

# Maths 271, Mathematical Methods, Solutions to Midterm Exam

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1. (a) Let  $\{x_n\}$  be a Cauchy sequence. Then, for  $\varepsilon = 1$ , there exists  $n_0 \in \mathbb{N}$  such that  $\|x_n - x_m\| < 1 \forall n, m \geq n_0$ . Consider some fixed  $m \geq n_0$ . Now

$$\begin{aligned}\|x_n\| &= \|x_n - x_m + x_m\| \\ &\leq \|x_n - x_m\| + \|x_m\| \\ &< 1 + \|x_m\| \quad \forall n \geq n_0.\end{aligned}$$

Let  $M = \max\{1 + \|x_m\|, \|x_1\|, \dots, \|x_{n_0-1}\|\}$ . Then  $\|x_n\| < M, \forall n \in \mathbb{N}$ .

- (b) Let  $\{x_n\}$  be a Cauchy sequence. By (a), there exists  $M \in \mathbb{R}$  such that  $\|x_n\| < M, \forall n \in \mathbb{N}$ . Let  $S$  be the set of points in  $\mathbb{R}^l$  with norm less than or equal to  $M$ .  $S$  is closed and bounded, and  $\{x_n\} \subset S$ . So by the Bolzano-Weierstrass Theorem,  $\{x_n\}$  has a convergent subsequence converging to a limit  $x \in S$ . We now show that  $x_n \rightarrow x$ .

Let  $\varepsilon > 0$ . There exists  $n_0 \in \mathbb{N}$  such that  $\|x_n - x_m\| < \varepsilon/2 \forall n, m \geq n_0$ . And since  $\{x_n\}$  has a subsequence converging to  $x$ , there must be a member of this subsequence, say  $x_k$ , with  $k > n_0$  and  $\|x_k - x\| < \varepsilon/2$ . Hence, for all  $n \geq n_0$ ,

$$\|x_n - x_k\| + \|x_k - x\| < \varepsilon.$$

By the triangle inequality,  $\|x_n - x\| < \varepsilon \quad \forall n \geq n_0$ . Therefore,  $x_n \rightarrow x$ .

2. Let  $\varepsilon > 0$ . Let  $\delta = \varepsilon$ . Then

$$\begin{aligned}|x - 0| < \delta &\Rightarrow |x - 0| < \varepsilon \\ &\Rightarrow |x| \left| \sin \frac{1}{x} \right| < \varepsilon \\ &\Rightarrow \left| x \sin \frac{1}{x} - 0 \right| < \varepsilon.\end{aligned}$$

So  $f$  is continuous at 0.

3. The orthogonal complement is the set of vectors in  $\mathbb{R}^3$  that solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 12 \\ -1 & -2 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reducing to echelon form, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So the solution is  $x_3 = 0, x_1 = -2x_2$ . Or

$$x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ which is the span of } \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

4. Let  $V \perp W$ , where  $V, W$  are subspaces of  $\mathbb{R}^n$ . Let  $\{v_1, \dots, v_k\}, \{w_1, \dots, w_m\}$  be bases for  $V, W$  respectively.

- (a) If  $w_1$  is a linear combination of  $v_1, \dots, v_k$ , then  $w_1 \in V \cap W$ . So  $w_1^T w_1 = 0$ . So  $w_1 = 0$ , which contradicts the fact that  $w_1$  is a basis element.
- (b) If  $\{v_1, \dots, v_k, w_1, \dots, w_m\}$  is not a linearly independent set, then there exist scalars  $a_i, b_j, \quad i = 1, \dots, k, \quad j = 1, \dots, m$ , not all zero, such that

$$\sum_{i=1}^k a_i v_i + \sum_{j=1}^m b_j w_j = 0,$$
$$\text{or } \sum_{i=1}^k a_i v_i = - \sum_{j=1}^m b_j w_j = u \text{ (say).}$$

$u \in V \cap W$ . So as in (a) above,  $u = 0$ . Since  $\{v_1, \dots, v_k\}, \{w_1, \dots, w_m\}$  are bases, we get  $a_i = b_j = 0$ , for all  $i = 1, \dots, k, \quad j = 1, \dots, m$ . This is a contradiction, so  $\{v_1, \dots, v_k, w_1, \dots, w_m\}$  is a linearly independent set. Since it clearly spans  $V + W$ , it is a basis for  $V + W$ . Therefore,  $\dim(V + W) = \dim V + \dim W$ .

5. (a)  $T$  is invertible with  $T^{-1}(v) = \frac{1}{5}v$ .
- (b) Not invertible since the range has dimension 2, so  $T$  cannot be one-one.
- (c) Invertible with  $T^{-1} = T$ .