

# AN ESTIMATE FOR THE LENGTH OF AN ARITHMETIC PROGRESSION THE PRODUCT OF WHOSE TERMS IS ALMOST SQUARE

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ABSTRACT. Erdős conjectured that

$$(1) \quad n(n+d) \cdots (n+(k-1)d) = y^2$$

in positive integers  $n, k \geq 3, d > 1, y$  with  $\gcd(n, d) = 1$ , implies that  $k$  is bounded by an absolute constant. Shorey and Tijdeman [16] showed that (1) implies that  $k$  is bounded by an effectively computable number depending only on  $\omega(d)$ , the number of distinct prime divisors of  $d$ . In this paper, an explicit bound for  $k$  in terms of  $\omega(d)$  is presented.

## 1. INTRODUCTION

For an integer  $x > 1$ , we denote by  $P(x)$  and  $\omega(x)$  the greatest prime factor of  $x$  and the number of distinct prime divisors of  $x$ , respectively. Further we put  $P(1) = 1$  and  $\omega(1) = 0$ . Let  $n, d, k, b, y$  be positive integers such that  $b$  is square free,  $d \geq 1, k \geq 3, P(b) \leq k$  and  $\gcd(n, d) = 1$ . We consider the equation

$$(2) \quad n(n+d) \cdots (n+(k-1)d) = by^2 \text{ in } n, d, k, b, y \text{ with } P(b) \leq k.$$

For a survey of results on (2), see [16], [4], [14] and [15]. Equation (2) with  $d = 1$  has been solved completely in [3] with  $P(b) < k$  and in [11] with  $P(b) = k$ . Therefore we assume from now onwards that  $d > 1$ . Marszalek [7] proved that (2) implies  $k$  is bounded by an effectively computable number  $k_0$  depending only on  $d$ . In fact the above assertion holds with  $k_0$  depending only on  $\omega(d)$ . This is due to Shorey and Tijdeman [16], who proved that  $2^{\omega(d)} > c \frac{k}{\log k}$  where  $c$  is an effectively computable absolute constant. However the bound  $k_0$  is very large. Further (2) with  $\omega(d) = 1$  and  $k \notin \{3, 5\}$  has been solved completely in [12] and [8]. Therefore we shall always assume that  $\omega(d) \geq 2$ . In this paper, we give an explicit bound for  $k$  in terms of  $\omega(d)$  whenever (2) holds.

For  $2 \leq \omega(d) \leq 11$ , we define  $\kappa_0 = \kappa_0(\omega(d))$  as in the Table below.

$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$	$\omega(d)$	$\kappa_0(d \text{ even})$	$\kappa_0(d \text{ odd})$
2	500	800	7	$2.643 \times 10^5$	$1.376 \times 10^6$
3	700	3400	8	$1.172 \times 10^6$	$6.061 \times 10^6$
4	2900	15300	9	$5.151 \times 10^6$	$2.649 \times 10^7$
5	13100	69000	10	$2.247 \times 10^7$	$1.149 \times 10^8$
6	59000	$3.096 \times 10^5$	11	$9.73 \times 10^7$	$4.95 \times 10^8$

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For  $\omega(d) \geq 12$ , we define  $\kappa_0 = \kappa_0(\omega(d))$  as

$$\kappa_0(\omega(d)) = \begin{cases} 2.25\omega(d)4^{\omega(d)} & \text{if } d \text{ is even} \\ 11\omega(d)4^{\omega(d)} & \text{if } d \text{ is odd.} \end{cases}$$

We prove

**Theorem 1.** *Equation (2) implies that*

$$(3) \quad k < \kappa_0.$$

Theorem 1 is a direct consequence of the following two propositions.

**Proposition 2.** *Let  $k \geq \kappa_0$ . Then (2) implies that*

$$(4) \quad d < 4c_1(k-1)^2,$$

$$(5) \quad n < c_1(k-1)^3$$

and hence

$$(6) \quad n + (k-1)d < 5c_1(k-1)^3$$

where

$$c_1 = \begin{cases} \frac{1}{16} & \text{if } d \text{ is odd} \\ \frac{1}{8} & \text{if } \text{ord}_2(d) = 1 \\ \frac{1}{4} & \text{if } \text{ord}_2(d) \geq 2. \end{cases}$$

**Proposition 3.** *Let  $k \geq \kappa_0$ . Then (2) implies that*

$$(7) \quad n + (k-1)d \geq 2^\delta \frac{5}{16} k^3$$

where

$$\delta = \min\{\text{ord}_2(d), 3\}.$$

## 2. NOTATION AND PRELIMINARIES

From (2), we have

$$(8) \quad n + id = A_i X_i^2$$

for  $0 \leq i < k$  with  $P(A_i) \leq k$  and  $(X_i, \prod_{p \leq k} p) = 1$ . Also we have

$$(9) \quad n + id = a_i x_i^2$$

for  $0 \leq i < k$  with  $a_i$  squarefree. Since  $\gcd(n, d) = 1$ , we see that

$$(10) \quad (A_i, d) = (a_i, d) = (X_i, d) = (x_i, d) = 1 \text{ for } 0 \leq i < k.$$

Let

$$T = \{i \mid 0 \leq i < k, X_i = 1\}, \quad T_1 = \{i \mid 0 \leq i < k, X_i \neq 1\}.$$

Note that  $X_i > k$  for  $i \in T_1$ . For  $0 \leq i < k$ , let

$$(11) \quad \nu(A_i) = |\{j \in T_1, A_j = A_i\}|.$$

We always suppose that there exist  $i_0 > i_1 > \dots > i_{\nu(A_i)-1}$  such that  $A_{i_0} = A_{i_1} = \dots = A_{i_{\nu(A_i)-1}}$ . Similarly we define

$$R = \{a_i \mid 0 \leq i < k\}$$

and

$$(12) \quad \nu(a_i) = |\{j \mid 0 \leq j < k, a_i = a_j\}|.$$

Define

$$(13) \quad \rho := \rho(d) = \begin{cases} 1 & \text{if } 3 \nmid d \\ 3 & \text{if } 3 \mid d. \end{cases}$$

The letter  $p$  always denotes a prime number and  $p_i$  the  $i$ -th prime number. Let  $P_1 < P_2 < \dots$  be odd prime divisors of  $d$ . Let  $r := r(d) \geq 0$  be the unique integer such that

$$(14) \quad P_1 P_2 \dots P_r < (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}} \text{ but } P_1 P_2 \dots P_{r+1} \geq (4c_1)^{\frac{1}{3}}(k-1)^{\frac{2}{3}}.$$

If  $r = 0$ , we understand that the product  $P_1 \dots P_r = 1$ .

Let  $d' \mid d$  and  $d'' = \frac{d}{d'}$  be such that  $\gcd(d', d'') = 1$ . We write

$$d'' = d_1 d_2, \gcd(d_1, d_2) = \begin{cases} 1 & \text{if } \text{ord}_2(d'') \leq 1 \\ 2 & \text{if } \text{ord}_2(d'') \geq 2 \end{cases}$$

and we always suppose that  $d_1$  is odd if  $\text{ord}_2(d'') = 1$ . We call such pairs  $(d_1, d_2)$  as partitions of  $d''$ .

We observe that the number of partitions of  $d''$  is  $2^{\omega(d'')-\theta_1}$  where

$$\theta_1 := \theta_1(d'') = \begin{cases} 1 & \text{if } \text{ord}_2(d'') = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

and we write  $\theta$  for  $\theta_1(d)$ . In particular, by taking  $d' = 1$  and  $d'' = d$ , the number of partitions of  $d$  is  $2^{\omega(d)-\theta}$ .

Suppose that  $A_i = A_j, i > j$ . Then from (8) and (10), we have

$$(15) \quad (i-j)d = A_i(X_i^2 - X_j^2) = A_i(X_i - X_j)(X_i + X_j)$$

such that  $\gcd(d, X_i - X_j, X_i + X_j) = 1$  if  $d$  is odd and 2 if  $d$  is even. Hence for any divisor  $d''$  of  $d$ , we have a partition  $(d_1, d_2)$  of  $d''$  corresponding to  $A_i = A_j$  such that  $d_1 \mid (X_i - X_j)$  and  $d_2 \mid (X_i + X_j)$  and it is the unique partition of  $d''$  corresponding to the pair  $(i, j)$ . Similarly, we have unique partition of  $d''$  corresponding to every pair  $(i, j)$  whenever  $a_i = a_j$ .

As in Shorey and Tijdeman [16], the proof depends on comparing an upper bound and a lower bound for  $n + (k-1)d$ . The upper bound of  $n + (k-1)d$  given by Proposition 2 is a consequence of Lemmas 5, 8, 11, 12, 13 which are refinements of results in [16], [1] and [12]. It is proved by counting the number of distinct  $a_i$ 's and looking at the number of partitions of  $d$ . The proof of Proposition 3 is by counting the number of  $X_i$ 's greater than  $k$  and calculating the maximal value of  $A_i$ . Proposition 3 is a consequence of Lemmas 4, 6, 7, 9, 10, 14. The new features of the paper are the refinement of the upper bound of

the multiplicities of  $A_i$  with respect to partitions of  $d$ , counting the number of  $A_i$ 's with multiplicity greater than 1 and the use of  $r$  to improve the lower bounds of the maximum of  $A_i$ 's.

We shall follow the notation of this section throughout the paper. We use *MATHEMATICA* for the computations in the paper. This is a part of my Master's thesis [6].

### 3. LEMMAS

We begin with some estimates from Prime number theory.

**Lemma 1.** *We have*

- (i)  $\pi(\nu) \leq \frac{\nu}{\log \nu} \left(1 + \frac{1.5}{\log \nu}\right)$  for  $\nu > 1$
- (ii)  $\pi(\nu) \geq \frac{\nu}{\log \nu} \left(1 + \frac{0.5}{\log \nu}\right)$  for  $\nu \geq 59$
- (iii)  $p_i \geq i \log i$  for  $i \geq 2$
- (iv)  $\sum_{p \leq \nu} \log p < 1.000081\nu$  for  $\nu > 0$
- (v)  $\text{ord}_p(k!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$  for  $p < k$ .

*Proof.* The estimates (i), (ii) and (iii) are due to Rosser and Schoenfeld [10]. For estimate (iv), see [13, p.360] and [2, Prop 1.7]. For a proof of (v), see [5, Lemma 2(i)].  $\square$

The next result is Stirling's formula, see [9].

**Lemma 2.** *For a positive integer  $\nu$ , we have*

$$\sqrt{2\pi\nu} e^{-\nu} \nu^\nu e^{\frac{1}{2\nu+1}} < \nu! < \sqrt{2\pi\nu} e^{-\nu} \nu^\nu e^{\frac{1}{12\nu}}.$$

**Lemma 3.** *Let  $\pi_d(k) \leq \pi(k) - 1$ . Then*

$$(16) \quad |T_1| > k - \frac{(k-1) \log(k-1)}{\log(n + (k-1)d) - \log 2} - \pi(k).$$

*Proof.* We use [12, Lemma 3] with  $t = k$ ,  $-\log \prod_{p|d} p^{-\text{ord}_p((k-1)!)} \geq 0$  and  $\pi_d(k) \leq \pi(k) - \omega(d) + 2$ . Let  $n \geq (k-1)d$ . Then  $\log n \geq \log(n + (k-1)d) - \log 2$ . This with [12, (4.2)] and Lemma 1 (i) gives (16). For  $n < (k-1)d$ , we have  $\log(k-1) + \log d > \log(n + (k-1)d) - \log 2$ . This with [12, (4.1)] and Lemma 1 (i) gives (16).  $\square$

**Lemma 4.** *Let  $d = d' d''$  with  $\gcd(d', d'') = 1$ . Let  $i_0 \in T_1$  be such that  $A_{i_0} \geq d'$ . Then*

$$(17) \quad \nu(A_{i_0}) \leq 2^{\omega(d'') - \theta_1(d'')}.$$

Proof. For simplicity, we write  $\theta_1 = \theta_1(d'')$ . Assume that  $\nu(A_{i_0}) > 2^{\omega(d'')-\theta_1}$ . Then there exists a sequence of indices  $i_0 > i_1 > \dots > i_{2^{\omega(d'')-\theta_1}}$  such that  $A_{i_0} = A_{i_1} = \dots = A_{i_{2^{\omega(d'')-\theta_1}}}$ . For each pair  $(i_0, i_r), r = 1, 2, \dots, 2^{\omega(d'')-\theta_1}$ , we have a unique partition corresponding to the pair. But there are at most  $2^{\omega(d'')-\theta_1}$  partitions of  $d''$ . Since  $(i_0 - i_r)d = A_{i_0}(X_{i_0} - X_{i_r})(X_{i_0} + X_{i_r})$  and  $A_{i_0} \geq d'$ , we have

$$k > i_0 - i_r = \frac{A_{i_0}}{d'} \left( \frac{X_{i_0} - X_{i_r}}{d_1} \right) \left( \frac{X_{i_0} + X_{i_r}}{d_2} \right) \geq \left( \frac{X_{i_0} - X_{i_r}}{d_1} \right) \left( \frac{X_{i_0} + X_{i_r}}{d_2} \right),$$

where  $(d_1, d_2)$  is the partition of  $d''$  corresponding to pair  $(i_0, i_r)$ . This shows that we cannot have the partition  $(\frac{d''}{2^{\theta_1}}, 2^{\theta_1})$  corresponding to any pair. Hence there can be at most  $2^{\omega(d'')-\theta_1} - 1$  partitions of  $d''$  with respect to  $2^{\omega(d'')-\theta_1}$  pairs of  $(i_0, i_r), r = 1, \dots, 2^{\omega(d'')-\theta_1}$ . Hence by Box Principle, there exist pairs  $(i_0, i_r), (i_0, i_s)$  with  $1 \leq r < s \leq 2^{\omega(d'')-\theta_1}$  and a partition  $(d_1, d_2)$  of  $d''$  corresponding to these pairs. Thus

$$d_1 \mid (X_{i_0} - X_{i_r}), d_2 \mid (X_{i_0} + X_{i_r}) \text{ and } d_1 \mid (X_{i_0} - X_{i_s}), d_2 \mid (X_{i_0} + X_{i_s})$$

so that  $\text{lcm}(d_1, d_2) \mid (X_{i_r} - X_{i_s})$ . Since  $A_{i_r} = A_{i_s} = A_{i_0}$  and  $\gcd(d_1, d_2) \leq 2$ , we have

$$k > (i_r - i_s) \frac{d'}{A_{i_0}} = \frac{(X_{i_r} - X_{i_s})}{\text{lcm}(d_1, d_2)} \frac{(X_{i_r} + X_{i_s})}{\gcd(d_1, d_2)} > \frac{(X_{i_r} + X_{i_s})}{2} > \frac{2k}{2} = k,$$

a contradiction.  $\square$

By taking  $d' = 1$  and  $d'' = d$ , the following result is immediate from Lemma 4 since  $\theta_1(d) = \theta$ .

**Corollary 1.** For  $i_0 \in T_1$ , we have  $\nu(A_{i_0}) \leq 2^{\omega(d)-\theta}$ .

**Lemma 5.** Let  $k \geq 17$ . Suppose  $n \geq c_1(k-1)^3$  or  $d \geq 4c_1(k-1)^2$ . Then for  $0 \leq i_0 < k$ , we have

$$(18) \quad \nu(a_{i_0}) \leq 2^{\omega(d)-\theta}.$$

Proof. Suppose that  $\nu(a_{i_0}) > 2^{\omega(d)-\theta}$ . We note that both  $x_i + x_j$  and  $x_i - x_j$  are even when  $d$  is even. Continuing as in the proof of (17) with  $d'' = d$ , we see that there exists  $i, j$  with  $i > j$  and

$$k > \frac{a_{i_0}(x_i + x_j)}{2}$$

where  $\frac{d}{2} \mid (x_i - x_0)$  if  $d$  is even and  $d \mid (x_i - x_0)$  if  $d$  is odd. We have  $x_i \geq x_j + \frac{d}{2}$  so that  $k > \frac{1}{2}a_{i_0}(x_i + x_j) \geq (a_j x_j^2)^{\frac{1}{2}} + \frac{d}{4} \geq n^{\frac{1}{2}} + \frac{d}{4}$  and hence

$$k > \begin{cases} 1 + c_1(k-1)^2 & \text{if } d \geq 4c_1(k-1)^2 \\ (c_1)^{\frac{1}{2}}(k-1)^{\frac{3}{2}} + 1 & \text{if } n \geq c_1(k-1)^3 \end{cases}$$

which is not true for  $k \geq 17$ .  $\square$

**Lemma 6.** Equation (2) implies that either

$$d \geq 4c_1(k-1)^2$$

or

$$r \geq \left\lceil \frac{\omega(d)}{3} \right\rceil.$$

Proof. If  $r+1 \leq \left\lceil \frac{\omega(d)}{3} \right\rceil$ , then  $\omega(d) \geq 3(r+1)$  giving  $d \geq 4c_1(k-1)^2$  by (14).  $\square$

**Lemma 7.** Let  $S \subseteq \{A_i | 0 \leq i < k\}$  and  $\min_{A_h \in S} A_h \geq U$ . Let  $t \geq 1$ . Assume that

$$(19) \quad |S| > Q_t \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_t-1}{2} \right)$$

where  $Q_t \geq 1$  is an integer. Then

$$(20) \quad \max_{A_h \in S} A_h \geq 2^\delta Q_t P_1 \cdots P_t + U.$$

Proof. For an odd  $p|d$ , we have

$$\left( \frac{A_h}{p} \right) = \left( \frac{A_h X_h^2}{p} \right) = \left( \frac{n}{p} \right)$$

where  $(\cdot)$  is Legendre symbol, so that  $A_h$  belongs to at most  $\frac{p-1}{2}$  distinct residue classes modulo  $p$  for each  $0 \leq h < k$ . If  $d$  is even, then  $A_h$  also belongs to a unique residue class modulo  $2^\delta$  for each  $0 \leq h < k$ . Hence by Chinese remainder theorem,  $A_h$  belongs to at most  $\left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_j-1}{2} \right)$  distinct residue classes modulo  $2^\delta P_1 \cdots P_j$  for each  $j$ ,  $1 \leq j \leq t$ . Assume that (20) does not hold. Then

$$\max_{A_h \in S} A_h - (U-1) \leq 2^\delta Q_t P_1 \cdots P_t.$$

Therefore

$$|S| \leq \frac{2^\delta Q_t P_1 \cdots P_t}{2^\delta P_1 \cdots P_t} \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_t-1}{2} \right),$$

contradicting (19).  $\square$

**Corollary 2.** Let  $S$  and  $U$  be as in Lemma 7. Let  $|S| \geq s > \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_t-1}{2} \right)$ , then

$$(21) \quad \max_{A_h \in S} A_h \geq \frac{3}{4} 2^{t+\delta} s + U.$$

Proof. Let  $(f-1) \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_{t-1}-1}{2} \right) < s - Q_t \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_t-1}{2} \right) \leq f \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_{t-1}-1}{2} \right)$  where  $Q_t \geq 1$  and  $1 \leq f \leq \frac{P_t-1}{2}$  is an integer. To see this, write  $s = Q \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_t-1}{2} \right) + Q' \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_{t-1}-1}{2} \right) + R$  where  $0 \leq Q' < \frac{P_t-1}{2}$  and  $0 \leq R < \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_{t-1}-1}{2} \right)$ . If  $R > 0$ , then take  $Q_t = Q$ ,  $f-1 = Q'$ ; if  $R = 0$  and  $Q' > 0$ , then take  $Q_t = Q$ ,  $f = Q'$ ; and if  $R = Q' = 0$ , then take  $Q_t = Q-1$  and  $f = \frac{P_t-1}{2}$ . We arrange the elements of  $S$  in increasing order and let  $S' \subseteq S$  be the first  $(f-1) \left( \frac{P_1-1}{2} \right) \cdots \left( \frac{P_{t-1}-1}{2} \right) + 1$  elements and

$S''$  consist of the remaining set. Then we see from Lemma 7 with  $t = t - 1$  and  $Q_t = f - 1$  that

$$\max_{A_h \in S'} A_h \geq 2^\delta (f - 1) P_1 P_2 \cdots P_{t-1} + U = U'.$$

Now we apply Lemma 7 with  $U = U'$  in  $S''$  to derive

$$\max_{A_h \in S} A_h \geq 2^\delta Q_t P_1 P_2 \cdots P_t + 2^\delta (f - 1) P_1 P_2 \cdots P_{t-1} + U.$$

Hence to derive (21), it is enough to prove

$$Q_t P_1 \cdots P_t + (f - 1) P_1 \cdots P_{t-1} \geq \frac{3}{4} \{Q_t (P_1 - 1) \cdots (P_t - 1) + 2f (P_1 - 1) \cdots (P_{t-1} - 1)\}.$$

By observing that

$$\begin{aligned} Q_t (P_1 - 1) \cdots (P_t - 1) &\leq Q_t P_1 \cdots P_t - Q_t P_1 \cdots P_{t-1}, \\ 2f (P_1 - 1) \cdots (P_{t-1} - 1) &\leq 2f P_1 \cdots P_{t-1} - 2f P_1 \cdots P_{t-2}, \end{aligned}$$

it suffices to show that

$$Q_t + \frac{3(Q_t - 1) - (2f + 1)}{P_t} + \frac{6f}{P_t P_{t-1}} \geq 0$$

which is true since  $Q_t \geq 1$  and  $1 \leq f \leq \frac{P_t - 1}{2}$ . □

**Lemma 8.** *Let  $s_i$  denote the  $i$ -th squarefree positive integer. Then*

$$(22) \quad s_i \geq 1.6i \text{ for } i \geq 78$$

and

$$(23) \quad \prod_{i=1}^l s_i \geq (1.6)^l l! \text{ for } l \geq 286.$$

Further let  $t_i$  be  $i$ -th odd squarefree positive integer. Then

$$(24) \quad t_i \geq 2.4i \text{ for } i \geq 51$$

and

$$(25) \quad \prod_{i=1}^l t_i \geq (2.4)^l l! \text{ for } l \geq 200.$$

**Proof.** The proof is similar to that of [12, (6.9)]. For (22) and (24), we check that  $s_i \geq 1.6i$  for  $78 \leq i \leq 286$  and  $t_i \geq 2.4i$  for  $51 \leq i \leq 132$ , respectively. Further we observe that in a given set of 144 consecutive integers, there are at most 90 squarefree integers and at most 60 odd squarefree integers by deleting multiples of 4, 9, 25, 49, 121 and 2, 9, 25, 49, respectively. Then we continue as in the proof of [12, (6.9)] to get (22) and (24). Further we check that (23) holds at  $l = 286$  and (25) holds at  $l = 200$ . Then we use (22) and (24) to obtain (23) and (25), respectively. □

**Lemma 9.** *Let  $X > 1$  be a positive integer. Then*

$$(26) \quad \sum_{i=1}^{X-1} 2^{\omega(i)} \leq \eta(X) X \log X$$

where

$$(27) \quad \eta := \eta(X) = \begin{cases} 1 & \text{if } X = 1 \\ \sum_{i=1}^{X-1} 2^{\omega(i)} & \\ \frac{i=1}{X \log X} & \text{if } 1 < X < 248 \\ 0.75 & \text{if } X \geq 248. \end{cases}$$

Proof. We check that (26) holds for  $1 < X < 11500$ . Thus we may assume  $X \geq 11500$ . Let  $s_j$  be the largest squarefree integer  $\leq X$ . Then by Lemma 8, we have  $1.6j \leq s_j \leq X$  so that  $j \leq \left\lfloor \frac{X}{1.6} \right\rfloor$ . We have  $2^{\omega(i)} = \sum_{e|i} |\mu(e)|$ . Therefore

$$\sum_{i=1}^{X-1} 2^{\omega(i)} = \sum_{i=1}^{X-1} \sum_{e|i} |\mu(e)| \leq \sum_{1 \leq e < X} \left\lfloor \frac{X-1}{e} \right\rfloor |\mu(e)| \leq (X-1) \sum_{1 \leq e < X} \frac{|\mu(e)|}{e} \leq X \sum_{i=1}^{\left\lfloor \frac{X}{1.6} \right\rfloor} \frac{1}{s_i}.$$

We check that there are 6990 squarefree integers upto 11500. By using (22), we have

$$\begin{aligned} \sum_{i=1}^{X-1} 2^{\omega(i)} &\leq X \left\{ \sum_{i=1}^{6990} \frac{1}{s_i} - \frac{1}{1.6} \sum_{i=1}^{6990} \frac{1}{i} + \frac{1}{1.6} \sum_{i=1}^{\left\lfloor \frac{X}{1.6} \right\rfloor} \frac{1}{i} \right\} \\ &\leq X \left\{ \sum_{i=1}^{6990} \frac{1}{s_i} - \frac{1}{1.6} \sum_{i=1}^{6990} \frac{1}{i} + \frac{1}{1.6} \left( 1 + \log \frac{X}{1.6} \right) \right\} \\ &\leq \frac{3}{4} X \log X \left\{ \frac{4}{3} \frac{1.1658}{\log X} + \frac{4}{3} \frac{1}{1.6} \right\}, \end{aligned}$$

implying (26). □

**Lemma 10.** Let  $c > 0$  be such that  $c2^{\omega(d)-3} > 1$ ,  $\mu \geq 2$  and

$$\mathfrak{C}_\mu = \{A_i \mid \nu(A_i) = \mu, A_i > \frac{\rho 2^{\delta k}}{3c2^{\omega(d)}}\}.$$

Then

$$(28) \quad \mathfrak{C} := \sum_{\mu \geq 2} \frac{\mu(\mu-1)}{2} |\mathfrak{C}_\mu| \leq \frac{c}{8} \eta(c2^{\omega(d)-3}) 2^{\omega(d)} (2^{\omega(d)-\theta} - 1) (\log c2^{\omega(d)-3}).$$

Proof. Let  $i_1 > i_2 > \dots > i_\mu$  be such that  $A_{i_1} = A_{i_2} = \dots = A_{i_\mu}$ . These give rise to  $\frac{\mu(\mu-1)}{2}$  pairs of  $(i, j)$ ,  $i > j$  with  $A_i = A_j$ . Therefore the total number of pairs  $(i, j)$  with  $i > j$  and  $A_i = A_j$  is  $\mathfrak{C}$ .

We know that there is a unique partition of  $d$  corresponding to each pair  $(i, j)$ ,  $i > j$  such that  $A_i = A_j$ . Hence by Box Principle, there exists at least  $\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1}$  pairs of  $(i, j)$ ,  $i > j$  with  $A_i = A_j$  and a partition  $(d_1, d_2)$  of  $d$  corresponding to these pairs. For every such pair  $(i, j)$ , we write  $X_i - X_j = d_1 r_{ij}$ ,  $X_i + X_j = d_2 s_{ij}$ . Then  $\gcd(X_i - X_j, X_i + X_j) = 2$  and  $24 \mid (X_i^2 - X_j^2)$ . Let  $r'_{ij}, s'_{ij}$  be such that  $r'_{ij} \mid r_{ij}$ ,  $s'_{ij} \mid s_{ij}$ ,  $\gcd(r'_{ij}, s'_{ij}) = 1$  and  $r_{ij} s_{ij} = \frac{24}{\rho 2^\delta} r'_{ij} s'_{ij}$ .



Then

$$r'_{ij}s'_{ij} = \frac{\rho 2^\delta}{24} r_{ij}s_{ij} = \frac{\rho 2^\delta}{24} \frac{X_i^2 - X_j^2}{d} = \frac{\rho 2^\delta}{24} \frac{i - j}{A_i} < \frac{\rho 2^\delta}{24} \frac{k}{A_i} < c 2^{\omega(d)-3}$$

since  $A_i > \frac{\rho 2^\delta k}{3c 2^{\omega(d)}}$ . There are at most  $\sum_{i=1}^{c 2^{\omega(d)-3}-1} 2^{\omega(i)}$  possible pairs of  $(r'_{ij}, s'_{ij})$ , and hence an equal number of possible pairs of  $(r_{ij}, s_{ij})$ . By Lemma 9, we estimate

$$\sum_{i=1}^{c 2^{\omega(d)-3}-1} 2^{\omega(i)} \leq \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}).$$

Thus if we have

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1} > \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}),$$

then there exist distinct pairs  $(i, j) \neq (g, h), i > j, g > h$  with  $A_i = A_j, A_g = A_h$  such that  $r_{ij} = r_{gh}, s_{ij} = s_{gh}$  giving

$$X_i - X_j = d_1 r_{ij} = X_g - X_h \text{ and } X_i + X_j = d_2 s_{ij} = X_g + X_h.$$

Thus  $X_i = X_g, X_j = X_h$  implying  $(i, j) = (g, h)$ , a contradiction. Hence

$$\frac{\mathfrak{C}}{2^{\omega(d)-\theta}-1} \leq \eta(c 2^{\omega(d)-3}) c 2^{\omega(d)-3} (\log c 2^{\omega(d)-3}),$$

implying (28). □

The following Lemma is a refinement of [16, Lemma 2].

**Lemma 11.** *Let  $i > j, g > h, 0 \leq i, j, g, h < k$  be such that*

$$(29) \quad a_i = a_j, \quad a_g = a_h$$

and

$$(30) \quad x_i - x_j = d_1 r_1, \quad x_i + x_j = d_2 r_2, \quad x_g - x_h = d_1 s_1, \quad x_g + x_h = d_2 s_2$$

where  $(d_1, d_2)$  is a partition of  $d$ ;  $r_1 \equiv s_1 \pmod{2}$ ,  $r_2 \equiv s_2 \pmod{2}$  when  $d$  is even; and either  $r_1 \equiv s_1 \pmod{2}$  and  $a_i \equiv a_g \pmod{4}$  or  $2 \mid \gcd(r_1, s_1)$  when  $d$  is odd. Then we have either

$$(31) \quad a_i = a_g, r_1 = s_1 \text{ or } a_i = a_g, r_2 = s_2$$

or (4) and (5) hold.

*Proof.* We follow the proof of [16, Lemma 2]. Suppose that (31) does not hold. Then

$$(32) \quad a_i r_1^2 - a_g s_1^2 \neq 0, \quad a_i r_2^2 - a_g s_2^2 \neq 0.$$

We proceed as in [16, Lemma 2] to conclude from  $d \mid (a_i x_i^2 - a_g x_g^2)$  that

$$(33) \quad d_1 d_2 = d \mid \frac{1}{4} \{ (a_i r_1^2 - a_g s_1^2) d_1^2 + (a_i r_2^2 - a_g s_2^2) d_2^2 + 2d(a_i r_1 r_2 - a_g s_1 s_2) \}.$$

Thus we have

$$(a_i r_1^2 - a_g s_1^2) d_1^2 = a_i (x_i - x_j)^2 - a_g (x_g - x_h)^2 \neq 0$$

and

$$(a_i r_2^2 - a_g s_2^2) d_2^2 = a_i (x_i + x_j)^2 - a_g (x_g + x_h)^2 \neq 0.$$

Since

$$n \leq a_j x_j^2 < a_i x_i x_j < a_i x_i^2 \leq n + (k-1)d$$

and

$$n \leq a_h x_h^2 < a_g x_g x_h < a_g x_g^2 \leq n + (k-1)d,$$

we have

$$(34) \quad |a_i x_i x_j - a_g x_g x_h| < (k-1)d.$$

Also

$$(35) \quad \begin{aligned} |a_i x_i^2 - a_g x_g^2| &= |i - g|d \leq (k-1)d \\ |a_j x_j^2 - a_h x_h^2| &= |j - h|d \leq (k-1)d \end{aligned}$$

and

$$(36) \quad n \leq \min \left\{ \frac{1}{4} a_i (x_i + x_j)^2, \frac{1}{4} a_g (x_g + x_h)^2 \right\}.$$

Hence we derive from (34), (35) and (36) that

$$(37) \quad |(a_i r_2^2 - a_g s_2^2) d_2^2| < 4(k-1)d$$

$$(38) \quad n |(a_i r_1^2 - a_g s_1^2) d_1^2| < \frac{1}{4} (k-1)^2 d^2$$

and further considering the cases  $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$ ,  $\{a_i r_1^2 > a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$ ,  $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 > a_g s_2^2\}$  and  $\{a_i r_1^2 < a_g s_1^2, a_i r_2^2 < a_g s_2^2\}$ , we derive

$$(39) \quad G(i, g) = |a_i r_1^2 - a_g s_1^2| d_1^2 + |a_i r_2^2 - a_g s_2^2| d_2^2 < 4(k-1)d.$$

Let  $d = d_1 d_2$  be odd,  $\gcd(d_1, d_2) = 1$ . We have either  $r_1, s_1$  are even and hence  $r_1, r_2, s_1, s_2$  are even, or  $a_i \equiv a_g \pmod{4}$  and  $r_1 \equiv s_1 \pmod{2}$  and hence  $r_2 \equiv s_2 \pmod{2}$ . Then reading modulo  $d_1$  and  $d_2$  separately in (33), we have

$$(40) \quad d_1 \left| \frac{1}{4} (a_i r_2^2 - a_g s_2^2) \right| \text{ and } d_2 \left| \frac{1}{4} (a_i r_1^2 - a_g s_1^2) \right|.$$

Therefore

$$(41) \quad 4dd_2 = 4d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2$$

and

$$(42) \quad 4dd_1 = 4d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2.$$

From (39), we have

$$4d(d_1 + d_2) \leq G(i, g) < 4(k-1)d$$

so that

$$d = d_1 d_2 \leq \left( \frac{d_1 + d_2}{2} \right)^2 < \frac{(k-1)^2}{4}.$$

This gives (4). Again from (42) and (38), we see that  $4n d d_1 < \frac{1}{4}(k-1)^2 d^2$ , i.e.,  $n < \frac{1}{16}(k-1)^2 d_2$ . From (41) and (37), we have  $4d d_2 < 4(k-1)d$ , i.e.,  $d_2 < (k-1)$ . Thus (5) is also valid.

Let  $d = d_1 d_2$  be even with  $\text{ord}_2(d) = 1$  and  $d_1$  odd. Then the  $x_i$ 's are odd and therefore both  $r_1$  and  $s_1$  is even. We see from (33) that

$$(43) \quad 4d_1 \mid (a_i r_2^2 - a_g s_2^2) d_2^2 \quad \text{and} \quad 4d_2 \mid (a_i r_1^2 - a_g s_1^2) d_1^2.$$

Since  $r_1 \equiv s_1 \pmod{2}$ ,  $r_2 \equiv s_2 \pmod{2}$ ,  $\gcd(d_1, d_2) = 1$  and  $d_1$  odd, we derive that

$$2d_1 \mid (a_i r_2^2 - a_g s_2^2), \quad 4d_2 \mid (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2d d_2 = 2d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 4d d_1 = 4d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (4) and (5).

Let  $d = d_1 d_2$  be even with  $\text{ord}_2(d) \geq 2$ ,  $\gcd(d_1, d_2) = 2$ . Then we see from (33) that (43) holds. Since  $\gcd(d_1, d_2) = 2$ ,  $r_1 \equiv s_1 \pmod{2}$  and  $r_2 \equiv s_2 \pmod{2}$ , we derive that

$$2d_1 \mid (a_i r_2^2 - a_g s_2^2), \quad 2d_2 \mid (a_i r_1^2 - a_g s_1^2).$$

Therefore

$$2d d_2 = 2d_1 d_2^2 \leq |a_i r_2^2 - a_g s_2^2| d_2^2, \quad 2d d_1 = 2d_1^2 d_2 \leq |a_i r_1^2 - a_g s_1^2| d_1^2.$$

Now we argue as above to conclude (4) and (5). □

**Lemma 12.** *For a prime  $p < k$ , let*

$$\gamma_p = \text{ord}_p \left( \prod_{a_i \in R} a_i \right), \quad \gamma'_p = 1 + \text{ord}_p((k-1)!).$$

*Let  $m > 1$  by any real number. Then*

$$(44) \quad \prod_{2 \leq p \leq m} p^{\gamma_p - \gamma'_p} \leq k^{1.5\pi(m)} \left( z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right) \left( z_2 \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right)^{-k}$$

*where  $(z_1, z_2) = (2^{\frac{4}{3}}, 2^{\frac{2}{3}})$  if  $d$  is odd and  $(z_1, z_2) = (4, 2)$  if  $d$  is even.*

**Proof.** The proof is the refinement of inequality [12, (6.4)]. Let  $p^h \leq k-1 < p^{h+1}$  where  $h$  is a positive integer. Then

$$(45) \quad \gamma'_p - 1 = \left\lfloor \frac{k-1}{p} \right\rfloor + \left\lfloor \frac{k-1}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p^h} \right\rfloor.$$

Let  $p \nmid d$ . Then we see that  $g_p$  is the number of terms in  $\{n, n+d, \dots, n+(k-1)d\}$  divisible by  $p$  to an odd power. After removing a term to which  $p$  appears to a maximal power, the number of terms in the remaining set divisible by  $p$  to an odd power is at most

$$\left\lfloor \frac{k-1}{p} \right\rfloor - \left( \left\lfloor \frac{k-1}{p^2} \right\rfloor - 1 \right) + \left\lfloor \frac{k-1}{p^3} \right\rfloor - \left( \left\lfloor \frac{k-1}{p^4} \right\rfloor - 1 \right) + \cdots + (-1)^\epsilon \left( \left\lfloor \frac{k-1}{p^h} \right\rfloor + (-1)^\epsilon \right)$$

where  $\epsilon = 1$  or  $0$  according as  $h$  is even or odd, respectively. We note that the above expression is always positive. Combining this with (45) and  $\left\lfloor \frac{k-1}{p^i} \right\rfloor \geq \frac{k-1}{p^i} - 1 + \frac{1}{p^i} = \frac{k}{p^i} - 1$ , we have

$$\begin{aligned} \gamma_p - \gamma'_p &\leq -2 \left\{ \left\lfloor \frac{k-1}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{k-1}{p^{h-1+\epsilon}} \right\rfloor \right\} + \frac{h-1+\epsilon}{2} \\ &\leq -2 \left\{ \frac{k}{p^2} + \cdots + \frac{k}{p^{h-1+\epsilon}} - \frac{h-1+\epsilon}{2} \right\} + \frac{h-1+\epsilon}{2} \\ &= -\frac{2k}{p^2(1-\frac{1}{p^2})} \left(1 - \frac{1}{p^{h-1+\epsilon}}\right) + 1.5(h-1+\epsilon). \end{aligned}$$

Since  $p^h \geq \frac{k}{p}$  and  $h < \frac{\log k}{\log p}$ , we get

$$\gamma_p - \gamma'_p < -\frac{2k}{p^2-1} + \frac{1.5 \log k}{\log p} + \frac{2p^{2-\epsilon}}{p^2-1} + 1.5\epsilon - 1.5 \leq -\frac{2k}{p^2-1} + \frac{1.5 \log k}{\log p} + \frac{2p}{p^2-1}.$$

When  $d$  is even, we have  $\gamma_2 - \gamma'_2 = -1 - \text{ord}_2(k-1) < -k + \frac{\log k}{\log 2} + 2$  by Lemma 1 (v). Now (44) follows immediately.  $\square$

**Lemma 13.** Suppose that  $n \geq c_1(k-1)^3$  or  $d \geq 4c_1(k-1)^2$  or both. Let  $1 \leq \varrho \leq 2^{\omega(d)-\theta}$  be the greatest integer such that  $R_\varrho = \{a_i \mid \nu(a_i) = \varrho\} \neq \emptyset$ . For  $k \geq \kappa_0$ , we have

$$\mathfrak{r} = |\{(i, j) \mid a_i = a_j, i > j\}| \geq g(\varrho) := \begin{cases} 4\varrho(2^{\omega(d)} - 1) & \text{if } d \text{ is odd} \\ 2\varrho(2^{\omega(d)-\theta} - 1) & \text{if } d \text{ is even.} \end{cases}$$

*Proof.* We have

$$k = \sum_{\mu=1}^{\varrho} \mu r_\mu \quad \text{and} \quad |R| = \sum_{\mu=1}^{\varrho} r_\mu$$

where  $r_\mu = |R_\mu| = |\{a_i \mid \nu(a_i) = \mu\}|$ . Each  $R_\mu$  gives rise to  $\frac{\mu(\mu-1)}{2} r_\mu$  pairs of  $i, j$  with  $i > j$  such that  $a_i = a_j$ . Then

$$\mathfrak{r} = \sum_{\mu=1}^{\varrho} \frac{\mu(\mu-1)}{2} r_\mu = k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu.$$

Suppose that the assertion of the Lemma 13 does not hold. Then  $g(\varrho) > k - |R| + \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu$ . We have

$$g(\varrho) - \sum_{\mu=1}^{\varrho} \frac{(\mu-1)(\mu-2)}{2} r_\mu \leq g(\varrho) - \frac{(\varrho-1)(\varrho-2)}{2} := g_0(\varrho).$$

We see that  $g_0(\varrho)$  is an increasing function of  $\varrho$ . Since  $\varrho \leq 2^{\omega(d)-\theta}$ , we find that

$$k - |R| < g_0(2^{\omega(d)-\theta}) = (2^{\omega(d)-\theta} - 1)(z_3 2^{\omega(d)-\theta} + 1) := g_1$$

where  $z_3 = \frac{7}{2}$  if  $d$  is odd and  $\frac{3}{2}$  if  $d$  is even. Thus  $|R| > k - g_1$ . Since the  $a_i$ 's are squarefree, we have by Lemma 8 that

$$\prod_{a_i \in R} a_i \geq z_4^{k-g_1} (k - g_1)!$$

where  $z_4 = 1.6$  if  $d$  is odd and  $2.4$  if  $d$  is even. Also, we have

$$\prod_{a_i \in R} a_i \mid (k-1)! \left( \prod_{p < k} p \right) \prod_{2 \leq p \leq m} p^{\gamma_p - \gamma'_p}$$

where  $\gamma_p, \gamma'_p$  and  $m$  are as in Lemma 12. This with (44) and Lemma 1 (iv) gives

$$\prod_{a_i \in R} a_i < k! k^{1.5\pi(m)-1} \left( z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right) \left( \frac{z_2}{2.7205} \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right)^{-k}.$$

Comparing the lower and upper bounds, we have

$$(46) \quad \frac{z_4^{g_1} k!}{(k - g_1)!} > k^{-1.5\pi(m)+1} \left( z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right)^{-1} \left( \frac{z_2 z_4}{2.7205} \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right)^k.$$

By Lemma 2, we have

$$\frac{z_4^{g_1} k!}{(k - g_1)!} < z_4^{g_1} e^{-g_1} k^{g_1} \left( \frac{k}{k - g_1} \right)^{k - g_1 + \frac{1}{2}} \frac{e^{\frac{1}{12k}}}{e^{\frac{1}{12(k - g_1) + 1}}}.$$

Since  $k \geq \kappa_0$ , we find that  $g_1 < \frac{k}{z_5}$  for  $\omega(d) \geq 12$  where  $z_5 = 37, 18$  for  $d$  odd and  $d$  even, respectively. Thus

$$\frac{z_4^{g_1} k!}{(k - g_1)!} < \begin{cases} \left( \frac{z_4(k - g_1)}{e} \right)^{g_1} \left( \frac{k}{k - g_1} \right)^{k + \frac{1}{2}} & \text{if } \omega(d) \leq 11 \\ \left( \frac{z_5}{z_5 - 1} \right)^{k + \frac{1}{2}} \left( \frac{z_4(z_5 - 1)k}{z_5 e} \right)^{g_1} & \text{if } \omega(d) \geq 12. \end{cases}$$

Hence we derive from (46) that

$$(47) \quad g_1 > \frac{k \log \left( \frac{z_2 z_4}{2.7205} \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right) + (k + \frac{1}{2}) \log(1 - \frac{g_1}{k})}{\log(k - g_1) - 1 + \log z_4} - \frac{(1.5\pi(m) - 1) \log k + \log \left( z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right)}{\log(k - g_1) - 1 + \log z_4}$$

for  $\omega(d) \leq 11$  and

$$(48) \quad g_1 > \frac{k \log \left( \frac{z_5 - 1}{z_5} \frac{z_2 z_4}{2.7205} \prod_{2 < p \leq m} p^{\frac{2}{p^2-1}} \right) - (1.5\pi(m) - 1) \log k - \log \left( \sqrt{\frac{z_5}{z_5 - 1}} z_1 \prod_{2 < p \leq m} p^{\frac{2p}{p^2-1}} \right)}{\log k - 1 + \log z_4(z_5 - 1) - \log z_5}$$

for  $\omega(d) \geq 12$ .

Let  $\omega(d) \leq 11$ . Taking  $m = \min(1000, \sqrt{\kappa_0})$  in (47), we observe that the right hand side of (47) is an increasing function of  $k$  and the inequality does not hold at  $k = \kappa_0$ . Hence (47) is not valid for all  $k \geq \kappa_0$ . For instance, when  $\omega(d) = 4$ ,  $d$  odd, we have  $\kappa_0 = 15700$  and  $g_1 = 855$ . With these values, we see that the right hand side of (47) exceeds 855 at  $k = 15300$ , a contradiction. Hence (47) is not valid for all  $k \geq 15300$ .

Let  $\omega(d) \geq 12$ . Taking  $m = 1000$  in (48), we derive that

$$g_1 > \begin{cases} 0.63104 \frac{k}{\log k} & \text{if } d \text{ is odd} \\ 1.183 \frac{k}{\log k} & \text{if } d \text{ is even.} \end{cases}$$

For  $d$  odd, we see that

$$\begin{aligned} 0.63104 \frac{k}{\log k} &\geq 0.63104 \frac{\kappa_0}{\log \kappa_0} = \frac{0.63104 \times 11\omega(d)4^{\omega(d)}}{\omega(d) \log 4 + \log 11 + \log \omega(d)} \\ &> \frac{7}{2} 4^{\omega(d)} > g_1, \end{aligned}$$

a contradiction. Similarly, we get a contradiction for  $d$  even.  $\square$

**Lemma 14.** *Let  $k \geq \kappa_0 = \kappa_0(\omega(d))$ . Assume that  $d < 4c_1(k-1)^2$ . Let  $T_1 = \{0 \leq i < k \mid X_i > 1\}$  defined in section 2 be such that*

$$|T_1| > C_1 := \begin{cases} \frac{k}{C_2} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2 \\ \frac{k}{C_2} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3, 4, 5 \\ \frac{k}{C_2} + \frac{k}{12} + \frac{k}{9} & \text{if } \omega(d) \geq 6 \end{cases}$$

where  $C_2 \leq 2k^{\frac{1}{3}}$  and  $C_3 = 39, 42, 195, 806$  for  $\omega(d) = 2, 3, 4, 5$ , respectively. Then

$$(49) \quad \max_{i \in T_1} A_i \geq 2^\delta C_0 \frac{k}{C_2} \text{ where } C_0 = C_0(\omega(d)) = \begin{cases} 1 & \text{if } \omega(d) = 2 \\ \frac{3}{4} 2^{\lceil \frac{\omega(d)}{3} \rceil} & \text{if } \omega(d) \geq 3. \end{cases}$$

*Proof.* We see that for  $\omega(d) \geq 6$ ,

$$\frac{k}{20 \cdot 2^{\omega(d)}} \geq (4c_1(k-1)^2)^{\frac{1}{\omega(d)}} > d^{\frac{1}{\omega(d)}}$$

where  $c_1$  is given by Proposition 2. Hence there exists a partition  $d = d_1 d_2$  of  $d$  with

$$d_1 < \frac{k}{20 \cdot 2^{\omega(d)}} \text{ with } \omega(d_1) \geq 1 \text{ and } \omega(d_2) \leq \omega(d) - 1.$$

Therefore

$$(50) \quad \nu(A_i) \leq 2^{\omega(d_2)} \leq 2^{\omega(d)-1} \text{ for } A_i \geq \frac{k}{20 \cdot 2^{\omega(d)}}$$

by Lemma 4.

Let

$$(51) \quad T_2 = \{i \in T_1 \mid A_i > \frac{2^\delta \rho k}{3c 2^{\omega(d)}}\}, \quad T_3 = T_1 - T_2,$$

where  $c = 16$  if  $\omega(d) = 2$ ,  $c = 4$  if  $\omega(d) = 3, 4, 5$  and  $c = 2$  if  $\omega(d) \geq 6$ . Further let

$$(52) \quad S_2 = \{A_i \mid i \in T_2\}, \quad S_3 = \{A_i \mid i \in T_3\}$$

and  $|S_3| = s$ . Then considering residue classes modulo  $2^\delta \rho$ , we derive that

$$\frac{2^\delta \rho k}{3c \cdot 2^{\omega(d)}} \geq \max_{A_i \in S_3} A_i \geq 2^\delta \rho(s-1) + 1$$

so that  $|S_3| = s \leq \frac{k}{3c2^{\omega(d)}} - \frac{1}{\rho} + 1 \leq \frac{k}{3c2^{\omega(d)}} + \frac{2}{3}$ . We see from Corollary 1, (50), (51) and (52) that

$$\begin{aligned} |T_3| &\leq \frac{k}{20 \cdot 2^{\omega(d)}} 2^{\omega(d)} + \left( \frac{k}{6 \cdot 2^{\omega(d)}} - \frac{k}{20 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)-1} \\ &\leq \frac{k}{20} + \left( \frac{k}{6} - \frac{k}{20} \right) 2^{-1} + \frac{2}{3} 2^{\omega(d)-1} \leq \frac{k}{12} + \frac{k}{40} + \frac{k}{6 \times 2^6} \leq \frac{k}{9} \end{aligned}$$

if  $\omega(d) \geq 6$  and

$$|T_3| \leq \begin{cases} \left( \frac{k}{48 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)} = \frac{k}{48} + \frac{8}{3} & \text{if } \omega(d) = 2 \\ \left( \frac{k}{12 \cdot 2^{\omega(d)}} + \frac{2}{3} \right) 2^{\omega(d)} = \frac{k}{12} + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3, 4, 5. \end{cases}$$

Therefore

$$|T_2| > C_1 - |T_3| \geq C_4 := \begin{cases} \frac{k}{C_2} + C_3 & \text{if } \omega(d) = 2, 3, 4, 5 \\ \frac{k}{C_2} + \frac{k}{4} & \text{if } \omega(d) \geq 6. \end{cases}$$

Let  $\mathfrak{C}, \mathfrak{C}_\mu$  be as in Lemma 10 with  $c = 16$  if  $\omega(d) = 2$ ,  $c = 4$  if  $\omega(d) = 3, 4, 5$  and  $c = 2$  if  $\omega(d) \geq 6$ . Then  $C_4 < |T_2| = |S_2| + \sum_{\mu \geq 2} (\mu - 1) |\mathfrak{C}_\mu|$ . Now we apply Lemma 10 and use  $k \geq \kappa_0 \geq \eta(2^{\omega(d)-2})(\log 2^{\omega(d)-2})2^{\omega(d)}(2^{\omega(d)-\theta} - 1)$  for  $\omega(d) \geq 6$  to get

$$C_4 < \begin{cases} |S_2| + C_3 & \text{if } 2 \leq \omega(d) \leq 5 \\ |S_2| + \frac{k}{12} & \text{if } \omega(d) \geq 6. \end{cases}$$

Thus

$$|S_2| > \frac{k}{C_2}.$$

Let  $\omega(d) = 2$ . Then considering the  $A_i$ 's modulo  $2^\delta$ , we see that

$$\max_{A_i \in S_2} A_i \geq 2^\delta \left\lfloor \frac{k}{C_2} \right\rfloor + \frac{2^\delta k}{48 \times 4} \geq 2^\delta \frac{k}{C_2}$$

which gives (49). Now we take  $\omega(d) \geq 3$ . Since  $d < 4c_1(k-1)^2$ , we have  $r \geq \lceil \frac{\omega(d)}{3} \rceil$  by

Lemma 6. By (14), we have  $\frac{k}{C_2} \geq \frac{k^{\frac{2}{3}}}{2} > \frac{1}{2^r} (4c_1(k-1)^2)^{\frac{1}{3}} > \prod_{j=1}^r \left( \frac{P_j - 1}{2} \right)$ . We now

apply Corollary 2 with  $s = \lceil \frac{k}{C_2} + 1 \rceil$  and  $U = 1$  to get

$$\max_{A_i \in S_2} A_i \geq \frac{3}{4} 2^{r+\delta} \left\lfloor \frac{k}{C_2} + 1 \right\rfloor \geq \frac{3}{4} 2^{\lceil \frac{\omega(d)}{3} \rceil + \delta} \frac{k}{C_2}$$

which yields (49).  $\square$

#### 4. PROOF OF PROPOSITION 2

We assume that either  $n \geq c_1(k-1)^3$  or  $d \geq 4c_1(k-1)^2$ . Then  $\nu(a_{i_0}) \leq 2^{\omega(d)-\theta}$  for  $0 \leq i_0 < k$  by Lemma 5. Let  $\varrho$  be as defined in the statement of Lemma 13. Then  $\nu(a_{i_0}) \leq \varrho$ . By Lemma 13, there are at least  $z\varrho(2^{\omega(d)} - 1)$  distinct pairs  $(i, j)$  with  $i > j$  and  $a_i = a_j$ , where  $z = 4$  if  $d$  is odd and  $2$  if  $d$  is even. Since there can be at most  $2^{\omega(d)-\theta} - 1$

possible partitions of  $d$ , by Box principle, there exists a partition  $(d_1, d_2)$  of  $d$  and at least  $z\varrho$  pairs of  $(i, j)$  with  $a_i = a_j, i > j$  corresponding to this partition. We write

$$x_i - x_j = d_1 r_1(i, j) \text{ and } x_i + x_j = d_2 r_2(i, j).$$

Let  $d$  be odd. Suppose there are at least  $\varrho$  distinct pairs  $(i_1, j_1), \dots, (i_\varrho, j_\varrho), \dots$  with the corresponding  $r_1(i, j)$  even. Then  $|\{i_1, \dots, i_\varrho, j_1, \dots, j_\varrho\}| > \varrho$ . Hence we can find  $1 \leq l, m \leq \varrho$  with  $(i_l, j_l) \neq (i_m, j_m)$ ,  $a_{i_l} = a_{j_l}, a_{i_m} = a_{j_m}$  and  $a_{i_l} \neq a_{i_m}$ . Now the result follows by Lemma 11. Thus we may assume that there are at most  $\varrho - 1$  pairs  $(i, j)$  with  $r_1(i, j)$  even. Then there are at least  $3\varrho + 1$  distinct pairs  $(i, j)$  with  $r_1(i, j)$  odd. Since  $a_i \equiv 1, 2, 3 \pmod{4}$ , we can find at least  $\varrho$  pairs with  $a_i \equiv a_g \pmod{4}$  for any two such pairs  $(i, j), (g, h)$ . Then there exist two distinct pairs  $(i, j), (g, h)$  with  $a_i = a_j, a_g = a_h$  and  $a_i \neq a_g$  from these pairs. Also  $r_1(i, j) \equiv r_1(g, h) \pmod{2}$ . This gives (4) and (5) by Lemma 11 which is a contradiction.

Let  $d$  be even. We observe that  $8|(x_i^2 - x_j^2)$  and  $\gcd(x_i - x_j, x_i + x_j) = 2$ . We claim that there are at least  $\varrho$  pairs with  $r_1(i, j) \equiv r_1(g, h) \pmod{2}$  and  $r_2(i, j) \equiv r_2(g, h) \pmod{2}$  for any two such distinct pairs  $(i, j)$  and  $(g, h)$ . If the claim is true, then there are two pairs  $(i, j) \neq (g, h)$  with  $i > j, g > h, a_i = a_j, a_g = a_h$  and  $a_i \neq a_g$  since  $\nu(a_i) \leq \varrho$ . This implies (4) and (5) by Lemma 11, contradicting our assumption. Let  $\text{ord}_2(d) = 1$ . Then  $d_1$  is odd, implying that  $r_1(i, j)$  is even. We can choose at least  $\varrho$  pairs whose  $r_2$ 's are of the same parity. Thus the claim is true in this case. Let  $\text{ord}_2(d) \geq 3$ . Then we have either  $\text{ord}_2(d_1) = 1$  implying that all the  $r_1$ 's are odd, or  $\text{ord}_2(d_2) = 1$  implying that all the  $r_2$ 's are odd. Thus the claim follows. Finally, let  $\text{ord}_2(d) = 2$ . Then  $2 \parallel d_1$  and  $2 \parallel d_2$  so that  $r_1$  and  $r_2$  are of the opposite parity for any pair and hence the claim holds.  $\square$

## 5. PROOF OF PROPOSITION 3

In this section, we assume that  $k \geq \kappa_0 = \kappa_0(\omega(d))$ . In view of Proposition 2, we may assume that  $d < 4c_1(k-1)^2$ . We may also assume that  $X_i$  is a prime for each  $i \in T_1$  in the proof of Proposition 3. Otherwise  $n + (k-1)d \geq (k+1)^4$ , which implies the assertion.

Since  $d < 4c_1(k-1)^2$ ,  $d$  has at least one prime divisor  $\leq k$  otherwise  $d > k^{\omega(d)} \geq k^2$ , giving a contradiction. Thus  $\pi_d(k) \leq \pi(k) - 1$ . Let  $n + (k-1)d \geq L$  for some  $L > 0$ . By Lemma 3 and Lemma 1 (i), we have

$$(53) \quad |T_1| > k - \frac{(k-1) \log(k-1)}{\log L - \log 2} - \frac{k}{\log k} \left( 1 + \frac{1.5}{\log k} \right).$$

We see from [5] that  $n(n+d) \cdots (n+(k-1)d)$  is divisible by at least  $\pi(2k) - \pi_d(k) \geq \pi(2k) - \pi(k) + 1$  primes exceeding  $k$ . Hence we have  $n + (k-1)d \geq 4k^2$ . Thus by taking  $L = 4k^2$  in (53), we get

$$|T_1| > k - \frac{(k-1) \log(k-1)}{\log(2k^2)} - \frac{k}{\log k} \left( 1 + \frac{1.5}{\log k} \right).$$



The right hand side of the above inequality is an increasing function of  $k$  and

$$(54) \quad |T_1| > \begin{cases} \frac{k}{5} + \frac{k}{48} + C_3 + \frac{8}{3} & \text{if } \omega(d) = 2 \\ \frac{k}{6} + \frac{k}{12} + C_3 + \frac{16}{3} & \text{if } \omega(d) = 3 \\ \frac{5}{24}k + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 4, 5 \\ \frac{5}{48}k + \frac{k}{12} + \frac{k}{9} & \text{if } \omega(d) \geq 6. \end{cases}$$

Now we see from Lemma 14 that (49) holds with

$$C_2 = \begin{cases} 5 & \text{if } \omega(d) = 2 \\ 6 & \text{if } \omega(d) = 3 \\ \frac{24}{5} & \text{if } \omega(d) = 4, 5 \\ \frac{48}{5} & \text{if } \omega(d) \geq 6. \end{cases}$$

This gives  $n + (k-1)d \geq \frac{C_0}{C_2}k^3$ . Hence (7) is valid for  $\omega(d) \geq 4$ . Now we take  $\omega(d) = 2, 3$ . Putting  $L = \frac{C_0}{5}k^3$  in (53), we derive that

$$|T_1| > \begin{cases} \frac{5k}{16} + \frac{k}{48} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 2 \\ \frac{5k}{24} + \frac{k}{12} + C_3 + \frac{2^{\omega(d)+1}}{3} & \text{if } \omega(d) = 3. \end{cases}$$

We apply Lemma 14 again to get  $\max_{i \in T_1} A_i \geq 2^\delta \frac{5}{16}k$  so that  $n + (k-1)d \geq 2^\delta \frac{5}{16}k^3$ , which implies (7). This completes the proof.  $\square$

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