Indian Statistical Institute, Delhi
Maths 271: Mathematical Methods
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Homework Assignment 8
Due 11.30 a.m. on 14 September 2009

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

   Determine the standard matrix A for T.

2. Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   Consider a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ with $L(e_1) = e_1, L(e_2) = e_1 + e_2, L(e_3) = e_2 + e_3$.

   (a) Determine the standard matrix A for L.

   (b) Does L map $\mathbb{R}^3$ onto $\mathbb{R}^3$? (Recall that an onto mapping is one for which the range is the whole co-domain.)

3. Find bases for the range (also called the *image*) and the null space (also called the *kernel*) of the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & 0 \\ 2 & 4 & 5 & 7 & 2 \\ -1 & -2 & 0 & 4 & 4 \end{bmatrix}$$

   from problem 1 of Assignment 7.

4. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ whose range has dimension n must be 1-1. (Recall that a function is 1-1 if every element in the range has a unique pre-image.)

5. Prove that a subspace V of an n-dimensional vector space W has dimension n if and only if $V = W$. Note that this implies that $T : \mathbb{R}^n \to \mathbb{R}^m$ is an onto function if and only if the image of T has dimension m. Note further that when $n = m$, then together with the previous problem, this implies that T is invertible iff $r = m = n$.

6. Consider the translation mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}.$$

   Is T a linear transformation? Why, or why not?

7. The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

   yields a shearing transformation, which leaves the y-axis unchanged. Sketch its effect on the x-axis by indicating what happens to (1,0), (2,0), and (-1,0), and how the whole axis is transformed.

8. On the 4-dimensional vector space P^3 of cubic polynomials in t, construct the 4 x 4 matrix that represents the linear transformation of differentiation $D : P^3 \to P^3$, using the standard basis $\{1, t, t^2, t^3\} = \{e_1, e_2, e_3, e_4\}$. (A polynomial is represented by its coefficient vector.) What is the nullspace and column space of the second derivative, D^2?