SQUARES IN PRODUCTS IN ARITHMETIC PROGRESSION WITH AT MOST TWO TERMS OMITTED AND COMMON DIFFERENCE A PRIME POWER

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Abstract. It is shown that a product of $k-2$ terms out of $k \geq 15$ terms in arithmetic progression with common difference a prime power > 1 is not a square. In fact it is not of the form by^2 where the greatest prime factor of b is less than k.

1. Introduction

For an integer $x > 1$, we denote by $P(x)$ and $\omega(x)$ the greatest prime factor of x and the number of distinct prime divisors of x, respectively. Further we put $P(1) = 1$ and $\omega(1) = 0$. Let p_i be the $i-$th prime number. Let $k \geq 4, t \geq k-2$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_t$ be integers with $0 \leq \gamma_i < k$ for $1 \leq i \leq t$. Thus $t \in \{k, k-1, k-2\}, \gamma_t \geq k-3$ and $\gamma_i = i-1$ for $1 \leq i \leq t$ if $t = k$. We put $\psi = k - t$. Let b be a positive squarefree integer and we shall always assume, unless otherwise specified, that $P(b) \leq k$. We consider the equation

$$\Delta = \Delta(n, d, k) = (n + \gamma_1 d) \cdots (n + \gamma_t d) = by^2 \tag{1.1}$$

in positive integers n, d, k, b, y, t. We prove

Theorem 1. *Let $\psi = 2, k \geq 15$ and $d \nmid n$. Assume that $P(b) < k$ if $k = 17, 19$. Then (1.1) with $\omega(d) = 1$ does not hold.*

From Theorem 1, we obtain the following results immediately.

Corollary 1. *Let $\psi = 1, k \geq 15$ and $d \nmid n$. Then (1.1) with $\omega(d) = 1$ does not hold.*

Corollary 2. *Let $\psi = 0, k \geq 15$ and $d \nmid n$. Assume that $P(b) \leq p_{\pi(k)+1}$ if $k = 17, 19$ and $P(b) \leq p_{\pi(k)+2}$ otherwise. Then (1.1) with $\omega(d) = 1$ does not hold.*

For the proof of Corollary 1, we may suppose $P(b) = k$ otherwise it follows from (2.1) and Theorem 1. Then we delete the term divisible by k on the left hand side of (1.1) and the assertion follows from Theorem 1. Further Corollary 2 also follow similarly from Theorem 1.

Let $\psi = 0$. If $d = 1$, then (1.1) has been completely solved for $P(b) < k$ by Erdős and Selfridge [ErSe75] and for $P(b) = k$ by Saradha [Sar97]. Let $d > 1$. We observe that (1.1) has infinitely many solutions if $k = 2, 3$ and $b = 1$. Also (1.1) with $k = 4$ and $b = 6$ has infinitely many solutions. It has been conjectured that (1.1) with $\gcd(n, d) = 1$ and $k \geq 5$ does not hold. Let $\omega(d) = 1$. It has been shown in [SaSh03a] for $k > 29$ and [MuSh03] for $4 \leq k \leq 29$ that (1.1) with $\gcd(n, d) = 1$ implies that either $k = 4, (n, d, b, y) = (75, 23, 6, 140)$ or

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$k = 5, P(b) = k$. In fact we shall derive the preceding result with $k \geq 10$ and $P(b) < k$ from Theorem 1, see Corollary 3.11. We refer to [LaSh06a] for results on (1.1) with $1 < \omega(d) \leq 4$.

Let $\psi = 1$. We may assume that $\gamma_1 = 0$ and $\gamma_t = k - 1$. It has been shown in [SaSh03b] that

$$\frac{6!}{5} = (12)^2, \frac{10!}{7} = (720)^2$$

are the only squares that are products of $k-1$ distinct integers out of k consecutive integers confirming a conjecture of Erdős and Selfridge [ErSe75]. This corresponds to the case $b = 1$ and $d = 1$ in (1.1). In general, it has been proved in [SaSh03b] that (1.1) with $d = 1$ and $k \geq 4$ implies that $(b, k, n) = (2, 4, 24)$ under the necessary assumption that the left hand side of (1.1) is divisible by a prime $> k$. Further it has been shown in [SaSh03a, Theorem 4] and [MuSh04a] that (1.1) with $d > 1$, $\gcd(n, d) = 1, \omega(d) = 1$ and $P(b) < k$ implies that $k \leq 8$. Thus we derive the preceding result with $k \geq 15$ from Corollary 1. Further the assumption $P(b) < k$ has been relaxed to $P(b) \leq k$ and the assumption $\gcd(n, d) = 1$ has been replaced by $d \nmid n$.

Let $\psi = 2$. Let $d = 1$. Then it has been shown in [MuSh04b, Corollary 3] that a product of $k-2$ distinct terms out of k consecutive positive integers is a square only if it is given by

$$\frac{6!}{1.5} = \frac{7!}{5.7} = 12^2, \ \frac{10!}{1.7} = \frac{11!}{7.11} = 720^2.$$

and

$$\begin{cases} \frac{4!}{2.3} = 2^2, \ \frac{6!}{4.5} = 6^2, \ \frac{8!}{2.5.7} = 24^2, \ \frac{10!}{2.3.4.6.7} = 60^2, \ \frac{9!}{2.5.7} = 72^2, \\ \frac{10!}{2.3.6.7} = 120^2, \ \frac{10!}{2.7.8} = 180^2, \ \frac{10!}{7.9} = 240^2, \ \frac{10!}{4.7} = 360^2, \\ \frac{21!}{13!.17.19} = 5040^2, \ \frac{14!}{2.3.4.11.13} = 5040^2, \ \frac{14!}{2.3.11.13} = 10080^2. \end{cases}$$

The above result corresponds to (1.1) with $b = 1$. For the general case, we have

Theorem 2. *Let $\psi = 2, d = 1$ and $k \geq 6$. Assume that the left hand side of* (1.1) *is divisible by a prime $> k$. Then* (1.1) *is not valid unless $k = 6$ and $n = 45, 240$.*

We observe that $n > k^2$ since the left hand side of (1.1) is divisible by a prime $> k$. Then the assertion follows immediately from [MuSh04b, Theorem 2].

Therefore we take $d > 1$ from now onwards in this paper. For the proof of Theorem 1, we show without loss of generality that $\gcd(n, d) = 1$. Let $\gcd(n, d) > 1$. Let $p^\beta = \gcd(n, d)$, $n' = \frac{n}{p^\beta}$ and $d' = \frac{d}{p^\beta}$. Then $d' > 1$ since $d \nmid n$. Now, by dividing $(p^\beta)^t$ on both sides of (1.1), we have

$$(n' + \gamma_1 d') \cdots (n' + \gamma_t d') = p^\epsilon b' y'^2 \tag{1.2}$$

where $y' > 0$ is an integer, b' squarefree, $P(b') < k$ when $k = 17$ and $\epsilon \in \{0, 1\}$. Since $p | d'$ and $\gcd(n', d') = 1$, we see that $p \nmid (n' + \gamma_1 d') \cdots (n' + \gamma_t d')$ giving $\epsilon = 0$ and assertion follows.

2. Notations and Preliminaries

We assume (1.1) with $\gcd(n, d) = 1$ in this section. Then we have

$$n + \gamma_i d = a_{\gamma_i} x_{\gamma_i}^2 \text{ for } 1 \leq i \leq t \tag{2.1}$$

with a_{γ_i} squarefree such that $P(a_{\gamma_i})\le \max(k-1,P(b))$. Thus (1.1) with b as the squarefree part of $a_{\gamma_1}\cdots a_{\gamma_t}$ is determined by the $t-$tuple $(a_{\gamma_1},\cdots,a_{\gamma_t})$. Also

$$n+\gamma_i d = A_{\gamma_i}X_{\gamma_i}^2 \text{ for } 1\le i\le t \tag{2.2}$$

with $P(A_{\gamma_i})\le k$ and $\gcd(X_{\gamma_i},\prod_{p\le k}p)=1$. Further we write

$$b_i=a_{\gamma_i},\ B_i=A_{\gamma_i},\ y_i=x_{\gamma_i},\ \ Y_i=X_{\gamma_i}.$$

Since $\gcd(n,d)=1$, we see from (2.1) and (2.2) that

$$(b_i,d)=(B_i,d)=(y_i,d)=(Y_i,d)=1 \ \text{ for } 1\le i\le t. \tag{2.3}$$

Let

$$R=\{b_i : 1\le i\le t\}.$$

For $b_{i_0}\in R$, let $\nu(b_{i_0})=|\{j:1\le j\le t, b_j=b_{i_0}\}|$. Let

$$T=\{1\le i\le t: Y_i=1\},\ T_1=\{1\le i\le t: Y_i>1\},\ S_1=\{B_i: i\in T_1\}.$$

Note that $Y_i>k$ for $i\in T_1$. For $i_0\in T_1$, we denote by $\nu(B_{i_0})=|\{j\in T_1: B_j=B_{i_0}\}|$.

Let

$$\delta=\min(3,\mathrm{ord}_2(d)),\ \delta'=\min(1,\mathrm{ord}_2(d)), \tag{2.4}$$

$$\eta=\begin{cases}1 & \text{if } \mathrm{ord}_2(d)\le 1,\\ 2 & \text{if } \mathrm{ord}_2(d)\ge 2,\end{cases} \tag{2.5}$$

$$\rho=\begin{cases}3 & \text{if } 3|d,\\ 1 & \text{if } 3\nmid d.\end{cases} \tag{2.6}$$

and

$$\theta=\begin{cases}1 & \text{if } d=2,4\\ 0 & \text{otherwise.}\end{cases} \tag{2.7}$$

Let $d=p^\alpha$. Then we say (d_1,d_2) is a partition of d if $d=d_1d_2$ and $\gcd(d_1,d_2)=\eta$ and we take $(1,2)$ as the partition of $d=2$. Further $(2,2)$ is the only partition if $d=4$. For $d\ne 2,4$, we see that $(\eta,\frac{d}{\eta})$ and $(\frac{d}{\eta},\eta)$ are the only distinct partitions of d. Let $b_i=b_j, i>j$. Then from (2.1) and (2.3), we have

$$\frac{(\gamma_i-\gamma_j)}{b_i}=\frac{y_i^2-y_j^2}{d}=\frac{(y_i-y_j)(y_i+y_j)}{d} \tag{2.8}$$

such that $\gcd(d,y_i-y_j,y_i+y_j)=2^{\delta'}$. Thus a pair (i,j) with $i>j$ and $b_i=b_j$ corresponds to a partition (d_1,d_2) of d such that $d_1|(y_i-y_j)$ and $d_2|(y_i+y_j)$ and this partition is unique. Similarly, we have unique partition of d corresponding to every pair (i,j) with $i>j, i,j\in T_1$ and $B_i=B_j$.

Let q be a prime $\leq k$ and coprime to d. Then the number of i's for which b_i are divisible by q is at most $\sigma_q = \lceil \frac{k}{q} \rceil$. Let $\sigma'_q = |\{b_i : q|b_i\}|$. Then $\sigma'_q \leq \sigma_q$. Let $r \geq 3$ be any positive integer. Define $F(k,r)$ and $F'(k,r)$ as

$$F(k,r) = |\{\gamma_i : P(b_i) > p_r\}| \text{ and } F'(k,r) = \sum_{i=r+1}^{\pi(k)} \sigma_{p_i}.$$

Then $|\{b_i : P(b_i) > p_r\}| \leq F(k,r) \leq F'(k,r) - \sum_{p|d, p>p_r} \sigma_p$. Let

$$\mathcal{B}_r = \{b_i : P(b_i) \leq p_r\},\ I_r = \{\gamma_i : b_i \in \mathcal{B}_r\} \text{ and } \xi_r = |I_r|.$$

We have

$$\xi_r \geq t - F(k,r) \geq t - F'(k,r) + \sum_{p|d, p>p_r} \sigma_p \tag{2.9}$$

and

$$t - |R| \geq t - |\{b_i : P(b_i) > p_r\}| - |\{b_i : P(b_i) \leq p_r\}| \tag{2.10}$$
$$\geq t - F(k,r) - |\{b_i : P(b_i) \leq p_r\}| \tag{2.11}$$
$$\geq t - F'(k,r) + \sum_{p|d, p>p_r} \sigma_p - |\{b_i : P(b_i) \leq p_r\}| \tag{2.12}$$
$$\geq t - F'(k,r) + \sum_{p|d, p>p_r} \sigma_p - 2^r. \tag{2.13}$$

We write $\mathcal{S} := \mathcal{S}(r)$ for the set of positive squarefree integers composed of primes $\leq p_r$. Let $p = 2^\delta$ if d is even and $p = P(d)$ if d is odd. Let $p = 2^\delta$. Then $b_i \equiv n(\text{mod } 2^\delta)$. Considering modulo 2^δ for elements of $\mathcal{S}(r)$, we see by induction on r that

$$|\{b_i : P(b_i) \leq p_r\}| \leq 2^{r-\delta} =: g_{2^\delta}. \tag{2.14}$$

Let $p = P(d)$. Then all b_i's are either quadratic residues mod p or non-quadratic residues mod p. We consider two sets

$$\begin{aligned} \mathcal{S}_1(p,r) &= \{s \in \mathcal{S} : \left(\frac{s}{p}\right) = 1\}, \\ \mathcal{S}_2(p,r) &= \{s \in \mathcal{S} : \left(\frac{s}{p}\right) = -1\} \end{aligned} \tag{2.15}$$

and define

$$g_p(r) = \max(|\mathcal{S}_1(p,r)|, |\mathcal{S}_2(p,r)|). \tag{2.16}$$

Then

$$|\{b_i : P(b_i) \leq p_r\}| \leq g_p. \tag{2.17}$$

In view of (2.14) and (2.17), the inequality (2.12) is improved as

$$t - |R| \geq k - \psi - F'(k,r) + \sum_{p|d, p>p_r} \sigma_p - g_p. \tag{2.18}$$

Let $r = 3, 4, 2 < p \le 220$. Then we calculate

$$g_p(r) = \begin{cases} 2^{r-2} & \text{if } p \le p_r \\ 2^{r-1} & \text{if } p > p_r \end{cases} \tag{2.19}$$

except when $r = 3, p \in \{71, 191\}$ where $g_p = 2^r$. We close this section with the following Lemmas which are independent of (1.1). The first Lemma is an estimate on $\pi(x)$ due to Dusart [Dus99].

Lemma 2.1. *We have*

$$\pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right) \text{ for } x > 1.$$

The following lemma is contained in [LaSh04, Theorem 1].

Lemma 2.2. *Let $k \ge 9$, $\gcd(n,d) = 1, n > k$ if $d = 2$ and $(n,d,k) \notin V$ where V is given by*

$$\begin{cases} n = 1,\ d = 3,\ k = 9, 10, 11, 12, 19, 22, 24, 31; \\ n = 2,\ d = 3,\ k = 12;\ n = 4,\ d = 3,\ k = 9, 10; \\ n = 2,\ d = 5,\ k = 9, 10; \\ n = 1,\ d = 7,\ k = 10. \end{cases} \tag{2.20}$$

Then

$$W(n(n+d)\cdots(n+(k-1)d)) := |\{i : 0 \le i < k, P(n+id) > k\}| \ge \pi(2k) - \pi_d(k). \tag{2.21}$$

Let $d = 2$ and $n \le k$. Then

$$W(n(n+d)\cdots(n+(k-1)d)) \ge \pi(2k) - \pi_d(k) - 1. \tag{2.22}$$

The following lemma is contained in [Lai06, Lemma 8].

Lemma 2.3. *Let s_i denote the i-th squarefree positive integer. Then*

$$\prod_{i=1}^{l} s_i \ge (1.6)^l l! \quad for \quad l \ge 286. \tag{2.23}$$

3. Lemmas for the equation (1.1)

All the lemmas in this section are under the assumption that (1.1) with $\omega(d) = 1$ is valid and we shall suppose it without reference.

Lemma 3.1. *Let ψ be fixed. Suppose that (1.1) with $P(b) \le k$ has no solution at $k = k_1$ with k_1 prime. Then (1.1) with $P(b) \le k$ and $k_1 \le k < k_2$ has no solution where k_1, k_2 are consecutive primes.*

Proof. Let k_1, k_2 be consecutive primes such that $k_1 \le k < k_2$. Suppose (n, d, b, y) is a solution of

$$(n + \gamma_1 d) \cdots (n + \gamma_t d) = by^2$$

with $P(b) \le k$. Then $P(b) \le k_1$. We observe that $\gamma_{k_1 - \psi} < k_1$ and by (2.1),

$$(n + \gamma_1 d) \cdots (n + \gamma_{k_1 - \psi} d) = b' y'^2$$

holds for some b' with $P(b') \le k_1$ giving a solution of (1.1) at $k = k_1$. This is a contradiction. □

In view of Lemma 3.1, there is no loss of generality in assuming that k is prime whenever $k \geq 23$ in the proof of Theorem 1. Therefore we suppose from now onward without reference that k is prime if $k \geq 23$. The following Lemma gives a lower bound for $|T_1|$, see [LaSh06a, Lemma 4.1].

Lemma 3.2. *Let $k \geq 4$. Then*

(3.1)
$$|T_1| > t - \frac{(k-1)\log\,(k-1) - \sum_{p|d,p<k} \max\left(0, \frac{(k-1-p)\log p}{p-1} - \log(k-2)\right)}{\log\,(n+(k-1)d)} - \pi_d(k) - 1.$$

We apply Lemmas 2.2 and 3.2 to derive the following result.

Corollary 3.3. *Let $k \geq 9$. Then we have*

(3.2)
$$|T_1| > 0.1754k \text{ for } k \geq 81.$$

and

(3.3)
$$n + \gamma_t d > \eta^2 k^2.$$

Proof. We observe that $\pi(2k) - \pi(k) > 2$ since $k \geq 9$. Therefore $P(\Delta) > k$ by Lemma 2.2. Now we see from (1.1) that

(3.4)
$$n + \gamma_t d > k^2.$$

By (3.1), $t \geq k-2$, $\pi_d(k) \leq \pi(k)$ and Lemma 2.1, we get

$$|T_1| > k - 3 - \frac{(k-1)\log k}{2\log k} - \frac{k}{\log k}\left(1 + \frac{1.2762}{\log k}\right).$$

Since the right hand side of the above inequality exceeds $0.1754k$ for $k \geq 81$, the assertion (3.2) follows.

Now we turn to the proof of (3.3). By (3.4), it suffices to consider $d = 2^\alpha$ with $\alpha > 1$. From Lemma 2.2 and (1.1), we have $n + (k-1)d > p^2_{\pi(2k)-2}$. Now we see from (3.1) that

(3.5)
$$|T_1| + \pi_d(k) - \pi(2k) > k - 3 - \frac{(k-1)\log(k-1) - (k-3)\log 2 + \log(k-2)}{2\log p_{\pi(2k)-2}} - \pi(2k)$$

and

$$|T_1| + \pi_d(k) - \pi(2k) > k - 3 - \frac{(k-1)\log k - (k-3)\log 2 + \log k}{2\log k} - \frac{2k}{\log 2k}\left(1 + \frac{1.2762}{\log 2k}\right)$$

by Lemma 2.1. When $k \geq 60$, we observe that the right hand side of the preceding inequality is positive. Therefore $|T_1| + \pi_d(k) > \pi(2k)$ implying $n + \gamma_t d > 4k^2$ for $k \geq 60$. Thus we may assume $k < 60$. Now we check that the right hand side of (3.5) is positive for $k \geq 33$. Therefore we may suppose that $k < 33$ and $n + (k-3)d \leq n + \gamma_t d \leq 4k^2$. Hence $d = 2^\alpha < \frac{4k^2}{k-3}$. For n, d, k satisfying $k < 33, d < \frac{4k^2}{k-3}, n + (k-3)d \leq 4k^2$ and $n + (k-1)d \geq p^2_{\pi(2k)-2}$, we check that there are at least three i with $0 \leq i < k$ such that $n + id$ is divisible by a prime $> k$ to the first power. This is not possible. $\square$

The next Lemma follows from (3.3) and [LaSh06a, Corollaries 3.5, 3.7].

Lemma 3.4. *For any pair (i,j) with $b_i=b_j$, the partition $(d\eta^{-1},\eta)$ of d is not possible. Further $\nu(b_{i_0})\le 2^{1-\theta}$ and $\nu(B_{i_0})\le 2^{1-\theta}$.*

The following Lemma follows from (3.3), Lemma 3.4 and [LaSh06a, Corollary 3.9].

Lemma 3.5. *Let $z_0\in\{2,3,5\}$. Assume that either d is odd or $8|d$ and $z_0=5$ if $8|d$. Further let $d=\theta_1(k-1)^2, n=\theta_2(k-1)^3$ with $\theta_1>0$ and $\theta_2>0$. Suppose that $t-|R|\ge z_0$. Then we have the partition $(\eta,d\eta^{-1})$ of d such that*

$$d\eta^{-1}<\frac{4(k-1)}{q_2} \tag{3.6}$$

and

$$\theta_2<\frac{1}{2}\left\{\frac{1}{q_1q_2}-\theta_1+\sqrt{\frac{1}{(q_1q_2)^2}+\frac{\theta_1}{q_1q_2}}\right\} \tag{3.7}$$

hold with $q_1\ge Q_1, q_2\ge Q_2$ where (Q_1,Q_2) is given by $(1,1),(2,2),(4,4)$ according as $z_0=2,3,5$, respectively when d is odd and $(Q_1,Q_2)=(2,8)$ when $z_0=5, 8|d$.

Lemma 3.6. *Let $z_1>1$ be a real number, $h_0>i_0\ge 0$ be integers such that $\prod_{b_i\in R}b_i\ge z_1^{|R|-i_0}(|R|-i_0)!$ for $|R|\ge h_0$. Suppose that $t-|R|<g$ and let $g_1=k-t+g-1+i_0$. For $k\ge h_0+g_1$ and for any real number $\mathfrak{m}>1$, we have*

$$\begin{aligned} g_1 > &\frac{k\log\left(\frac{z_1\mathfrak{n}_0}{2.71851}\prod\limits_{p\le\mathfrak{m}}p^{\frac{2}{p^2-1}}\right)+(k+\frac{1}{2})\log(1-\frac{g_1}{k})}{\log(k-g_1)-1+\log z_1} - \\ &\frac{(1.5\pi(\mathfrak{m})-.5\ell-1)\log k+\log\left(\mathfrak{n}_1^{-1}\mathfrak{n}_2\prod\limits_{p\le\mathfrak{m}}p^{.5+\frac{2}{p^2-1}}\right)}{\log(k-g_1)-1+\log z_1}\end{aligned} \tag{3.8}$$

where

$$\ell=|\{p\le\mathfrak{m}:p|d\}|,\ \mathfrak{n}_0=\prod_{\substack{p|d\\ p\le\mathfrak{m}}}p^{\frac{1}{p+1}},\ \mathfrak{n}_1=\prod_{\substack{p|d\\ p\le\mathfrak{m}}}p^{\frac{p-1}{2(p+1)}}\text{ and }\mathfrak{n}_2=\begin{cases}2^{\frac{1}{6}} & \text{if } 2\nmid d\\ 1 & \text{otherwise.}\end{cases}$$

For a proof, see [LaSh06a, Lemma 5.3]. The assumption $\omega(d)=1$ is not necessary for Lemmas 3.1, 3.2, 3.6 and Corollary 3.3.

Lemma 3.7. *We have*

$$t-|R|\ge\begin{cases}5 \text{ for } k\ge 81\\ 5-\psi \text{ for } k\ge 55\\ 4-\psi \text{ for } k\ge 28, k\ne 31\\ 3-\psi \text{ for } k=31.\end{cases} \tag{3.9}$$

Proof. Suppose $t-|R|<5$ and $k\ge 292$. Then $|R|\ge 286$ since $t\ge k-2$ and $\prod_{b_i\in R}b_i\ge(1.6)^{|R|}(|R|)!$ by (2.23). We observe that (3.8) hold for $k\ge 292$ with $i_0=0, h_0=286$, $z_1=1.6, g_1=6, \mathfrak{m}=17, \ell=0, \mathfrak{n}_0=1, \mathfrak{n}_1=1$ and $\mathfrak{n}_2=2^{\frac{1}{6}}$. We check that the right hand side of (3.8) is an increasing function of k and it exceeds g_1 at $k=292$ which is a contradiction.

Therefore $t-|R|\geq 5$ for $k\geq 292$. Thus we may assume that $k<292$. By taking $r=3$ for $k<50$, $r=4$ for $50\leq k\leq 181$ and $r=5$ for $181<k<292$ in (2.11) and (2.13), we get $t-|R|\geq k-\psi-F'(k,r)-2^r\geq 7-\psi, 5-\psi, 4-\psi$ for $k\geq 81, 55, 28$, respectively except at $k=29,31,43,47$ where $t-|R|\geq k-\psi-F(k,r)-2^r\geq k-\psi-F'(k,r)-2^r=3-\psi$. We may suppose that $k=29,43,47$, $t-|R|=3-\psi$ and $F(k,r)=F'(k,r)$. Further we may assume that for each prime $7\leq p\leq k$, there are exactly σ_p number of i's for which $p|b_i$ and for any i, $pq\nmid b_i$ whenever $7\leq q\leq k, q\neq p$. Now we get a contradiction by considering the i's for which b_i's are divisible by primes $7,13; 7,41; 23,11$ when $k=29,43,47$, respectively. For instance let $k=29$. Then $7|b_i$ for $i\in\{0,7,14,21,28\}$. Then $13|b_i$ for $i\in\{h+13j : 0\leq j\leq 2\}$ with $h=0,1,2$. This is not possible. $\square$

Lemma 3.8. *Let $9\leq k\leq 23$ and d odd. Suppose that $t-|R|\geq 3$ for $k=23$ and $t-|R|\geq 2$ for $k<23$. Then* (1.1) *does not hold.*

Proof. Suppose (1.1) holds. Let $Q=2$ if $k=23$ and $Q=1$ if $k<23$. We now apply Lemma 3.5 with $z_0=3$ for $k=23$ and $z_0=2$ for $k<23$ to get $d<\frac{4}{Q}(k-1)$, $\theta_1<\frac{4}{Q(k-1)}$ and

$$\theta_1+\theta_2<\frac{1}{2}\left\{\frac{1}{Q^2}+\frac{4}{Q(k-1)}+\sqrt{\frac{1}{Q^4}+\frac{4}{Q^3(k-1)}}\right\}=:\Omega(k-1).$$

Further from (2.21), we have $n+(k-1)d\geq n+\gamma_t d\geq p^2_{\pi(2k)-2}$. Therefore $p^\alpha=d<\frac{4}{Q}(k-1)$ and $p^2_{\pi(2k)-2}\leq n+(k-1)d<(k-1)^3\Omega(k-1)$. For these possibilities of n,d and k, we check that there are at least three i with $0\leq i<k$ such that $n+id$ is divisible by a prime $>k$ to an odd power. This contradicts (1.1). $\square$

Lemma 3.9. *Equation* (1.1) *with $k\geq 9$ implies that $t-|R|\leq 1$.*

Proof. Assume that $k\geq 9$ and $t-|R|\geq 2$. Let $d=2,4$. Then $|R|\leq t-2$ contradicting $|R|=t$ by Lemma 3.4. Thus $d\neq 2,4$. By Lemma 3.4, we have $\nu(b_{i_0})\leq 2$ and $\nu(B_{i_0})\leq 2$.

Let $k\geq 81$. Then $t-|R|\geq 5$ by Lemma 3.7. Now we derive from Lemma 3.5 with $z_0=5$ that $d<k-1$ giving $\theta_1<\frac{1}{k-1}$ and hence

$$n+(k-1)d=(\theta_1+\theta_2)(k-1)^3<\frac{(k-1)^3}{2}\left\{\frac{1}{16}+\frac{1}{k-1}+\sqrt{\frac{1}{(16)^2}+\frac{1}{16(k-1)}}\right\}.$$

On the other hand, we get from (3.2) and $\nu(B_{i_0})\leq 2$ that $n+(k-1)d\geq\frac{0.1754k}{2}k^2\geq 0.1754\frac{k^3}{2}$. Comparing the upper and lower bounds of $n+(k-1)d$, we obtain

$$0.1754<\left\{\frac{1}{16}+\frac{1}{k-1}+\sqrt{\frac{1}{(16)^2}+\frac{1}{16(k-1)}}\right\}\leq 0.144$$

since $k\geq 81$. This is a contradiction.

Thus $k<81$. Let d be even. Then $8|d$ and we see from $\nu(a_i)\leq 2$ and (2.14) that $\xi_r\leq 2g_{2^\delta}\leq 2^{r-2}$. Let $r=3$. From (2.9), we get $k-2-F'(k,r)\leq\xi_r\leq 2^{r-2}$. We find $k-2-F'(k,r)>2^{r-2}$ by computation. This is a contradiction.

Thus d is odd. Since $\psi\leq 2$, we get from Lemmas 3.7 and 3.5 with $z_0=3,2$ that $d<2(k-1)$ if $k\geq 55$ and $d<4(k-1)$ if $k<55$. Since $g_p(r)\leq 2^{r-1}$ for $r=4, p<220$ by (2.19), we get from (2.18) with $r=4$ that $t-|R|\geq k-2-F'(k,r)-2^{r-1}$ which is ≥ 5 for $k\geq 29$ and ≥ 3 for $k=23$.

Let $k\geq 29$. Then we get from Lemma 3.5 with $z_0=5$ that $d<k-1$. By taking $r=3$ for $k<53$ and $r=4$ for $53\leq k<81$, we derive from (2.17), (2.19), $\nu(a_i)\leq 2$ and (2.9) that $k-2-F'(k,r)\leq \xi_r\leq 2g_p\leq 2^r$. On the other hand, we check by computation that $k-2-F'(k,r)>2^r$. This is a contradiction.

Thus $k\leq 23$. Then $t-|R|\geq 3$ for $k=23$ and $t-|R|\geq 2$ for $k<23$. By Lemma 3.8, this is not possible. □

Corollary 3.10. *Let $k\geq 9$. Equation* (1.1) *implies that either $k\leq 23$ or $k=31$. Also $P(d)>k$.*

Proof. By Lemmas 3.7 and 3.9, we see that either $k\leq 23$ or $k=31$. Suppose that $P(d)\leq k$. Since $g_{P(d)}(r)\leq 2^{r-1}$ for $r=3$ by (2.19), we get from (2.18) with $r=3$ that $t-|R|\geq k-2-F'(k,r)-2^{r-1}\geq 2$ except at $k=9$ where $t-|R|=1$. This contradicts Lemma 3.9 for $k>9$. Let $k=9$. By taking $r=4$, we get from $g_{P(d)}(r)\leq 2^{r-2}$ by (2.19) and (2.18) that $t-|R|\geq k-2-F'(k,4)-2^{4-2}\geq 2$. This contradicts Lemma 3.9. □

As a consequence, we derive the following Corollary which is [SaSh03a, Theorem 1 (*ii*)].

Corollary 3.11. *Let $\psi=0$. Equation* (1.1) *with $P(b)<k$ implies that $k\leq 9$.*

Proof. Let $k\geq 10$. By Corollary 3.10, we see that either $k\leq 23$ or $k=31$. Let $k=10$. Then we get from (2.13) with $r=2$ that $t-|R|\geq k-F'(k,r)-2^r=2$ contradicting Lemma 3.9. Thus (1.1) does not hold at $k=10$. By induction, we may assume $k\in\{12,14,18,20\}$ and further there is at most one i for which $p|a_i$ with $p=k-1$. We take $r=2$ for $k=12,14$ and $r=3$ for $k=18,20$. Now we get from $|\{b_i : P(b_i)>p_r\}|\leq F'(k,r)-1$ and (2.10) that $t-|R|\geq k-F'(k,r)+1-2^r\geq 2$. This contradicts Lemma 3.9. □

4. Proof of Theorem 1

Suppose that the assumptions of Theorem 1 are satisfied and assume (1.1) with $\omega(d)=1$. By Corollary 3.10, we have $P(d)>k$ and further we restrict to $k\leq 23$ and $k=31$. Also $t-|R|\leq 1$ by Lemma 3.9. Further it suffices to prove the assertion for $k\in\{15,18,20,23,31\}$ since the cases $k=16,17$; $k=19$ and $k=21,22$ follows from those of $k=15,18$ and 20, respectively.

We shall arrive at a contradiction by showing $t-|R|\geq 2$. For a prime $p\leq k$, we observe that $p\nmid d$ and let i_p be such that $0\leq i_p<p$ and $p|n+i_pd$. For any subset $\mathcal{I}\subseteq[0,k)\cap\mathbb{Z}$ and primes p_1 and p_2, we define

$$\mathcal{I}_1=\{i\in\mathcal{I}:\left(\frac{i-i_{p_1}}{p_1}\right)=\left(\frac{i-i_{p_2}}{p_2}\right)\}\text{ and }\mathcal{I}_2=\{i\in\mathcal{I}:\left(\frac{i-i_{p_1}}{p_1}\right)\neq\left(\frac{i-i_{p_2}}{p_2}\right)\}.$$

Then from $\left(\frac{a_i}{p}\right)=\left(\frac{i-i_p}{p}\right)\left(\frac{d}{p}\right)$, we see that either

$$\left(\frac{a_i}{p_1}\right)\neq\left(\frac{a_i}{p_2}\right)\text{ for all } i\in\mathcal{I}_1\text{ and }\left(\frac{a_i}{p_1}\right)=\left(\frac{a_i}{p_2}\right)\text{ for all } i\in\mathcal{I}_2 \tag{4.1}$$

or

$$\left(\frac{a_i}{p_1}\right)\neq\left(\frac{a_i}{p_2}\right)\text{ for all } i\in\mathcal{I}_2\text{ and }\left(\frac{a_i}{p_1}\right)=\left(\frac{a_i}{p_2}\right)\text{ for all } i\in\mathcal{I}_1. \tag{4.2}$$

We define $(\mathcal{M},\mathcal{B}) = (\mathcal{I}_1,\mathcal{I}_2)$ in the case (4.1) and $(\mathcal{M},\mathcal{B}) = (\mathcal{I}_2,\mathcal{I}_1)$ in the case (4.2). We call $(\mathcal{I}_1,\mathcal{I}_2,\mathcal{M},\mathcal{B}) = (\mathcal{I}_1^k,\mathcal{I}_2^k,\mathcal{M}^k,\mathcal{B}^k)$ when $\mathcal{I} = [0,k)\cap\mathbb{Z}$. Then for any $\mathcal{I}\subseteq[0,k)\cap\mathbb{Z}$, we have

$$\mathcal{I}_1\subseteq\mathcal{I}_1^k, \mathcal{I}_2\subseteq\mathcal{I}_2^k, \mathcal{M}\subseteq\mathcal{M}^k, \mathcal{B}\subseteq\mathcal{B}^k$$

and

$$|\mathcal{M}|\geq|\mathcal{M}^k|-(k-|\mathcal{I}|),\ |\mathcal{B}|\geq|\mathcal{B}^k|-(k-|\mathcal{I}|). \tag{4.3}$$

By taking $m = n+\gamma_t d$ and $\gamma_i' = \gamma_t-\gamma_{t-i+1}$, we re-write (1.1) as

$$(m-\gamma_1' d)\cdots(m-\gamma_t' d) = by^2. \tag{4.4}$$

The equation (4.4) is called the mirror image of (1.1). The corresponding t-tuple $(a_{\gamma_1'}, a_{\gamma_2'},\cdots,a_{\gamma_t'})$ is called the mirror image of $(a_{\gamma_1},\cdots,a_{\gamma_t})$.

4.1. **The case** $k = 15$. Then $\sigma_7' = 3$ implies that $7|a_{7j}$ for $j = 0,1,2$ and $\sigma_7'\leq 2$ if $7\nmid a_0a_7a_{14}$. Similarly $\sigma_{13}' = 2$ implies $13|a_0, 13|a_{13}$ or $13|a_1, 13|a_{14}$ and $\sigma_{13}'\leq 1$ otherwise. Thus $|\{a_i : 7|a_i \text{ or } 13|a_i\}|\leq 4$. It suffices to have

$$|\{a_i : p|a_i \text{ for } 5\leq p\leq 13\}|\leq 7 \tag{4.5}$$

since then $t-|R|\geq k-2-|\{a_i : p|a_i \text{ for } 5\leq p\leq 13\}|-4\geq 2$ by (2.10) with $r = 2$, a contradiction.

Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{\gamma_1,\gamma_2,\cdots,\gamma_t\}$. We observe that $P(a_i)\leq 7$ for $i\in\mathcal{M}\cup\mathcal{B}$. Since $\left(\frac{5}{11}\right)\neq\left(\frac{5}{13}\right)$ but $\left(\frac{q}{11}\right) = \left(\frac{q}{13}\right)$ for a prime $q<k$ other than $5,11,13$, we observe that $5|a_i$ whenever $i\in\mathcal{M}$. Since $\sigma_5\leq 3$ and $|\mathcal{I}| = k-2$, we obtain from (4.3) that $|\mathcal{M}^k|\leq 5$ and $5|a_i$ for at least $|\mathcal{M}^k|-2$ i's with $i\in\mathcal{M}^k$. Further $5\nmid a_i$ for $i\in\mathcal{B}$.

By taking the mirror image (4.4) of (1.1), we may suppose that $0\leq i_{13}\leq 7$. For each possibility $0\leq i_{11}<11$ and $0\leq i_{13}\leq 7$, we compute $|\mathcal{I}_1^k|, |\mathcal{I}_2^k|$ and restrict to those pairs (i_{11},i_{13}) with $\min(|\mathcal{I}_1^k|,|\mathcal{I}_2^k|)\leq 5$. We see from $\max(|\mathcal{I}_1^k|,|\mathcal{I}_2^k|)\geq 6$ that $\mathcal{M}^k$ is exactly one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality and hence $\mathcal{B}^k$ is the other. Now we restrict to those pairs (i_{11},i_{13}) for which there are at most two elements $i\in\mathcal{M}^k$ such that $5\nmid a_i$. There are 31 such pairs. By counting the multiples of 11 and 13 and also the maximum multiples of 5 in $\mathcal{M}^k$ and the maximum number of multiples of 7 in $\mathcal{B}^k$, we again restrict to those pairs (i_{11},i_{13}) which do not satisfy (4.5). With this procedure, all pairs (i_{11},i_{13}) are excluded other than

$$(0,6),(1,3),(2,4),(3,5),(4,6),(5,3). \tag{4.6}$$

We first explain the procedure by showing how $(i_{11},i_{13}) = (0,0)$ is excluded. Now $\mathcal{M}^k = \{5,10\}$ and $\mathcal{B}^k = \{1,2,3,4,6,7,8,9,12,14\}$. Then there are 3 multiples of 11 and 13, at most 2 multiples of 5 in $\mathcal{M}^k$ and at most 2 multiples of 7 in $\mathcal{B}^k$ implying (4.5). Thus $(i_{11},i_{13}) = (0,0)$ is excluded.

Let $(i_{11},i_{13}) = (5,3)$. Then $\mathcal{M}^k = \{1,6,11\}$ and $\mathcal{B}^k = \{0,2,4,7,8,9,10,12,13,14\}$ giving $i_5 = 1$ and $5|a_1a_6a_{11}$. We may assume that $7|a_i$ for $i\in\{0,7,14\}$ otherwise (4.5) holds. By taking $p_1 = 5, p_2 = 11$ and $\mathcal{I} = \mathcal{B}^k$, we get $\mathcal{I}_1 = \{4,10,13\}$ and $\mathcal{I}_2 = \{0,2,7,8,9,12,14\}$. Since $\left(\frac{2}{5}\right) = \left(\frac{2}{11}\right)$, $\left(\frac{7}{5}\right) = \left(\frac{7}{11}\right)$ and $\left(\frac{3}{5}\right)\neq\left(\frac{3}{11}\right)$, we observe that $3|a_i$ for $i\in\mathcal{I}_1\cap\mathcal{B}$ and $3\nmid a_i$ for $i\in\mathcal{I}_2\cap\mathcal{B}$. Thus $a_i\in\{3,6\}$ for $i\in\mathcal{I}_1\cap\mathcal{B}$ and $a_i\in\{1,2,7,14\}$ for $i\in\mathcal{I}_2\cap\mathcal{B}$. Now from $\left(\frac{a_i}{7}\right) = \left(\frac{i-0}{7}\right)\left(\frac{d}{7}\right)$ and $\left(\frac{3}{7}\right) = \left(\frac{6}{7}\right)$, we see that at least one of $4,10,13$ is not in $\mathcal{B}$ implying $i\notin\mathcal{B}$ for at most one $i\in\mathcal{I}_2$. Therefore there are distinct pairs (i_1,i_2) and (j_1,j_2) with

$i_1,i_2,j_1,j_2\in\mathcal{I}_2\cap\mathcal{B}$ such that $a_{i_1}=a_{i_2}, i_1>i_2$ and $a_{j_1}=a_{j_2}, j_1>j_2$ giving $t-|R|\geq 2$. This is a contradiction. Similarly, all other pairs (i_{11},i_{13}) in (4.6) are excluded.

4.2. **The case $k=18$.** We may assume that $\sigma'_{17}=1$ and $17\nmid a_0a_1a_2a_{15}a_{16}a_{17}$ otherwise the assertion follows the case $k=15$. If $|\{a_i:P(a_i)=5\}|=4$, we see from $\{a_i:P(a_i)=5\}\subseteq\{5,10,15,30\}$ that $a_{i_5}a_{i_5+5}a_{i_5+10}a_{i_5+15}=(150)^2$ implying $(n+i_5d)(n+(i_5+5)d)(n+(i_5+10)d)(n+(i_5+15)d)$ is a square, contradicting Eulers' result for $k=4$. Thus we have $|\{a_i:P(a_i)=5\}|\leq 3$. Further for each prime $7\leq p\leq 13$, we may also assume that $\sigma'_p=\sigma_p$ and for any i, $pq\nmid a_i$ whenever $7\leq q\leq 17, q\neq p$ otherwise $t-|R|\geq k-2-\sum_{7\leq p\leq 17}\sigma'_p-3-4\geq 2$ by (2.10) with $r=2$.

Let $p_1=11$, $p_2=13$ and $\mathcal{I}=\{\gamma_1,\gamma_2,\cdots,\gamma_t\}$. Since $\left(\frac{5}{11}\right)\neq\left(\frac{5}{13}\right)$ and $\left(\frac{17}{11}\right)\neq\left(\frac{17}{13}\right)$ but $\left(\frac{q}{11}\right)=\left(\frac{q}{13}\right)$ for $q<k, q\neq 5,17,11,13$, we observe that for $i\in\mathcal{M}$, exactly one of $5|a_i$ or $17|a_i$ holds. Thus $5\cdot 17\nmid a_i$ whenever $i\in\mathcal{M}$. For $i\in\mathcal{B}$, either $5\nmid a_i, 17\nmid a_i$ or $5|a_i, 17|a_i$. Thus for $i\in\mathcal{B}$, we have $P(a_i)\leq 7$ except possibly for one i for which $5\cdot 17|a_i$. Since $\sigma_5\leq 4$ and $\sigma'_{17}\leq 1$, we obtain $|\mathcal{M}^k|\leq 7$ and $5|a_i$ for at least $|\mathcal{M}^k|-3$ i's with $i\in\mathcal{M}^k$. Hence $|\mathcal{M}^k|=7$ implies that either

$$\{a+5j:0\leq j\leq 3\}\subseteq\mathcal{I}_1^k \text{ or } \{b+5j:0\leq j\leq 3\}\subseteq\mathcal{I}_2^k \tag{4.7}$$

for some $a,b\in\{0,1,2\}$.

Since $\sigma'_{11}=2$ and $\sigma'_{13}=2$, we may suppose that $0\leq i_{11}\leq 6$ and $0\leq i_{13}\leq 4$. Further $i_{11}\neq i_{13}$ and $i_{11}+11\neq i_{13}+13$. We observe that either $\min(|\mathcal{I}_1^k|,|\mathcal{I}_2^k|)\leq 6$ or $|\mathcal{I}_1^k|=|\mathcal{I}_2^k|=7$. For pairs (i_{11},i_{13}) with $|\mathcal{I}_1^k|=|\mathcal{I}_2^k|=7$, we check that (4.7) is not valid. Thus we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|,|\mathcal{I}_2^k|)\leq 6$. There are 16 such pairs. Further we see from $\max(|\mathcal{I}_1^k|,|\mathcal{I}_2^k|)\geq 8$ that $\mathcal{M}^k$ is exactly one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality and hence $\mathcal{B}^k$ is the other one. Now we restrict to those pairs (i_{11},i_{13}) for which $5|a_i$ for at least 3 elements $i\in\mathcal{M}^k$ otherwise $t-|R|\geq k-2-\sum_{7\leq p\leq 17}\sigma'_p-2-4\geq 2$ by (2.10) with $r=2$. We find that $(i_{11},i_{13})\in\{(1,3),(2,4),(4,0),(5,1)\}$. For these pairs (i_{11},i_{13}), we check that there are at most 4 multiples a_i of 5 and 17 with $i\in\mathcal{M}^k\cup\mathcal{B}^k$. Thus if $|\{i:i\in\mathcal{B},7|a_i\}|\leq 2$, then $t-|R|\geq 2$ by (2.10) with $r=2$. Therefore we may assume that $|\{i:i\in\mathcal{B},7|a_i\}|=3$ and hence $|\{i:i\in\mathcal{B}^k,7|a_i\}|=3$. We now restrict to those pairs (i_{11},i_{13}) for which $|\{i:i\in\mathcal{B}^k,7|a_i\}|=3$. They are given by $(i_{11},i_{13})\in\{(2,4),(4,0)\}$.

Let $(i_{11},i_{13})=(2,4)$. Then by taking $p_1=11$ and $p_2=13$ as above, we have $\mathcal{M}^k=\{1,6,8,11\}$ and $\mathcal{B}^k=\{0,3,5,7,9,10,12,14,15,16\}$ giving $i_5=1$ and $5|a_1a_6a_{11}$. We may assume that $17|a_8$ since $17\nmid a_{16}$. Hence $P(a_i)\leq 7$ for $i\in\mathcal{B}$. Consequently $P(a_i)\leq 7$ for exactly 8 elements $i\in\mathcal{B}^k$ and other 2 elements are not in $\mathcal{B}$. Further $7|a_i$ for $i\in\{0,7,14\}$ and $0,7,14\in\mathcal{B}$. Now we take $p_1=5, p_2=11$ and $\mathcal{I}=\mathcal{B}^k$ to get $\mathcal{I}_1=\{0,5,7,9\}$ and $\mathcal{I}_2=\{3,10,12,14,15\}$. Since $\left(\frac{2}{5}\right)=\left(\frac{2}{11}\right)$, $\left(\frac{7}{5}\right)=\left(\frac{7}{11}\right)$ and $\left(\frac{3}{5}\right)\neq\left(\frac{3}{11}\right)$, we observe that either $3|a_i$ for $i\in\mathcal{I}_1\cap\mathcal{B}$ or $3|a_i$ for $i\in\mathcal{I}_2\cap\mathcal{B}$. The former possibility is excluded since $0,7\in\mathcal{I}_1\cap\mathcal{B}$ and the latter is not possible since $14\in\mathcal{I}_2\cap\mathcal{B}$. The other case $(i_{11},i_{13})=(4,0)$ is excluded similarly.

4.3. **The case $k=20$.** We may assume that $\sigma'_{19}=1$ and $19\nmid a_0a_{19}$ otherwise the assertion follows from the case $k=18$. Also we have $|\{a_i:P(a_i)=5\}|\leq 3$ by Eulers' result for $k=4$. Further for each prime $7\leq p\leq 17$, we may also assume that $\sigma'_p=\sigma_p$ and for any i, $pq\nmid a_i$ whenever $7\leq p<q\leq 19$ otherwise $t-|R|\geq k-2-\sum_{7\leq p\leq 17}\sigma'_p-3-4\geq 2$ by (2.10) with $r=2$.

Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$. Then as in the case $k = 18$, we observe that for $i \in \mathcal{M}$, exactly one of $5|a_i$ or $17|a_i$ holds but $5 \cdot 17 \nmid a_i$. For $i \in \mathcal{B}$, either $5 \nmid a_i, 17 \nmid a_i$ or $5|a_i, 17|a_i$. Since $\sigma_5 \le 4$ and $\sigma_{17} \le 2$, we obtain $|\mathcal{M}^k| \le 8$ and $5|a_i$ for at least $|\mathcal{M}^k| - 4$ i's with $i \in \mathcal{M}^k$. Hence $|\mathcal{M}^k| = 8$ implies that either

$$\{a + 5j : 0 \le j \le 3\} \subseteq \mathcal{I}_1^k \text{ or } \{b + 5j : 0 \le j \le 3\} \subseteq \mathcal{I}_2^k \tag{4.8}$$

for some $a, b \in \{0, 1, 2, 3, 4\}$.

Since $\sigma'_{11} = 2$ and $\sigma'_{13} = 2$, we may suppose that $0 \le i_{11} \le 8$ and $0 \le i_{13} \le 6$. Further $i_{11} \ne i_{13}$ and $i_{11}+11 \ne i_{13}+13$. We observe that either $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \le 7$ or $|\mathcal{I}_1^k| = |\mathcal{I}_2^k| = 8$. For pairs (i_{11}, i_{13}) with $|\mathcal{I}_1^k| = |\mathcal{I}_2^k| = 8$, we check that (4.8) is not valid. Thus we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \le 7$. There are 40 such pairs. Further we see from $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \ge 8$ that $\mathcal{M}^k$ is the one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality and hence $\mathcal{B}^k$ is the other. Now we restrict to those pairs (i_{11}, i_{13}) for which $5|a_i$ for at least 3 elements $i \in \mathcal{M}^k$ otherwise $t - |R| \ge k - 2 - 1 - \sum_{7 \le p \le 17} \sigma'_p - 2 - 4 \ge 2$ by (2.10) with $r = 2$. We are left with 22 such pairs. Further by (4.3) and $|\mathcal{I}| = k - 2$, we restrict to those pairs (i_{11}, i_{13}) for which there are at least $|\mathcal{M}^k| - 2$ elements $i \in \mathcal{M}^k$ such that $5|a_i$ or $17|a_i$. There are 12 such pairs (i_{11}, i_{13}) and for these pairs, we check that there are at most 4 multiples a_i of 5 and 17 with $i \in \mathcal{M}^k \cup \mathcal{B}^k$. This implies $t - |R| \ge k - 2 - 1 - 4 - \sum_{11 \le p \le 13} \sigma'_p - 4 \ge 2$ by (2.10) with $r = 2$. For instance, let $(i_{11}, i_{13}) = (3, 5)$. Then $\mathcal{M}^k = \{2, 7, 9, 12\}$ and $\mathcal{B}^k = \{0, 1, 4, 6, 8, 10, 11, 13, 15, 16, 17, 19\}$. Since $5|a_i$ for at least three elements $i \in \mathcal{M}^k$, we get $5|a_i$ for $i \in \{2, 7, 12\}$ giving $i_5 = 2$. Further $17|a_9$ or $5 \cdot 17|a_{17}$ giving 4 multiples a_i of 5 and 17 with $i \in \mathcal{M}^k \cup \mathcal{B}^k$. Thus $t - |R| \ge 2$ as above.

4.4. **The case** $k = 23$**.** We may assume that $\sigma'_{23} = 1$ and $23 \nmid a_i$ for $0 \le i \le 2$ and $20 \le i < 23$ otherwise the assertion follows from the case $k = 20$. We have $\sigma'_{11} = 3$ if $11|a_{11j}$ with $j = 0, 1, 2$ and $\sigma'_{11} \le 2$ if $11 \nmid a_0 a_{11} a_{22}$. Also $\sigma'_7 = 4$ implies that $7|a_{7j}$ or $7|a_{1+7j}$ with $0 \le j \le 3$ and $\sigma'_7 \le 3$ otherwise. Thus $|\{a_i : 7|a_i \text{ or } 11|a_i\}| \le 6$. Further by Eulers result for $k = 4$, we obtain $|\{a_i : P(a_i) = 5\}| \le 4$. If

$$|\{a_i : p|a_i, 5 \le p \le 23\} \le 4 + \sum_{7 \le p \le 23} \sigma_p - 1 - 2 = 15,$$

then we get from (2.10) with $r = 2$ that $t - |R| \ge k - 2 - 15 - 4 = 2$, a contradiction. Therefore we have

$$4 + \sum_{7 \le p \le 23} \sigma_p - 2 \le |\{a_i : p|a_i, 5 \le p \le 23\} \le 4 + \sum_{7 \le p \le 19} \sigma_p - 1. \tag{4.9}$$

Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$. Then as in the case $k = 18$, we observe that for $i \in \mathcal{M}$, exactly one of $5|a_i$ or $17|a_i$ holds but $5 \cdot 17 \nmid a_i$. Further for $i \in \mathcal{B}$, either $5 \nmid a_i, 17 \nmid a_i$ or $5 \cdot 17|a_i$. Since $\sigma_5 \le 5$ and $\sigma_{17} \le 2$, we obtain $|\mathcal{M}^k| \le 9$ and $5|a_i$ for at least $|\mathcal{M}^k| - 4$ i's with $i \in \mathcal{M}^k$.

By taking the mirror image (4.4) of (1.1), we may suppose that $0 \le i_{11} < 11$ and $0 \le i_{13} \le 11$. For each of these pairs (i_{11}, i_{13}), we compute $|\mathcal{I}_1^k|, |\mathcal{I}_2^k|$ and check that $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) > 9$. First we restrict to those pairs (i_{11}, i_{13}) for which $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \le 9$. Therefore $\mathcal{M}^k$ is exactly one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality and hence $\mathcal{B}^k$ is the other set. Now we restrict to those pairs (i_{11}, i_{13}) for which there are at least $|\mathcal{M}^k| - 2$ elements $i \in \mathcal{M}^k$ such that either $5|a_i$ or $17|a_i$. There are 31 such pairs. Next we count the number of multiples of $11, 13$, maximum multiples of $5, 17$ in $\mathcal{M}^k \cup \mathcal{B}^k$ and $7, 19$ in $\mathcal{B}^k$ to check that (4.9) is not valid. This is a contradiction. For example, let $(i_{11}, i_{13}) = (0, 2)$. Then $\mathcal{M}^k = \{4, 6, 9, 18, 19, 20\}$ and $\mathcal{B}^k = \{1, 3, 5, 7, 8, 10, 12, 13, 14, 16, 17, 21\}$ giving $5|a_i$ for $i \in \{4, 9, 19\}$, $i_5 = 4$. Further $17|a_i$ for exactly one $i \in \{6, 18, 20\}$ and other two i's in $\{6, 18, 20\}$ deleted. Thus $5 \cdot 17 \nmid a_{14}$ so that (4.9) is not valid. For another example, let $(i_{11}, i_{13}) = (4, 0)$. Then $\mathcal{M}^k = \{6, 9, 11, 16, 21\}$ and $\mathcal{B}^k = \{1, 2, 3, 5, 7, 8, 10, 12, 14, 17, 18, 19, 20, 22\}$ giving $5|a_i$ for $i \in \{6, 11, 16, 21\}$, $i_5 = 1$. Further we have either $17|a_9$, $\gcd(5 \cdot 17, a_1) = 1$ or $9 \notin \mathcal{M}, 5 \cdot 17|a_1$. Now $7|a_i$ for at most 3 elements $i \in \mathcal{B}^k$ so that (4.9) is not satisfied. This is a contradiction.

4.5. **The case $k = 31$.** From $t - |R| \geq k - 2 - \sum_{7 \leq p \leq 31} \sigma'_p - 8 \geq k - 2 - \sum_{7 \leq p \leq 31} \sigma_p - 8 = 1$ by (2.10) and (2.13) with $r = 3$, we may assume for each prime $7 \leq p \leq 31$ that $\sigma'_p = \sigma_p$ and for any i, $pq \nmid a_i$ whenever $7 \leq p < q \leq 31$. Let $\mathcal{I} = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$. By taking the mirror image (4.4) of (1.1) and $\sigma_{19} = \sigma_{29} = 2$, we may assume that $i_{29} = 0$ and $1 \leq i_{19} \leq 11, i_{19} \neq 10$. For $p \leq 31$ with $p \neq 19, 29$, since $\left(\frac{p}{19}\right) \neq \left(\frac{p}{29}\right)$ if and only if $p = 11, 13, 17$, we observe that for $i \in \mathcal{M}$, either $11|a_i$ or $13|a_i$ or $17|a_i$. Since $\sigma_{11} + \sigma_{13} + \sigma_{17} \leq 8$, we obtain $|\mathcal{M}^k| \leq 10$ and $p|a_i$ for at least $|\mathcal{M}^k| - 2$ elements $i \in \mathcal{M}^k$ and $p \in \{11, 13, 17\}$. Now for each of the pair (i_{19}, i_{29}) given by $i_{29} = 0, 1 \leq i_{19} \leq 11, i_{19} \neq 10$, we compute $|\mathcal{I}_1^k|, |\mathcal{I}_2^k|$. Since $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \geq 14$, we restrict to those pairs (i_{19}, i_{29}) with $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 10$. Then we are left with the only pair $(i_{19}, i_{29}) = (1, 0)$. Further noticing that $\mathcal{M}^k$ is exactly one of $\mathcal{I}_1^k$ or $\mathcal{I}_2^k$ with minimum cardinality, we get $\mathcal{M}^k = \{3, 5, 6, 7, 11, 14, 15, 19, 24, 25\}$ and $\mathcal{B}^k = \{2, 4, 8, 9, 10, 12, 13, 16, 17, 18, 21, 22, 23, 26, 27, 28, 30\}$. We find that there are at most 7 elements $i \in \mathcal{M}^k$ for which either $11|a_i$ or $13|a_i$ or $17|a_i$. This is not possible. $\square$

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