $3 \le k \le 6$. Then $P(\Delta(m,2,k)) \le 31$. Therefore $m,\ m+2 \in S_{31}$. For every n with both n and $n+2 \in S_{31}$ and $3 \le k \le 6$, we check that $P(\Delta(n,2,k)) > 6k$ except for $k=3,\ n=117$ and $k=4,\ n=115$. Thus $(m,k) \in \{(117,3),(115,4)\}$.

Let $7 \le k \le 27$ and $m > 10^6$. For every prime with $100 < p < 6k$, we delete a term in $\{m, m+2, \ldots, m+2(k-1)\}$ divisible by p. Then the number of remaining terms is at least $k - \pi(6k) + \pi(100) > \lceil \frac{k}{2} \rceil$. Hence there is some i_0 with $0 \le i_0 \le k-2$ such that $P((m+2i_0)(m+2(i_0+1))) \le 100$. Then $m + 2i_0 = n \in S_{100}$. We check that

$$P\left(\prod_{i=1}^{2}(n+2+2i)\right) > 162 \ge 6k \text{ and } P\left(\prod_{i=1}^{2}(n-2i)\right) > 162$$

for each $n \in S_{100}$. This is a contradiction.

Let $28 \le k \le 38$ and $m > 10^6$. We observe that $k - \pi(6k) + \pi(100) > \lceil \frac{k}{3} \rceil$ and we argue as above to derive that one of the following holds.

(i) There exists $0 \le i_0 \le k-2$ such that $n = m + 2i_0 \in S_{100}$
(ii) There exists $0 \le i_0' \le k-3$ such that $n = m + 2i_0' \in S_{100}'$.

Assume (i). We check that $P\left(\prod_{i=1}^{3}(n+2+2i)\right) > 228 \ge 6k$ and $P\left(\prod_{i=1}^{4}(n-2i)\right) > 228 \ge 6k$ for each $n \in S_{100}$, with $n > 10^6$. Assume (ii). We check that $P\left(\prod_{i=1}^{3}(n+4+4i)\right) > 228$ and $P\left(\prod_{i=1}^{4}(n-4i)\right) > 228$ for all $n \in S_{100}'$, with $n > 10^6$. This is a contradiction.

Thus it remains to consider the cases $7 \le k \le 38$ and $m \le 10^6$. We check that $P(\Delta(m,2,6)) > 228 \ge 6k$ for $m \ge 50000$. Thus $m < 50000$. Since $m > 6k$, we check that $P(\Delta(m,2,k)) > 48$, $P(\Delta(m,2,k)) > 66$, $P(\Delta(m,2,k)) > 84$, $P(\Delta(m,2,k)) > 108$, $P(\Delta(m,2,k)) > 150$, $P(\Delta(m,2,k)) > 198$ and $P(\Delta(m,2,k)) > 228$ for $7 \le k \le 8$, $9 \le k \le 11$, $12 \le k \le 14$, $15 \le k \le 18$, $19 \le k \le 25$, $26 \le k \le 33$ and $34 \le k \le 38$, respectively. Hence the result. □

Lemma 2.8. *Let $m > 6k$ and $2 \le k \le 75$. Then $P(\Delta(m,2,k)) > 5k$ unless $(m,k) \in \{(25,2),(243,2)\}$.*

Proof. For the exceptions given in Lemma 2.7 we check that $P(\Delta(m,2,k)) > 5k$ except for $(m,k) \in \{(25,2),(243,2)\}$. Therefore we derive from Lemma 2.7 that $k \ge 39$. Let $P(\Delta(m,2,k)) < 5k$.

Let $39 \le k \le 75$ and $m > 10^6$. Now $k - \pi(5k) + \pi(100) > \lceil \frac{k}{3} \rceil$. Then one of the following holds:

(i) There exists $0 \le i_0 \le k-2$ such that $n = m + 2i_0 \in S_{100}$
(ii) There exists $0 \le i_0' \le k-3$ such that $n = m + 2i_0' \in S_{100}'$.

Assume (i). We check that $P\left(\prod_{i=1}^{3}(n+2+2i)\right) > 375 \geq 5k$ and $P\left(\prod_{i=1}^{4}(n-2i)\right) \geq 375 \geq 5k$ for each $n \in S_{100}$ with $n > 10^6$. Assume (ii). We check that $P\left(\prod_{i=1}^{3}(n+4+4i)\right) > 375$ and $P\left(\prod_{i=1}^{4}(n-4i)\right) > 375 \geq 5k$ for all $n \in S'_{100}$. This is a contradiction.

Thus it remains to consider the cases $39 \leq k \leq 75$ and $m \leq 10^6$. We check that $P(\Delta(m,2,7)) > 375 \geq 5k$ for $m > 50000$. Thus we may assume that $m \leq 50000$. Since $m > 5k$, we check that $P(\Delta(m,2,k)) > 240$, $P(\Delta(m,2,k)) > 335$ and $P(\Delta(m,2,k)) > 375$ for $39 \leq k \leq 48$, $49 \leq k \leq 67$ and $68 \leq k \leq 75$, respectively. Hence the assertion. □

3. Proof of Theorem 4

For more details upto inequality (6), we refer to [LaSh12, Section 3] with $d = 2$. Let $D > 1$ be a real number. Let $v = \frac{m}{2k}$. Assume that $P(\Delta(m,2,k)) \leq Dk$. Then

$$\omega(\Delta(m,2,k)) \leq \pi(Dk) - 1.$$

For every prime $p \leq Dk$ dividing Δ, we delete a term $m+2i_p$ such that $\mathrm{ord}_p(m+2i_p)$ is maximal. Then we are left with a set T with $1+t := |T| \geq k - \pi(Dk) + 1 := 1 + t_0$ with $t_0 = k - \pi(Dk)$. Let $t_0 \geq 0$ so that T is non-empty. We arrange the elements of T as $m + 2i_0 < m + 2i_1 < \cdots < m + 2i_{t_0} < \cdots < m + 2i_t$. Let

$$\mathfrak{P} = \prod_{\nu=0}^{t_0}(m+2i_\nu) \geq 2^{k-\pi(Dk)+1} \prod_{i=0}^{k-\pi(Dk)} (vk+i).$$

For a prime p dividing $\mathfrak{P}$, we observe that $p > 2$ and

$$\mathrm{ord}_p(\mathfrak{P}) \leq \mathrm{ord}_p((k-1-i_p)!i_p!) \leq \mathrm{ord}_p((k-1)!).$$

Therefore

$$\mathfrak{P} \leq (k-1)!2^{-\mathrm{ord}_2((k-1)!)}.$$

By comparing the upper and lower bound for $\mathfrak{P}$, we get

$$2^{\pi(Dk)} \geq \frac{2^{k+1}2^{\mathrm{ord}_2((k-1)!)}(vk)^{k+1-\pi(Dk)}}{(k-1)!}. \tag{6}$$

By using the estimates for $\mathrm{ord}_2((k-1)!)$ and $(k-1)!$ given in Lemma 2.1, we obtain

$$\begin{aligned}(2vk)^{\pi(Dk)} &> \frac{(2vk)^{k+1}2^{k-2}(k-1)^{-1}}{\sqrt{2\pi(k-1)}(\frac{k-1}{e})^{k-1}\exp\left(\frac{1}{12(k-1)}\right)} \\ &= \left(4ev\frac{k}{k-1}\right)^k \frac{v\sqrt{k}}{2e\sqrt{2\pi}}\sqrt{\frac{k}{k-1}}\exp\left(-\frac{1}{12(k-1)}\right).\end{aligned}$$

By taking logarithms on both sides and using the estimates for $\pi(\nu)$ given in Lemma 2.1 (i) and $\frac{\log(2vk)}{\log(Dk)} = 1 + \frac{\log\frac{2v}{D}}{\log(Dk)}$, we derive

$$(7)\qquad \begin{aligned} 0 > {} & \frac{1}{2}\log\frac{v^2k}{8\pi e^2} - \frac{1}{12(k-1)} \\ & + k\left(\log(4ev) - D\left(1 + \frac{\log\frac{2v}{D}}{\log(Dk)}\right)\left(1 + \frac{1.2762}{\log(Dk)}\right)\right). \end{aligned}$$

Let v be fixed with $2v \geq D$. Then

$$F(k,v) := \log(4ev) - D\left(1 + \frac{\log\frac{2v}{D}}{\log(Dk)}\right)\left(1 + \frac{1.2762}{\log(Dk)}\right)$$

is an increasing function of k. Let $k_1 := k_1(v)$ be such that $F(k,v) > 0$ for all $k \geq k_1$. Then we observe that the right hand side of (7) is an increasing function for $k \geq k_1$. Let $k_0 := k_0(v) \geq k_1$ be such that right hand side of (7) is positive. Thus (7) is not valid for $k \geq k_0$ implying (6) is not valid for $k \geq k_0$. Also for a fixed k, if (7) is not valid at some $v = v_0$, then (6) is also not valid at $v = v_0$. Observe that for a fixed k, if (6) is not valid at some $v = v_0$, then (6) is also not valid when $v \geq v_0$. Therefore for a given $v = v_0$ with $2v_0 \geq D$, there exists $k_0(v_0)$ such that (6) is not valid for (k, v) with $k \geq k_0(v_0)$ and $v \geq v_0$.

Let $D = 5$ so that $P(\Delta(m, 2, k) \leq 5k$. The assertion of Theorem 4 with $k \geq 2$ and $m < 6k$ follows from Lemma 2.4. Thus $m > 6k$. By Lemmas 2.3, 2.6, 2.8 and 2.7, we may assume that $k \geq 76$, and further

$$(8)\qquad m > \max\{M_0, 131 \times 2k\}.$$

By taking $v = 131$ in (7), we obtain $k_0 = 53000$. Therefore (6) is not valid for all $k_0 \geq 53000$ and for all $v \geq 131$. This implies $k < 53000$. Then we see from (8) that $v = \frac{m}{2k} \geq \frac{M_0}{2\cdot 53000} \geq 10^5$. By taking $v = 10^5$ in (7), we obtain $k_0 = 426$. Therefore (6) is not valid for $k \geq 426$ and $v \geq 10^5$. Thus $k < 426$. Then $v \geq \frac{M_0}{2\cdot 426}$ and further we get $k_0 \leq 231$. Therefore $k < 231$ as above. For each $180 \leq k < 231$, we find that (6) is not valid at $v = \frac{M_0}{2k}$ and hence for all $v \geq \frac{M_0}{2k}$. Thus we may assume that $k < 180$.

Let $150 \leq k < 180$. We continue as in [LaSh09, Section 3] with $d = 2$ to obtain

$$(9)\qquad m \leq \Big((k-1)! \prod_{p \leq p_l} p^{L_0(p)}\Big)^{\frac{1}{k+1-\pi(Dk)}}$$

for every $l \geq 1$ where

$$L_0(p) = \begin{cases} \min\left(0, h_p(k+1-\pi(Dk)) - \sum_{u=1}^{h_p}\lfloor\frac{k-1}{p^u}\rfloor\right) & \text{if } p \nmid d \\ -\mathrm{ord}_p((k-1)!) & \text{if } p \mid d \end{cases}$$

with $h_p \geq 0$ such that $\lfloor\frac{k-1}{p^{h_p+1}}\rfloor \leq k + 1 - \pi(Dk) < \lfloor\frac{k-1}{p^{h_p}}\rfloor$. Taking $l = 3$ and $D = 5$ in (9), we find that $m < M_0$ when $150 \leq k < 180$ except for $k = 152, 153, 154, 155$. Thus $76 \leq k \leq 149$ or $152 \leq k \leq 155$ by (8).

Let $76 \leq k \leq 149$ or $152 \leq k \leq 155$ where it remains to show that $P(\Delta(m, 2, k)) > 4.7k$. Now we take $D = 4.7$. For each $112 \leq k \leq 149$ and $152 \leq k \leq 155$, we check that (6) is not valid at $v = \frac{M_0}{2k}$ and hence for all $v \geq \frac{M_0}{2k}$ which is a contradiction. Thus $k < 113$. Further taking $l = 4$ and $D = 4.7$ in (9), we find that $m < M_0$ when $101 \leq k < 113$. Thus $k \leq 100$.

Let $76 \leq k < 100$. We observe that $k - \pi(4.7k) + \pi(100) > \lceil \frac{k}{3} \rceil$. Therefore one of the following holds:
(i) There exists $0 \leq i_0 \leq k-2$ such that $n = m + 2i_0 \in S_{100}$
(ii) There exists $0 \leq i_0^{'} \leq k-3$ such that $n = m + 2i_0^{'} \in S_{100}^{'}$.

Thus $n \geq m > M_0$ by (8). For all $n \in S_{100}$ with $n > M_0$, we check that $P\left(\prod_{i=1}^{3}(n+2+2i)\right) > 466 > 4.7k$ and $P\left(\prod_{i=1}^{2}(n-2i)\right) > 466$. Also for $n \in S'_{100}$ with $n > M_0$, we check that $P\left(\prod_{i=1}^{3}(n+4+4i)\right) > 466$ and $P\left(\prod_{i=1}^{4}(n-4i)\right) > 466$. Therefore $P(\Delta(m,2,k)) > 466 > 4.7k$ for $76 \leq k < 100$.

Now we take $k = 100$. For $m > M_0$ and $1 \leq i < 100$ such that $m + 2i = n$ with $n \in S_{100}$, we check that $P(\Delta(m,2,k)) > 4.7k$. Hence we may assume that for every m and $1 \leq i < 100$, $m + 2i \notin S_{100}$. For every prime $100 < p \leq 4.7k$, we delete a term in $\{m, m+2, \ldots, m+2(k-1)\}$ divisible by p. Let $0 \leq i_1 < i_2 < \cdots < i_l \leq 99$ be such that $m + 2i_j$ is in the remaining set where $l \geq k - (\pi(4.7k) - \pi(100))$. Since $m + 2i \notin S_{100}$ for each $1 \leq i < 100$, we observe that $i_{j+1} - i_j \geq 3$ implying $k - 1 \geq i_l - i_1 \geq 3(l-1) \geq 3(k - \pi(4.7k) + 24)$. Since $k = 100$, we find that $l = k - (\pi(4.7k) - \pi(100)) = 34$ otherwise $3(l-1) \geq 102 > k-1$ which is a contradiction. Thus $i_{34} - i_1 = 99$ implying $i_1 = 0, i_{j+1} = i_j + 3$ for each $1 \leq j \leq 33$. In other words $P(m(m+6)(m+12)\cdots(m+6\cdot 33)) \leq 100$.

Set $X = m - 6$. Since m is odd, X is odd. We have $P((X+6)\cdots(X+6\cdot 34)) \leq 100$. Suppose $3|X$. Then putting $Y = \frac{X}{3}$, we get $P((Y+2)\cdots(Y+2\cdot 34)) \leq 100$ which implies $Y + 2 < 6\cdot 34$ by Lemma 2.8 with $k = 34$. This is not possible since $3(Y+2) = X + 6 = m > M_0$. Hence we may assume that $3 \nmid X$. Then $3 \nmid (X+6)\cdots(X+6\cdot 34)$. After deleting terms $X + 6i$ divisible by primes $37 \leq p \leq 100$, we are left with at least 20 terms divisible by primes $5 \leq p \leq 31$. Out of these, there are at most two terms each divisible by prime $p \in \{17, 19, 23, 29, 31\}$, at most 3 terms divisible by 13 and at most 4 terms divisible by 11. After deleting these terms divisible by a prime $p \geq 11$, we are left with at least 3 terms divisible by 5 and 7. Out of these, there is a term $X + 6i$ where $\mathrm{ord}_5(X+6i) \leq 2$ and $\mathrm{ord}_7(X+6i) \leq 1$. Since $3 \nmid X + 6i$, we get $X + 6 \leq 5^2 \times 7 = 175$. This contradicts $X + 6i \geq m > M_0$. $\square$

4. Newton Polygons

Let $f(x) = \sum_{j=0}^{m} a_j x^j \in \mathbb{Z}[x]$ with $a_0 a_m \neq 0$ and let p be a prime. Let S be the following set of points in the extended plane

$$S = \{(0, \nu(a_m)), (1, \nu(a_{m-1})), (2, \nu(a_{m-2})), \ldots, (m, \nu(a_0))\}.$$

Consider the lower edges along the convex hull of these points. The left most endpoint is $(0, \nu(a_m))$ and the right most endpoint is $(m, \nu(a_0))$. The endpoints of each edge belong to S and the slopes of the edges increase from left to right. When referring to the edges of a Newton polygon, we shall not allow two different edges to have the same slope. The polygonal path formed by these edges is called the Newton polygon of $f(x)$ with respect to the prime p and we denote it by $NP_p(f)$. The endpoints of the edges on $NP_p(f)$ are called the vertices of $NP_p(f)$. We denote the Newton function of f with respect to the prime p as the real function $f_p(x)$ on the interval $[0, m]$ which has the polygonal path formed by these edges as its graph. Hence $f_p(i) = \nu(a_{m-i})$ for $i = 0$, m and at all points i such that $(i, \nu(a_{m-i}))$ is a vertex of $NP_p(f)$. We need the following lemma proved in [ShTi10, Lemma 2.13].

Lemma 4.1. *Let k, m and r be integers with $m \geq 2k > 0$. Let $g(x) = \sum_{j=0}^{m} b_j x^j \in \mathbb{Z}[x]$ and let p be a prime such that $p \nmid b_m$. Let denote by $g_p(x)$ the Newton function of $g(x)$ with respect to p. Let $a_0, a_1, \ldots, a_m$ be integers with $p \nmid a_0 a_m$. Put $f(x) = \sum_{j=0}^{m} a_j b_j x^j \in \mathbb{Z}[x]$. If $g_p(k) > r$ and $g_p(m) - g_p(m-k) < r + 1$, then $f(x)$ cannot have a factor of degree k.*

Lemma 4.1 implies the following result of [Fil94, Lemma 2] where the condition $|a_0 a_m| = 1$ is replaced by $p \nmid a_0 a_m$.

Corollary 4.2. *Let l, k, m be integers with $m \geq 2k > 2l \geq 0$. Suppose $g(x) = \sum_{j=0}^{m} b_j x^j \in \mathbb{Z}[x]$ and p be a prime such that $p \nmid b_m$ and $p \mid b_j$ for $0 \leq j \leq m - l - 1$ and the right most edge of the $NP_p(g)$ has slope $< \frac{1}{k}$. Then for any integers $a_0, a_1, \ldots, a_m$ with $p \nmid a_0 a_m$, the polynomial $f(x) = \sum_{j=0}^{m} a_j b_j x^j$ cannot have a factor with degree in $[l + 1, k]$.*

Proof. Since $p | b_j$ for $0 \leq j \leq m - l - 1$, we have $g_p(K) > 0$ for $K \in [l + 1, k]$. Let $(m_1, g_p(m_1))$ be the starting point of the right most edge of $NP_p(g)$. Then

$$\frac{1}{m - m_1} \leq \frac{g_p(m) - g_p(m_1)}{m - m_1} < \frac{1}{k}$$

giving $m_1 < m-k \le m-K$ for $K \le k$. Hence for $K \in [l+1, k]$, we observe that $(m-K, g_p(m-K))$ lie on the right most edge implying $\frac{g_p(m)-g_p(m-K)}{K} < \frac{1}{k} \le \frac{1}{K}$. Thus $g_p(m)-g_p(m-K) < 1$. Now we apply Lemma 4.1 with $r=0$ to get the assertion. □

5. Irreducibility Criterion

Lemma 5.1. *For any odd y and a prime p, we have*

$$\nu_y := \nu_p(3\cdot 5\cdots y) \le \frac{y-1}{2(p-1)} + \frac{\log y}{2\log p}.$$

Proof. We may assume that $p>2$, otherwise the assertion follows immediately. By Lemma 2.2,

$$\begin{aligned}\nu_y &= \nu(y!) - \nu((\frac{y-1}{2})!)\\ &= \frac{y-\sigma_p(y)}{p-1} - \frac{(\frac{y-1}{2}-\sigma_p(\frac{y-1}{2}))}{p-1} = \frac{\frac{y+1}{2}-\sigma_p(y)+\sigma_p(\frac{y-1}{2})}{p-1}.\end{aligned} \tag{10}$$

Now we write y in base p as

$$y = a_sp^s + a_{s-1}p^{s-1} + \cdots + a_1p + a_0, \quad 0 \le a_j < p \text{ for } 0 \le j \le s, a_s > 0.$$

Write $a_j = 2b_j + \delta_j$ with $\delta_j \in \{0,1\}$ for $0 \le j \le s$. Since y is odd, $\delta_j = 1$ for an odd number of j's, say $j_1 > j_2 > \cdots > j_{2t-1}, t \ge 1$. Then

$$\sigma_p(y) = 2\sum_{j=0}^{s} b_j + 2t - 1 \tag{11}$$

and

$$\begin{aligned}\frac{y-1}{2} &= \sum_{j=0}^{s} b_jp^j + \frac{\sum_{l=1}^{2t-1} p^{j_l} - 1}{2}\\ &= \sum_{j=0}^{s} b_jp^j + \frac{\sum_{l=1}^{t-1}\{p^{j_{2l}}(p^{j_{2l-1}-j_{2l}}-1)+2p^{j_{2l}}\} + (p^{j_{2t-1}}-1)}{2}\\ &= \sum_{j=0}^{s} b_jp^j + \sum_{l=1}^{t-1}\left(\frac{p-1}{2}p^{j_{2l}}(1+p+\cdots+p^{j_{2l-1}-j_{2l}-1}) + p^{j_{2l}}\right) + \frac{p-1}{2}(1+\cdots+p^{j_{2t-1}-1}).\end{aligned}$$

Therefore

$$\begin{aligned}\sigma_p(\frac{y-1}{2}) &= \sum_{j=0}^{s} b_j + t - 1 + \frac{p-1}{2}\left(\sum_{l=1}^{t-1}(j_{2l-1}-j_{2l}) + j_{2t-1}\right)\\ &\le \sum_{j=0}^{s} b_j + t - 1 + \frac{s(p-1)}{2}\end{aligned}$$

since $\sum_{l=1}^{t-1}(j_{2l-1}-j_{2l})+j_{2t-1}=j_1-(j_2-j_3)-\cdots-(j_{2t-2}-j_{2t-1})\le j_1\le s$. This together with (11) gives

$$\sigma_p(\frac{y-1}{2})-\sigma(y)\le \frac{s(p-1)}{2}-t-\sum_{j=0}^{s}b_j\le \frac{s(p-1)}{2}-1\le \frac{(p-1)\log y}{2\log p}-1$$

since $p^s\le y$ and $t\ge 1$. Now the assertion follows from (10). $\square$

Lemma 5.2. *Let* $1\le k\le \frac{n}{2}$ *and* $a_0,a_1,\ldots,a_n\in\mathbb{Z}$.

(i) *Suppose there is a prime* p *with*

$$p\Big|\prod_{l=0}^{k-1}(1+2u+2(n-l)),\quad p\nmid\prod_{l=1}^{k}(1+2u+2l) \tag{12}$$

satisfying

$$p>\max(2k,1+\sqrt{2(u+1)})\ \textit{and}\ p\nmid a_0a_n. \tag{13}$$

Then $G_\alpha(x^2)$ *does not have a factor of degree in* $\{2k-1,2k\}$. *Further when* n *is odd and* $k=\frac{n-1}{2}$, $G_\alpha(x^2)$ *does not have a factor of degree* $n=2k+1$.

(ii) *Let* $u\le 45$ *and* $k=2$. *Suppose there is a prime* $p\ge 5$ *satisfying* (12) *and* $p\nmid a_0a_n$ *and further* $p\ne 5$ *when* $u\in\{8,9,33,34\}$ *and* $p\ne 7$ *when* $u\in\{20,21\}$. *Then* $G_\alpha(x^2)$ *does not have a factor of degree* 3 *and* 4.

Proof. (*i*) We use Corollary 4.2. We write $(m,k,l)=(2n,2k,2(k-1))$ and

$$g(x)=\sum_{j=0}^{2n}b_jx^j \text{ where } b_j=\begin{cases}\displaystyle\prod_{i=l+1}^{n}(1+2(u+i)) & \text{if } j=2l\\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

Then $p\nmid b_{2n}$ and $p|b_j$ for $0\le j\le 2n-2k$ by (12) and $p|b_{2n-2k+1}$ since $b_{2n-2k+1}=0$. Consider the Newton polygon $NP_p(g)$ with respect to p. The slope of the right most edge of $NP_p(g)$ is given by

$$\max_{1\le j\le n}\left\{\frac{\nu_p(b_0)-\nu_p(b_{2j})}{2n-(2n-2j)}\right\}.$$

Let

$$\Delta_j=\prod_{l=1}^{j}(1+2u+2l)=\frac{b_0}{b_{2j}}. \tag{15}$$

Then by Corollary 4.2, it is enough to show that

$$\nu(\Delta_j)<\frac{2j}{2k}=\frac{j}{k}\text{ for } 1\le j\le n \tag{16}$$

to conclude that $G_\alpha(x^2)$ cannot have a factor of degree in $\{2k-1,2k\}$.

Now let n be odd and $k = \frac{n-1}{2}$. We apply Corollary 4.2 where (m, k, l) is given by $(2n, 2k+1, 2k)$ and $g(x)$ by (14). Then $p \nmid b_{2n}$ and $p | b_{2n-2k}$. Since $p | b_j$ for $0 \le j \le 2n-2k-1$, we derive from Corollary 4.2, as above, that it suffices to show that

$$\nu(\Delta_j) < \frac{2j}{2k+1} \text{ for } 1 \le j \le n. \tag{17}$$

We observe that (16) will follow from (17). Therefore it is enough to prove (17) to conclude the assertion of Lemma 5.2(i).

Let j_0 be the least positive integer j such that $p | (1+2u+2j)$ and write $1+2u+2j_0 = pl_0$. Then $j_0 \ge k+1$ since $p \nmid \Delta_k$ by (12). Also $j_0 \le p$. Hence $pl_0 \le 1+2u+2p$ implying $p(p-l_0) = p^2 - pl_0 \ge p^2 - 2p + 1 - 2(u+1) = (p-1)^2 - 2(u+1) > 0$ since $p > 1 + \sqrt{2(u+1)}$ by (13). Therefore $l_0 < p$ and hence $\nu(\Delta_{j_0}) = 1 < \frac{2j_0}{2k+1}$. Also observe that $\nu(\Delta_j) = 0 < \frac{2j}{2k+1}$ for each $1 \le j < j_0$.

Hence we consider $j > j_0$. To show $\nu(\Delta_j) < \frac{2j}{2k+1}$, we can restrict to those j such that $p | (1+2u+2j)$. Then $p | (j - j_0)$. Put $j = j_0 + pt$ with $t \ge 1$. Then $1 + 2u + 2j = p(l_0 + 2t)$ and

$$\nu(\Delta_j) = \nu(pl_0(p(l_0+2)) \cdots p(l_0+2t)).$$

Therefore

$$\nu(\Delta_j) = t + 1 + \nu(l_0(l_0+2) \cdots (l_0+2t)) \tag{18}$$

implying

$$\nu(\Delta_j) \le t + 1 + \nu(3 \cdot 5 \cdots (l_0+2t)). \tag{19}$$

Recall that

$$\frac{2j}{2k+1} = \frac{2(j_0+pt)}{2k+1} \ge \frac{2(k+1+pt)}{2k+1} \ge 1 + \frac{1}{2k+1} + \frac{2pt}{2k+1}. \tag{20}$$

We consider two cases:

Case I: Let $l_0 + 2t < p^2$. Let $t \ge 3$. Then since $l_0 < p$ and $p \ge 2k+1$ by (13), we get

$$\begin{aligned}
\nu(\Delta_j) &\le t + 1 + \left(1 + \frac{l_0+2t}{2p}\right) < t + 2 + \left(\frac{p+2t}{2p}\right) \\
&< 3 + t(1 + \frac{1}{p}) \le 3 + t(1 + \frac{1}{2k+1}) \\
&\le 1 + 2t < 1 + \frac{1}{2k+1} + \frac{2pt}{2k+1} \le \frac{2j}{2k+1}
\end{aligned}$$

by (20). Further $\nu(\Delta_j) \le 3$ if $t = 1$ and $\nu(\Delta_j) \le 4$ if $t = 2$ by (19) and $l_0 < p$. Therefore for $t \in \{1, 2\}$, we have

$$\nu(\Delta_j) \le 1 + 2t \le 1 + \frac{2pt}{2k+1} < \frac{2j}{2k+1}$$

by (20).

Case II: Let $l_0+2t \geq p^2$. Then $\nu(\Delta_j) \leq t+1+\frac{l_0+2t-1}{2(p-1)}+\frac{\log(l_0+2t)}{2\log p}$ by (19) and Lemma 5.1. Since $\frac{2j}{2k+1} > 1+\frac{2pt}{2k+1}$ by (20), it suffices to show that $\frac{2p}{2k+1} \geq 1+\frac{l_0+2t-1}{2t(p-1)}+\frac{\log(l_0+2t)}{2t\log p} = 1+\frac{1}{p-1}+\frac{l_0-1}{2t(p-1)}+\frac{\log(l_0+2t)}{2t\log p}$. Since $l_0 < p$, we prove

$$\frac{1}{2t}+\frac{\log(l_0+2t)}{2t\log p} \leq \frac{2p}{2k+1}-1-\frac{1}{p-1}.$$

By observing that the left hand side is a decreasing function of t and using $l_0+2t \geq p^2, l_0 < p$ and $p \geq 2k+1 \geq 3$, we obtain

$$\begin{aligned}\frac{1}{2t}+\frac{\log(l_0+2t)}{2t\log p} &\leq \frac{1}{p^2-l_0}+\frac{\log p^2}{(p^2-l_0)\log p} = \frac{3}{p^2-l_0}\\ &\leq \frac{3}{p^2-p} \leq \frac{3}{3^2-3} = \frac{1}{2} \leq \frac{2p}{2k+1}-1-\frac{1}{p-1}.\end{aligned}$$

This proves (i).

Remarks:(a) We observe that the condition $p > 1+\sqrt{2(u+1)}$ is required only in deriving $l_0 < p$ in the above proof. This observation is used in the proof of (ii).
(b) If n is odd and $k = \frac{n-1}{2}$, we observe that the assumption (12) can be relaxed to $p | \prod_{l=0}^{k}(1+2u+2(n-l))$.

For the proof of (ii), we may suppose that $p \in \{5,7\}$ since $u \leq 45$ and $k = 2$ otherwise the assertion follows from (i). As in the proof of (i), it suffices to show that $\nu(\Delta_j) < \frac{j}{2}$. [See (16).] Let j_0, l_0 be as defined in (i) given by $pl_0 = 1+2u+2j_0$ and we may assume that $l_0 \geq p$ otherwise the assertion follows as in (i); see the above remark. Also $\nu(\Delta_j) = 0$ for $1 \leq j < j_0$. Further from $pl_0 \leq 1+2u+2p$, we get $l_0 \leq 2+\frac{2u+1}{p} \leq 2+\frac{91}{5} < 21$. Since l_0 is odd, we get $l_0 \leq 19$. Also $3 = k+1 \leq j_0 \leq p \leq 7$. Thus $1+2u+2j_0 \leq 105$. If $p \nmid l_0$, $\nu(\Delta_{j_0}) = 1 < \frac{3}{2} \leq \frac{j_0}{2}$. If $p|l_0$, then $p = 5$ implies $1+2u+2j_0 \in \{25,75\}$ and $p = 7$ implies $1+2u+2j_0 \in \{49\}$. Let $j_0 \in \{3,4\}$. Then $p = 5$ implies $u \in \{8,9,33,34\}$ and $p = 7$ implies $u \in \{20,21\}$. This is not possible by our assumption. Thus we obtain $j_0 \geq 5$ whenever $p|l_0$ and thus $\nu(\Delta_{j_0}) = 2 < \frac{5}{2} \leq \frac{j_0}{2}$ in this case. Hence we may assume that $j > j_0$ and we may restrict to those j such that $p|(1+2u+2j)$. As in (i), we have $j = j_0+pt \geq 3+5t$. Since $l_0 \leq 19$, we have $\nu(l_0(l_0+2)\cdots(l_0+2t) \leq 1$ for $t \leq 2$ and $\nu(l_0(l_0+2)\cdots(l_0+2t) \leq 2$ for $3 \leq t \leq 4$. Thus in all these cases, we derive from (18) that $\nu(\Delta_j) < \frac{3+5t}{2} \leq \frac{j}{2}$. Therefore $t \geq 5$. To show $\nu(\Delta_j) < \frac{j}{2}$, it suffices to show $2t+2+2\nu(3\cdot 5\cdots(l_0+2t)) < 3+5t$ by (19). Since by Lemma 2.2 and $t \geq 5$

$$2\nu(3\cdot 5\cdots(l_0+2t)) \leq 2\nu((l_0+2t)!) < \frac{2(l_0+2t)}{p-1} \leq \frac{2(19+2t)}{4} < 1+3t,$$

the assertion follows. □

Corollary 5.3. *Let* $1 \leq k \leq \frac{n}{2}$. *Suppose there is a prime* p *satisfying* (12) *and* (13). *Then* $G_\alpha(x)$ *does not have a factor of degree* k.

Proof. Assume that $G_\alpha(x)$ has a factor of degree $1 \le k \le \frac{n}{2}$. Then $G_\alpha(x^2)$ has a factor of degree $2k$. Since there is a prime satisfying (12) and (13), the assumptions in Lemma 5.2(i) are satisfied and we conclude that $G_\alpha(x^2)$ has no factor of degree $2k$. This is a contradiction. □

6. Proof of Theorem 2

First, we continue with some lemmas for the proof of Theorem 2 in this section. We always asssume that $1 \le u \le 45$ and $\alpha = u + \frac{1}{2}$ in these lemmas. Let $1 \le k \le \frac{n}{2}$. Since the degree of $G_\alpha(x^2)$ is equal to $2n$, it suffices to consider the factors of $G_\alpha(x^2)$ of degrees in $[3, n]$. A factor of degree $\le n$ of $G_\alpha(x^2)$ has degree in $\{2k-1, 2k\}$ or $2k+1$ when $n = 2k+1$ for some $1 \le k \le \frac{n}{2}$.

Let $2 \le k \le \frac{n}{2}$. We write

$$m = 1 + 2(u+1) + 2(n-k). \tag{21}$$

Assume that $G_\alpha(x^2)$ has a factor of degree in $\{2k-1, 2k\}$ or $G_\alpha(x^2)$ has a factor of degree $2k+1$ and $k = \frac{n-1}{2}$. By Lemma 5.2(i), we may restrict to those m and k such that

$$p > \max(2k, 1 + \sqrt{2(u+1)}), p | \Delta(m, 2, k) \text{ imply } p | \prod_{l=1}^{k} (1 + 2u + 2l). \tag{22}$$

For a pair (m, k), we say that (22) is valid if it holds for every prime p and (22) is not valid if there exists a prime p which does not satisfy (22). Therefore

$$P := P(\Delta(m, 2, k)) < 2k + 2(u+1) \tag{23}$$

whenever (22) holds. Since $n \ge 2k$, we have $P < m$ by (21) and (23). In particular, each of integers $m, m+2, \dots, m+2(k-1)$ is composite.

Lemma 6.1. *Let $2 \le k \le \frac{n}{2}$. Assume that $G_\alpha(x^2)$ has a factor of degree in $\{2k-1, 2k\}$ or $G_\alpha(x^2)$ has a factor of degree $2k+1$ when $n = 2k+1$. Then either $k \le 22$, $2k \le u \le 45$ or $(u, n, k) \in \{(1, 12, 2), (1, 121, 2)\}$.*

Proof. Let m be given by (21). Let the assumptions in Lemma 6.1 be satisfied. Then (22) and (23) are valid. Let $(m, k) \in T$ where T is given in Theorem 4. We consider all the triples (u, m, k) with $1 \le u \le 45$ and $(m, k) \in T$. First we exclude those triples (u, m, k) for which (23) is not satisfied. For each of the remaining pairs (u, m, k), we check that there exists a prime p which does not satisfy (22) unless $k = 2$ and $(u, m) \in \{(1, 25), (1, 243)\}$. Thus $(u, n, k) \in \{(1, 12, 2), (1, 121, 2)\}$ by (21). Therefore we may assume that $(m, k) \notin T$. Since $m > 2k$, we obtain from Theorem 4 and (23) that $3.5k \le P < 2k + 2(u+1)$ implying $1.5k < 2(u+1)$. This gives $m > 3.5k$ by (21) and $k < 62$ since $u \le 45$. By using Theorem 4 and (23) again, we get $4.7k \le P < 2k + 2(u+1)$ implying $u + 1 > 1.35k$ and $k \le 34$. Hence by Theorem 4 and (23), we obtain $5k \le P < 2k + 2(u+1)$ implying $u + 1 > 1.5k$. This gives

$m > 5k$. Finally, by using Theorem 4 and (23) again, we obtain $6k \le P < 2k+2(u+1)$ implying $u+1 > 2k$. Hence

$$k \le 22 \quad \text{and} \quad 2k \le u \le 45.$$

□

Lemma 6.2. *Let $2 \le k \le \frac{n}{2}$ such that $n = 2k+1$. Then $G_\alpha(x^2)$ has no factor of degree $2k+1$.*

Proof. Assume that $G_\alpha(x^2)$ has a factor of degree $2k+1 = n$. Then $m = 2(u+k)+5$ by (21). By Lemma 6.1, we have $k \le 22$ and $2k \le u \le 45$ since the possibilities $(u,n,k) \in \{(1,12,2),(1,121,2)\}$ are excluded by $2k+1 = n$. For each $k \le 22$ and $2k \le u \le 45$, we check that (23) is not satisfied except for $(u,k) = (8,2)$. This gives $(u,n) = (8,5)$ and in this case, the vertices of Newton polygon with respect to $p = 23$ are given by $\{(0,0),(4,0),(10,1)\}$ and the slope of the right most edge is $\frac{1}{6} < \frac{1}{5}$. Thus $G_\alpha(x^2)$ does not have a factor of degree 5 by Corollary 4.2 when $(u,n) = (8,5)$. □

We recall the set S given in Theorem 2.

Lemma 6.3. *Let $2 \le k \le \frac{n}{2}$ and $G_\alpha(x^2)$ has a factor in degree $\{2k-1, 2k\}$. Then either $(u,n) \in S \cup \{(1,12),(5,8),(6,7),(8,5)\}$, $k = 2$ or $(u,n,k) \in (44,79,3)$.*

Proof. Let $2 \le k \le \frac{n}{2}$ and we assume that $G_\alpha(x^2)$ has a factor of degree $\{2k-1, 2k\}$. Then (22) and (23) are valid. Now by Lemma 6.1, we have $k \le 22$, $2k \le u \le 45$ or $(u,n,k) \in \{(1,12,2),(1,121,2)\}$. The latter possibility is excluded since $(1,121) \in S$.

Let $5 \le k \le 22$. Let $m > 10^6$. By (23), we have $P(\Delta(m,2,k)) < 2k+2(u+1)$. For $k \le 22$ and $2k \le u \le 45$, we check that $k - \pi(2k+2(u+1)+\pi(100) > \lceil \frac{k}{2} \rceil$. Therefore after deleting terms in $m, m+2, \ldots, m+2(k-1)$ divisible by primes > 100, there is at least one $0 \le i < k$ such that $P((m+2i)(m+2i+2)) \le 100$. Hence $m+2i \in S_{100}$. Then for $N \in S_{100}$, and $N > 10^6$, we check that $P((N+2)(N+4)(N+6)) > 136$ and $P((N-2)(N-4)) > 136$. This is a contradiction since $2k+2(u+1) \le 136$. Thus $m \le 10^6$. For each $3500 < m \le 10^6$, we check that $P(\Delta(m,2,7)) > 136$ and further $P(\Delta(m,2,6))) > 104$ and $P(\Delta(m,2,5)) > 102$. Hence we may assume that $m \le 3500$. For $2k+2(u+1) < m \le 3500$, we check that (22) is not valid for each $k \le 22$ and $2k \le u \le 45$.

Let $3 \le k \le 4$. Then $2k+2(u+1) \le 100$ which implies $m \in S_{100}$ by (23). Let $m > 10^6$. We check that for $N \in S_{100}$ and $N > 10^6$ that $P((N+2)(N+4)(N+6)) > 100$ and further $P((N+2)(N+4)) > 100$ except for $N = 5337421$. For $m = 5337421$ and $k = 3$, we can choose $p = 97$ for $6 \le u \le 44$ and $p = 67$ for $u = 45$ so that (22) is not valid. This is a contradiction. Thus $m \le 10^6$. For each $3 \le k \le 4$, we first compute

$$U(k) = \{100 < m \le 10^6 : P(m(m+2)\cdots(m+2(k-1))) \le 100\}$$

and then compute

$$V(k,u) = \{2k+2(u+1) < m \le 100 \text{ or } m \in U(k) : (22) \text{ is valid}\}$$

for each $2k \le u \le 45$. We are left with only $(k,u,m) = (3,44,243)$. This implies $(k,u,n) = (3,44,79)$ by (21).

Let $k = 2$. Then $u \ge 4$ and by (23), we get $P(m(m+2)) \le 4 + 2(u+1) < 100$. Thus $m \in S_{4+2(u+1)} \subset S_{100}$. For each $m \in S_{4+2(u+1)}$ with $m > 10^6$ and $4 \le u \le 45$, we check that there is a prime $p \ge 11$ such that $p|m(m+2)$ but $p \nmid (2u+3)(2u+5)$. Hence we may suppose by (22) that $m \le 10^6$. For each $4 \le u \le 45$, we compute

$$W(u) = \{m : 2k + 2(u+1) < m \le 10^6,\ P(m(m+2)) < 100\}$$
$$W_0(u) = \{m : m \in W(u), \exists\, p \ge 11, p|m(m+2), p \nmid (2u+3)(2u+5)\}$$

and further

$$W_1(u) = \begin{cases} \{m \in W(u) \setminus W_0(u) : \exists\, p \ge 5,\ p|m(m+2), p \nmid (2u+3)(2u+5)\} \text{ if } u \notin \\ \{8,9,20,21,33,34\} \\ \{m \in W(u) \setminus W_0(u) : 7|m(m+2), 7 \nmid (2u+3)(2u+5)\} \text{ if } u \in \{8,9,33,34\} \\ \{m \in W(u) \setminus W_0(u) : 5|m(m+2), 5 \nmid (2u+3)(2u+5)\} \text{ if } u \in \{20,21\}. \end{cases}$$

We denote by $W_2(u)$ the complement of $W_0(u) \cup W_1(u)$ in $W(u)$. By (22) and Lemma 5.2(ii), we need to consider only $W_2(u)$. For $4 \le u \le 45$, we have $(u,m) \in \{(5,25), (6,25), (8,25), (8,133), (8,243), (9,25), (9,243), (9,343), (9,1125), (15,75), (15,243), (16,243), (20,243), (21,243), (26,361), (26,625), (30,243), (36,243), (43,2401), (44,1519)\}$ where $m \in W_2(u)$. By (21) with $k = 2$, we have $(u,n) = (u, \frac{m+1}{2} - u)$ which we use to conclude that the set of above pairs is equal to the set of pairs (u,n) consisting of $S \cup \{(5,8),(6,7),(8,5)\}$ other than $\{(1,121)\}$. □

Lemma 6.4. *(a) Let $(u,n) \in S \cup \{(1,12),(5,8),(6,7),(8,5)\}$. If $G_\alpha(x^2)$ has a factor of degree 3, then $(u,n) \in \{(1,12),(6,7),(9,113),(21,101)\}$.*
(b) $G_\alpha(x^2)$ has no factor of degree 4 when $(u,n) \in \{(1,12),(5,8),(6,7),(8,5)\}$.
(c) $G_\alpha(x^2)$ has no factor of degree 5 when $(u,n) = (44,79)$.

Proof. (a) We may assume that $(u,n) \notin \{(1,12),(6,7),(9,113),(21,101)\}$. Let $(u,n) = (9,4)$. We apply Lemma 4.1 with $p = 3, r = 1$ to $g(x)$ given by (14) to conclude that $G_\alpha(x^2)$ cannot have a factor of degree 3 in this case. Now we show that for the remaining pairs $G_\alpha(x^2)$ has no factor of degree 3. Let p be the largest prime dividing $(1+2u+2(n-1))(1+2u+2n)$ with $p \nmid (2u+3)$. We take $p = 3$ for $(u,n) = (16,106)$, $p = 5$ for $(u,n) \in \{(5,8),(8,5),(30,92)\}$, $p = 7$ for $(u,n) \in \{(1,121),(8,59),(8,114),(15,23),(15,107),(20,102),(36,86),(43,1158)\}$, $p = 19$ for $(u,n) = \{(26,155),(26,287)\}$, $p = 23$ for $(u,n) \in \{(9,163),(9,554)\}$ and $p = 31$ for $(u,n) = (44,716)$. As in the proof of Lemma 5.2, we derive from Corollary 4.2 with $k = 3$ and $l = 2$ that it suffices to show $\nu(\Delta_j) < \frac{2j}{3}$ for $1 \le j \le n$ where Δ_j is given by (15). Clearly this is true for $j = 1$ since $p \nmid (2u+3)$. Let $1 \le j_1 \le j$ be the such that $\nu(1+2u+2j_1)$ is maximal. Observe that $1+2u+2j_1 \le 1+2u+2n \le 1+2\cdot3+2\cdot1158 < 5^5$ if $(u,n) \ne (16,106)$ and $1+2u+2j_1 \le 245 < 3^6$ if $(u,n) = (16,106)$. Therefore by Lemma 2.2, we get

$$\nu(\Delta_j) \le \nu(1+2u+2j_1) + \nu((j-1)!) < 5 + \frac{j-1}{p-1} < 5 + \frac{j-1}{2} < \frac{2j}{3}$$

for $j > 27$. Thus $j \le 27$. For $2 \le j \le \min(27,n)$, we check that $\nu(\Delta_j) < \frac{2j}{3}$.

(b) Let $(u,n)=(1,12)$. We apply Lemma 4.1 with $p=3, r=1$ to $g(x)$ given by (14). Then $g_3(4)=\frac{10}{9}>1, g_3(24)-g_3(20)=\frac{4}{3}<2$. Thus $G_\alpha(x^2)$ does not have a factor of degree 4. Similarly the cases $(u,n)\in\{(5,8),(6,7),(8,5)\}$ are excluded by Lemma 4.1 with $p=3$ and $r=1$.

(c) Let $(u,n)=(44,79)$. We use Corollary 4.2 with $k=5, l=4$. Taking $p=19$, as in the proof of Lemma 5.2, it suffices to show $\nu(\Delta_j)=\nu(\prod_{l=1}^{j}(91+2(l-1)))<\frac{2j}{5}$ for $1\le j\le 79$. This is true for $1\le j\le 5$. Since $19^2\nmid(91+2(l-1)$ for $1\le l\le 79$, we see that $\nu(\Delta_j)\le 1+\nu((j-1)!)<1+\frac{j-1}{18}<\frac{2j}{5}$ for $j>5$. Hence $G_\alpha(x^2)$ does not have a factor of degree 5.

□

Proof of Theorem 2. Assume that $G_\alpha(x^2)$ has a factor of degree $3\le t\le n$. We take

$$k=\begin{cases}\frac{t-1}{2} & \text{if } t=n \text{ odd}\\ \frac{t}{2} & \text{if } t \text{ is even}\\ \frac{t+1}{2} & \text{if } t \text{ is odd and } t\neq n.\end{cases}$$

Thus $k\le\frac{n}{2}$ since $t\le n$. Also $k\ge 2$ unless $t=n=3$. Let $(t,n)\neq(3,3)$. Then $k\ge 2$. By Lemmas 6.2, we may assume that $G_\alpha(x^2)$ has a factor of degree in $\{2k-1,2k\}$. Now we derive from Lemma 6.3 that $G_\alpha(x^2)$ may have factors only of degrees $\{3,4\}$ or $\{5,6\}$ at $S\cup\{(1,12),(5,8),(6,7),(8,5)\}$ or $(44,79)$, respectively. By Lemma 6.4(a) and 6.4(b), we see that $G_\alpha(x^2)$ may have a factor of degree 3 only when $(u,n)\in\{(1,12),(6,7),(9,113),(21,101)\}$ and a factor of degree 4 only when $(u,n)\in S$. Finally, we conclude from Lemma 6.4(c) that $G_\alpha(x^2)$ does not have a factor of degree 5 and hence, it may have a factor of degree 6 when $(u,n)=(44,79)$.

It remains to consider the case $t=n=3$. Then $k=1$. By Lemma 5.2(i) if we can find a prime p satisfying $p|(2u+7)$, $p\nmid(2u+3)$ and (13), then $G_\alpha(x^2)$ does not have a factor of degree 3. For $u\le 45$, we check that we can always find such a prime p except for $u\in\{1,9,10,19,21,28,34,37\}$. We apply Corollary 4.2 with $(m,k,l)=(6,3,2)$, $g(x)$ given by (14) and $(u,p)\in\{(1,7),(9,23),(19,43),(21,47),(28,61),(34,73),(37,79)\}$. We find that the slope of the right most edge of $NP_p(g)$ is $<\frac{1}{3}$ in each case. Hence $G_\alpha(x^2)$ has no factor of degree 3 for the above values of u other than $u=10$. □

Proof of Corollary 1.2 : If $G_\alpha(x)$ has a factor of degree $k\ge 2$, then $G_\alpha(x^2)$ has a factor of degree $2k\ge 4$. Then the assertion follows from Theorem 2.

7. Proof of Theorem 1

Let $(u,n)\neq(10,3)$ and $\alpha=u+\frac{1}{2}$. We consider

$$\mathcal{L}_n^{(\alpha)}(x^2)=\sum_{j=0}^{n}\binom{n}{j}\prod_{i=j+1}^{n}(1+2(u+i))x^{2j}$$

and we observe that $L_n^{(\alpha)}(\frac{-x^2}{2}) = \frac{\mathcal{L}_n^{(\alpha)}(x^2)}{2^n n!}$ by (3). By Remark (ii) after the statement of Theorem 2 in Section 1, it suffices to show that $L_n^{(\alpha)}(\frac{-x^2}{2})$ does not have a factor of degree in $\{1, 2\}$. Therefore it suffices to prove that $\mathcal{L}_n^{(\alpha)}(x^2)$ does not have a factor of degree in $\{1, 2\}$. By Lemma 5.2, we may suppose that

$$p|(2n+2u+1) \Rightarrow p|(2u+3) \text{ or } p \le 1+\sqrt{2(u+1)}. \tag{24}$$

Let

$$p|n(2n+2u+1) \text{ and } p \text{ odd} \tag{25}$$

and we denote by $j_1 := j_1(p) = 1, 2, 3, 5$ according as $p \ge 11, p = 7, p = 5, p = 3$, respectively. First we prove three lemmas for the proof of Theorem 1.

Lemma 7.1. *Let p be a prime satisfying* (25) *and assume that*

$$\nu(\Delta_j) - \nu\left(\binom{n}{j}\right) < j \text{ for } 1 \le j \le j_1. \tag{26}$$

Then $\mathcal{L}_n^{(\alpha)}(x^2)$ does not have a factor of degree in $\{1, 2\}$.

Proof. If $p|n$, then $p|\binom{n}{j}$ for $1 \le j < p$ which together with $p|\Delta_j$ for $j \ge p$ implies $p|\binom{n}{j}\Delta_j$ for $j \ge 1$ where Δ_j is given by (15). This is also true when $p|(2n+2u+1)$. Now we apply Corollary 4.2 with $g(x) = \mathcal{L}_n^{(\alpha)}(x^2)$, $k = 2$ and $l = 0$ to conclude that it suffices to show

$$\nu(\Delta_j) - \nu\left(\binom{n}{j}\right) < j \quad \text{for} \quad 1 \le j \le n. \tag{27}$$

It suffices to prove (27) for $j > j_1$ by (26) and in this case, we show that $\nu(\Delta_j) < j$ which implies (27). Since $u \le 45$ and $p \ge 3$, we have

$$\begin{aligned}\nu(\Delta_j) &\le \frac{\log(2u+1+2j)}{\log p} + \nu((j-1)!) \\ &\le \frac{\log(91+2j)}{\log p} + \frac{j-2}{p-1} \le \frac{\log(91+2j)}{\log 3} + \frac{j-2}{2} \le \frac{j}{2} + \frac{\log(30+j)}{\log 3} < j\end{aligned}$$

for $j \ge 7$ by Lemma 2.2. Thus (27) is valid for $j \ge 7$. Let $j \le 6$. Then $2u+1+2j \le 103$ implying $\nu_p(\Delta_j) \le 1, 2, 3, 5$ according as $p \ge 11, p = 7, p = 5, p = 3$, respectively. Thus $\nu(\Delta_j) \le j_1 < j$. $\square$

Since $\nu(\binom{n}{j}) \ge \nu(n)$ for $1 \le j < p$, the following result ia an immediate consequence of Lemma 7.1.

Corollary 7.2. *Let p be a prime satisfying* (25) *and assume that*

$$\begin{cases}\nu(\Delta_j) - \nu(n) < j & \text{for } \ 1 \le j < \min(j_1+1, p) \\ \nu(\Delta_j) < j & \text{for } \ \min(j_1+1, p) \le j \le j_1.\end{cases} \tag{28}$$

Then $\mathcal{L}_n^{(\alpha)}(x^2)$ does not have a factor of degree in $\{1, 2\}$.

Lemma 7.3. *Let $P(n) = 2$. Then $\mathcal{L}_n^{(\alpha)}(x^2)$ does not have a factor of degree in $\{1, 2\}$.*

Proof. Assume that $P(n) = 2$ and we write $2n = 2^a$. Then $a \geq 2$ since $n > 1$. Let $p|(2n+2u+1)$. Then it is clear that (25) holds so that it suffices to prove (26). First, we derive from (24) that

$$p|(2u+3) \text{ or } p \in \{3, 5, 7\}$$

since $u \leq 45$. Now we exclude the latter possibility.

Let $p \in \{5, 7\}$, $p|(2n+2u+1)$ and $p \nmid (2u+3)$. Then $j_1 \leq 3$ and (26) is valid for p except when $2u+5 \in \{25, 75\}$, $5|(2n+2u+1)$, $7 \nmid (2n+2u+1)$ and $2u+5 = 49$,$7|(2n+2u+1)$, $5 \nmid (2n+2u+1)$. Hence we have following possibilities:

$$\begin{aligned}
&2u+5 = 25,\ 2^a + 21 = 5^b \cdot 23^c \\
&2u+5 = 75,\ 2^a + 71 = 3^b \cdot 5^c \cdot 73^d \\
&2u+5 = 49,\ 2^a + 45 = 7^b \cdot 47^e.
\end{aligned}$$

The last equation is not possible by modulo 8 unless $a = 2$. Then $(u, n) = (22, 2)$. Consider the first equation $2^a + 21 = 5^b \cdot 23^c$. The solutions for this equation can be found by transforming it into Thue equations of the form $5^{\beta_1}23^{\gamma_1}X^3 - 2^{\alpha_1}Y^3 = 21$ where $a = 3\alpha + \alpha_1, b = 3\beta + \beta_1, c = 3\gamma + \gamma_1$, $0 \leq \alpha_1, \beta_1, \gamma_1 \leq 2$, $X = 5^\beta 23^\gamma$ and $Y = 2^\alpha$, We solve these Thue equations using SAGE and we use this method of solving equations without any reference. The solutions for these equations give $(u, n) = (10, 2)$. Similarly we solve the second equation which implies $(u, n) \in \{(35, 2), (35, 2^5), (35, 2^9)\}$. For these values of (u, n), we check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible. Thus for $p \in \{5, 7\}$, $p|(2n+2u+1)$ implies $p|(2u+3)$.

Let $p = 3$, $p|(2n+2u+1)$ but $p \nmid (2u+3)$. Since $3 \nmid n$, we get $3 \nmid (2u+1)$ and hence $3 \mid (2u+5)$. For these values of u, $p = 3$ and $j_1 = 5$ we check that (26) holds except when $2u+5 \in \{9, 27, 45, 63, 75, 81\}$. Then we get from (24) the following possibilities.

$$\begin{aligned}
&2u+5 = 9, \quad 2^a + 5 = 3^b \cdot 7^c \\
&2u+5 = 27, \quad 2^a + 23 = 3^b \cdot 5^c \\
&2u+5 = 45, \quad 2^a + 41 = 3^b \cdot 43^c \\
&2u+5 = 63, \quad 2^a + 59 = 3^b \cdot 61^c \\
&2u+5 = 81, \quad 2^a + 77 = 3^b \cdot 79^c \\
&2u+5 = 75, \quad 2^a + 71 = 3^b \cdot 73^d.
\end{aligned}$$

We find that the solutions of the above equations are given by

$$(u, n) \in \{(2, 2), (2, 8), (2, 2^9), (11, 2), (38, 2)\}$$

and we check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible for each of the above pairs of (u, n).

Hence we conclude that $p|(2n+2u+1)$ implies $p|(2u+3)$. Further by using $\omega(2u+3) \le 2$ since $u \le 45$, we get the following equations:

$$p^b - 2^a = 2u+1 \text{ if } \omega(2u+3) = 1 \text{ and } p|2u+3$$
$$p^b q^c - 2^a = 2u+1 \text{ if } \omega(2u+3) = 2 \text{ and } pq|(2u+3).$$

The solutions of these equations are given by $(u,n) \in \{(6,2^4),(9,4),(9,2^6),(16,2^3),(21,2^4),(24,2^4),(30,2^6),(36,4),(44,2^{12})\}$. We check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible for these values of (u,n). □

Lemma 7.4. *Assume that $\mathcal{L}_n^{(\alpha)}(x^2)$ has a factor of degree in $\{1,2\}$. Then $P(n) = 3$.*

Proof. We have $P(n) \ge 3$ by Lemma 7.3. Assume that $P(n) > 3$ and $\mathcal{L}_n^{(\alpha)}(x^2)$ has a factor of degree in $\{1,2\}$. We shall contradict Corollary 7.2 by satisfying (28) for some prime such that (25) is valid. For a prime p, let $j \le j_1(p) = j_1$.

Since $P(n) > 3$, there exists a prime $p \ge 5$ dividing n. If $p \ge 11$, then $j_1 = j_1(p) = 1$, $\nu(\Delta_j) \le 1$ and (28) follows since $\nu(n) > 0$. Let $p = 7$. Then $j_1 = 2$, $\nu(\Delta_j) \le 2$ and (28) follows for $j = 2$ since $\nu(n) > 0$. Thus we may suppose that $j = 1$. Then $\nu(2u+3) = 2$ otherwise (28) holds since $\nu(n) > 0$. Now we have $2u+3 = 49$ implying $u = 23$. Let q be a prime dividing $2n+2u+1$. Then, since $u = 23$, we derive from (24) that $q \le 7$ and further $q \ne 7$ since $7|n$. Thus $q \in \{3,5\}, j_1(q) \le 5$ and we check that $\nu_q(\Delta_j) < j$ for $j \le 5$ implying (28) with p replaced by q. Hence we conclude that $P(n) \le 5$.

It remains to consider the case $p = 5$. Then $j_1 = 3$ and $\nu(\Delta_j) \le 2$. Therefore (28) is valid with $j = 3$. Thus $j \le 2$ and (28) is satisfied for $j = 2$ since $\nu(n) > 0$. Further, we observe that $\nu_5(n) = 1$ otherwise (28) is valid. Thus $j = 1$ and we may suppose that $\nu(2u+3) = 2$ since $\nu(n) > 0$ otherwise (28) is satisfied. Therefore $2u+3 \in \{25,75\}$ implying $u \in \{11,36\}$. Let q be a prime divisor of $2n+2u+1$. Then $q \ne 5$ since $5|n$ and we derive from (24) that $(q,u) = (3,11),(3,36)$ or $(7,36)$. The last possibility is excluded by checking that $\nu_q(\Delta_j) < j$ for $1 \le j \le 2$ implying (28). Therefore $2n+2u+1 = 3^a$ where a is a positive integer. Thus $3 \nmid n$ and $2n = 5 \cdot 2^b$ for some positive integer b since $\nu_5(n) = 1$. We obtain the equation $3^a - 5 \cdot 2^b = 2u+1$. Taking modulo 5, we obtain a is odd. If $b \ge 3$, we get a contradiction modulo 8. Hence $b \le 2$ and for these possibilities of b, the equation does not have a solution. □

Proof of Theorem 1. By Lemma 7.4, it remains to consider the case $P(n) = 3$. Let $p = 3$ and $p|n$. Thus $j_1 = 5$ and let $j \le j_1$. Then $\nu(\Delta_j) \le 5$ and (26) is valid for $j = 5$ since $\nu(\binom{n}{j}) \ge \nu(n) \ge 1$. Thus we may assume that $j \le 4$. Further we show that we need to consider the following possibilities with respect to prime $p = 3$.

$$(i)\ j=1: \ \nu(2u+3) = a,\ \nu(n) \le a-1,\ 2 \le a \le 4;$$
$$(ii)\ j=2: \ \nu(2u+5) = a,\ \nu(n) \le a-2,\ 3 \le a \le 4;$$
$$(iii)\ j=3: \ \nu(2u+7) = a,\ \nu(n) \le a-2,\ 3 \le a \le 4;$$
$$(iv)\ j=4: \ \nu((2u+3)(2u+9)) = 5,\ \nu(n) = 1.$$

Let $j = 1$. By (26) we need to consider $\nu(2u+3) > \nu(n)$ which is listed in (i). Let $j = 2$. Then $\nu(\binom{n}{2}) = \nu(n)$. By (26), we need to consider only when $\nu(\Delta_j) > 1+\nu(n)$. Therefore we have $\nu(\Delta_j) = a,\ \nu(n) \leq a-2,\ 3 \leq a \leq 4$. Since 3 divides exactly one of the terms $2u+3$ or $2u+5$, these cases are covered in (i) and (ii). Let $j = 3$. By (26) and since $\nu(\binom{n}{3}) = \nu(n) - 1$, we need to consider when $\nu(\Delta_j) \geq 2 + \nu(n)$. Therefore we have $\nu(\Delta_j) = a$, $\nu(n) \leq a-2$, $3 \leq a \leq 4$. In view of the cases $j = 1$ and $j = 2$, it remains to consider $\nu(2u+7) = a$, $\nu(n) \leq a-2$, $3 \leq a \leq 4$. Let $j = 4$. By (26) and since $\nu(\binom{n}{4}) = \nu(n) + \nu(n-3) - 1$, we need to consider when $\nu(\Delta_j) \geq 3 + \nu(n) + \nu(n-3) \geq 5$. Therefore $\nu(\Delta_j) = \nu((2u+3)(2u+9)) = 5$ and $\nu(n) = \nu(n-3) = 1$.

Consider (i). We have $2u+3 \in \{9, 27, 45, 63, 81\}$ and hence $3|u$. Let q be a prime dividing $2n+2u+1$. Then $q \neq 3$ since $3|u$ and $3|n$. We derive from (24) that $q \in \{5, 7\}$ and we check that (28) is valid with $q \in \{5, 7\}$ except when

$$2u+3 = 45,\ \ 5|(2n+2u+1), 7 \nmid (2n+2u+1)$$
$$2u+3 = 63,\ \ 7|(2n+2u+1), 5 \nmid (2n+2u+1).$$

In both cases, $\nu_3(n) = 1$ and hence $2n = 3 \cdot 2^r$ for some integer $r > 0$. We have

$$2u+3 = 45,\ 2n = 3 \cdot 2^r,\ 2n+43 = 5^\alpha \ \text{ implying } 5^\alpha - 3 \cdot 2^r = 43$$
$$2u+3 = 63,\ 2n = 3 \cdot 2^r,\ 2n+61 = 7^\beta \ \text{ implying } 7^\beta - 3 \cdot 2^r = 61.$$

If $r \geq 3$, then modulo 8 gives a contradiction. Hence $r \leq 2$ and we check that there are no solutions.

Consider (ii). Then $2u+5 \in \{27, 81\}$, and hence $3 \nmid (2n+2u+1)$ since $3|n$. From (24), we get $q|(2n+2u+1)$ implies $q = 5$ if $2u+5 = 27$ and $q \in \{5, 7, 79\}$ if $2u+5 = 81$. For these values of q, we see that (28) is valid except when

$$2u+5 = 27,\ \ q|(2n+2u+1) \text{ implying } q = 5$$
$$2u+5 = 81,\ \ q|(2n+2u+1) \text{ implying } q = 79.$$

Also $\nu_3(n) = 1$ if $2u+5 = 27$ and $\nu_3(n) \in \{1, 2\}$ if $2u+5 = 81$ by (ii). Hence we get an equation $5^\alpha - 3 \cdot 2^r = 23$ in the first case and $79^\beta - 3^k \cdot 2^r = 77$, $k \in \{1, 2\}$ in the latter case. These are excluded by modulo 8.

Consider (iii). We have $2u+7 \in \{27, 81\}$, $\nu_3(n) = 1$ if $2u+7 = 27$ and $\nu_3(n) \in \{1, 2\}$ if $2u+7 = 81$. Let $2u+7 = 27$. By (24), $q|(2n+2u+1)$ implying $q \in \{3, 5, 23\}$. Thus $2n+2u+1 = 3^\alpha \cdot 5^\beta \cdot 23^\gamma$ and $2n = 3 \cdot 2^r$ implying $3^\alpha \cdot 5^\beta \cdot 23^\gamma - 3 \cdot 2^r = 21$. Since $\alpha \geq 1$, $3^{\alpha-1} \cdot 5^\beta \cdot 23^\gamma - 2^r = 7$. This gives $(u, n) \in \{(10, 3), (10, 12), (10, 24), (10, 192)\}$. We check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible except for $(u, n) = (10, 3)$ which is already excluded in the begining of Section 7. Let $2u+7 = 81$. Then $u = 37$ and by (24), $q|(2n+2u+1)$ implying $q \in \{3, 5, 7, 11\}$. If $5|(2n+75)$, then $5|n$ which is not possible. Thus $5 \nmid (2n+75)$ and we have $2n+2u+1 = 2n+75 = 3^\alpha \cdot 7^\beta \cdot 11^\gamma$ and $2n = 3^k \cdot 2^r, k \in \{1, 2\}$ implying $3^\alpha \cdot 7^\beta \cdot 11^\gamma - 3^k \cdot 2^r = 75$. Let $k = 1$. If $\beta \geq 1$, then modulo 7 gives a contradiction. Thus $3^{\alpha-1} \cdot 11^\gamma - 2^r = 25$. It has the only solution $3 \cdot 11 - 2^3 = 25$ giving $n = 12$. We check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible when $(u, n) = (37, 12)$. Let $k = 2$. Then $3^\alpha \cdot 7^\beta \cdot 11^\gamma - 3^2 \cdot 2^r = 75$ implying $\alpha = 1$ and we have

$7^\beta \cdot 11^\gamma - 3 \cdot 2^r = 25$. The solutions of this equation give $(u, n) \in \{(37, 36), (37, 144)\}$ in which cases we check that $\mathcal{L}_n^{(\alpha)}(x^2)$ is irreducible.

Now we consider (iv). We have $2u + 3 \in \{75, 81\}$. Then $3 \nmid (2n + 2u + 1)$ as $\nu_3(n) = 1$. By (24), $q|(2n + 2u + 1)$ implies $q \in \{5, 7\}$. We check that (28) is valid for $2u + 3 = 81$ with $q \in \{5, 7\}$. Let $2u + 3 = 75$. If $7|(2n + 2u + 1)$, we see that (28) is valid with $q = 7$. Hence $2n + 2u + 1 = 5^\alpha$ which, together with $2n = 3 \cdot 2^r$, gives $5^\alpha - 3 \cdot 2^r = 73$. There are no solutions for this equation. □

8. Proof of Theorem 3

Assume that $G_\alpha(x^2)$ has a factor of degree l with $3 \le l \le n$. Since $(u, n) \notin \{(1, 12), (1, 121)\}$, the assertion of Theorem 3 holds for all exceptions in Theorem 2. Therefore $u \ge 46$ by Theorem 2.

Case (i) Let $l = n$ and n odd. Let $u \le \frac{1.35l}{2} - 1.2$ and we take $k = \frac{n-1}{2} = \frac{l-1}{2}$. Since $u \ge 46$, we have $l \ge 70$ implying $k \ge 34$. Further (21), (22), (23) are valid by Lemma 5.2 as in the proof of Theorem 2. Thus $m = 2u + 2k + 5 > 2k$ and we get from Theorem 4 that $P(\Delta(m, 2, k)) > 3.5k$ which, together with (23), gives $u > .75k - 1$. Therefore $m > 3.5k$. Now we apply Theorem 4 again to derive $P(\Delta(m, 2, k)) > 4.7k$ which, along with (23), implies $u > 1.35k - \frac{1}{2} > \frac{1.35l}{2} - 1.2$ since $k = \frac{l-1}{2}$.

Case (ii) Let $u \le \frac{1.35l}{2} - 0.5$. Then $l \ge 69$ since $u \ge 46$. If l is even, we take $k = \frac{l}{2}$. Since $l \le n$, we have $n \ge 2k$. If l is odd, we have $l \le n - 1$. We take $k = \frac{l+1}{2}$ and then $n \ge 2k$.

Hence $n \ge 2k$ and $k \ge 35$. Since $G_\alpha(x^2)$ has a factor of degree in $\{2k - 1, 2k\}$, we see that (21),(22) and (23) are valid. Then $m > 2k$ by (21). Now we apply Theorem 4 twice as in Case (i) to conclude that $u > 1.35k - \frac{1}{2} \ge \frac{1.35l}{2} - \frac{1}{2}$. □

Proof of Corollary 1.3: Assume that $G_\alpha(x)$ has a factor of degree $l \ge 2$ and $u \le 1.35l - 0.5$ Then $G_\alpha(x^2)$ has a factor of degree $2l \ge 4$. Thus $(u, n) \ne (1, 12)$ by Theorem 2 and then the assertion follows from Theorem 3(ii). □

9. Factorization of $G_\alpha(x)$

The factorizations for some Laguerre polynomials have been obtained in [ShTi10] without using computers and therefore it has not been possible to factorize Laguerre polynomials of large degree in [ShTi10]. We thank Professor Michael Filaseta for explaining us a method to carry out the computations on the computer and we explain this method for finding the factorization in the case $(u, n) = (15, 23)$. For convenience, we write $f(x) = G_\alpha(x)$.

Let $(u, n) = (15, 23)$. Initially we assume all a_j's to be 1. Then the coefficients of x^n, x^{n-1} and x^{n-2} are $1, 77$ and $77 \cdot 75$, respectively, which are composed of $3, 5, 7$ and

11. Now we consider the Newton polygon of $f(x)$ with respect to primes $3, 5, 7, 11$ which are given by

$$
\begin{aligned}
NP_3(f) =&\{(0,0),(1,0),(7,2),(16,6),(22,9),(23,10)\}\\
NP_5(f) =&\{(0,0),(1,0),(21,5),(23,6)\}\\
NP_7(f) =&\{(0,0),(14,2),(21,4),(23,5)\}\\
NP_{11}(f) =&\{(0,0),(22,2),(23,3)\}.
\end{aligned}
$$

Each of $NP_3(f), NP_5(f), NP_7(f)$ have edges of length 2 and slope $\frac{1}{2}$. By choosing a_1 to be a multiple of 11, we modify the vertices of $NP_{11}(f)$ to $\{(0,0),(21,2),(23,3)\}$ to get an edge of slope $\frac{1}{2}$ without any changes in $NP_3(f), NP_5(f), NP_7(f)$. Hence a quadratic factor $q(x) = x^2 + Ax + B$ of $f(x)$ may have a Newton polygon with respect to $3, 5, 7, 11$ having an edge of length 2 and slope $\frac{1}{2}$ if $3|A, 3||B, 5|A, 5||B, 7|A, 7||B, 11|A, 11||B$. So a possible quadratic factor can be $q(x) = x^2 + 1155$. Equating remainder obtained by dividing $f(x)$ with $q(x)$ to be 0 and solving the system of equations we get the following values for the coefficients a_j's. Thus for $(u, n) = (15, 23)$, $G(x)$ with $\{a_0, a_1 \dots a_n\} = \{1, 11, 1, 269, 0, 18, 0, 142, 0, 255, 0, 6, 0, 38, 0, 2, 0, 356, 0, 869, 0, 1449167, 0, 1\}$ has a quadratic factor $x^2 + 1155$.

Let $(u, n) = (8, 59)$. Then $G_\alpha(x)$ with $\{a_0, a_1, \dots, a_n\} = \{1,$ 1, 80, 70, 653, 271, 576, 2, 2, 540, 55, 427, 5, 2, 85, 35, 316, 17, 93, 18, 514, 10, 8, 32, 603, 22, 108, 102, 10, 60, 585, 161, 69, 127, 480, 74, 7, 16, 5, 418, 198, 1198, 7, 1, 638, 318, 79, 23, 97, 2, 34, 36, 67, 173, 217, 500, 153, 3182477123845074506664566503, 5284872575202700721255121305, 1$\}$ has a quadratic factor $x^2 + 105x + 1995$.

Let $(u, n) = (9, 4)$. A factorization has already been given in [FiSa10, p.4].

Let $(u, n) = (30, 92)$. Then $G_\alpha(x)$ with $\{a_0 = 1, a_1 = 7, a_2 = 13, a_{88} = 15, a_{89} = 15, a_{90} = -7522, a_{91} = -71267, a_{92} = 1\}$ and all other a_j's to be 0 has a quadratic factor $x^2 + 35x + 315$.

Let $(u, n) = (36, 86)$. Then $G_\alpha(x) = \sum_{j=0}^{n} a_j(\prod_{i=j+1}^{n}(1 + 2(u + i)))x^j$ with $a_0 = 1, a_1 = 5, a_2 = 11, a_{79} = 35, a_{80} = 42, a_{81} = 127, a_{82} = 477, a_{83} = 52, a_{84} = -5348598, a_{85} = -25850555, a_{86} = 1$ and all other $a_j's$ to be 0 has a quadratic factor $x^2 + 35x + 525$.

Let $(u, n) = (44, 716)$. Then $G_\alpha(x)$ with $\{a_0 = 1, a_1 = 13, a_2 = 1, a_{710} = 2821, a_{711} = 2418, a_{712} = 2341, a_{713} = 231, a_{714} = -6042090470827, a_{715} = -7635814082870, a_{716} = 1\}$ and all other a_j's to be 0 has a quadratic factor $x^2 + 1209x + 8463$.

For the remaining pairs (u, n) we give the details of the quadratic factor $q(x)$ of $G_\alpha(x^2)$. Here suitable integer coefficients $a_j's$ of $G_\alpha(x^2)$ are obtained by equating the remainder obtained by dividing $G_\alpha(x^2)$ with $q(x)$ to be 0 and solving the system of equations.

(u,n)	$q(x)$	(u,n)	$q(x)$
$(1,121)$	$x^2+105x+105$	$(16,106)$	$x^2+105x+105$
$(8,114)$	$x^2+105x+105$	$(20,102)$	$x^2+35x+105$
$(9,113)$	$x^2+35x+105$	$(21,101)$	$x^2+35x+315$
$(9,163)$	$x^2+345x+2415$	$(26,155)$	$x^2+627x+627$
$(9,554)$	$x^2+2415x+2415$	$(26,287)$	$x^2+3135x+3135$
$(15,107)$	$x^2+35x+105$	$(43,1158)$	x^2+1869

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