Indian Statistical Institute, Delhi
Maths 271: Mathematical Methods
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Homework Assignment 1
Due 11.30 a.m. on 3 August 2009

1. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

   (a) Show that d is a metric on $\mathbb{R}^2$.

   (b) Draw an open ball in $\mathbb{R}^2$ in this metric.

   (c) Prove that the open sets of the metric spaces $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, \rho)$ are the same, where ρ is the Euclidean metric in $\mathbb{R}^2$.

2. Let $\{O_\alpha\}_{\alpha \in A}$ be a (possibly infinite) collection of open sets (in a metric space X).

   (a) Show that $\cup_{\alpha \in A} O_\alpha \equiv \{x | x \in O_\alpha$ for at least one $\alpha \in A\}$ is an open set.

   (b) Is $\cap_{\alpha \in A} O_\alpha \equiv \{x | x \in O_\alpha \forall \alpha \in A\}$ necessarily an open set? Prove or disprove. What happens if A is finite?

3. Let S be a subset of $\mathbb{R}$. Define a metric on S to be the usual (Euclidean) metric d restricted to S (meaning that the domain of d is $S \times S$ instead of $\mathbb{R} \times \mathbb{R}$).

   (a) If $O \subset S$ is an open set in $(\mathbb{R}, d)$, must it also be an open set in (S, d)? Recall that an open ball in a metric space (X, d) is a subset of X of the form $\{x \in X | d(x, x_0) < \varepsilon\}$ where $x_0 \in X$ and $\varepsilon > 0$.

   (b) Is every open set in the metric space (S, d) also an open set in the metric space $(\mathbb{R}, d)$? Prove or disprove.

   (c) Does the answer to (b) change if S is an open set in $\mathbb{R}$?

   (d) What if S is an open ball in $\mathbb{R}$?