Some bivariate notions of IFR and DMRL and related properties

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April 24, 2002
isid/ms/2002/07

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AMS classification: 62N05, 60G09, 90B25.

Key words : Residual lifetimes. Mean residual life function. Multivariate extensions of IFR and DMRL. Log-concavity. Schur-concavity. Aging properties of mixtures.

Abstract

Recently Bassan and Spizzichino (1999) have given some new concepts of multivariate aging for exchangeable random variables, such as a special type of bivariate IFR, by comparing distributions of residual lifetimes of dependent components of different ages. In the same vein, we further study some properties of concepts of IFR in the bivariate case. Then we introduce concepts of bivariate DMRL aging and we develop a treatment that parallels the one developed for bivariate IFR. For both concepts of IFR and DMRL, we analyze a weak and a strong version, and discuss some of the differences between them.

1 Introduction

Univariate concepts of aging like IFR (increasing failure rate), NBU (new better than used), DMRL (decreasing mean residual life) have played an important role in survival analysis, reliability theory, maintenance policies, operations research and many other areas of applied probability. They have also been found useful in getting bounds and inequalities on efficiencies of estimates and tests.

A complex system usually consists of several components which are working under the same environment and hence their lifetimes are, generally, dependent. In the literature several attempts have been made to extend the concepts of univariate aging to the multivariate case. Some important references are Brindley and Thompson (1972), Arjas (1981), Savits (1985), Shaked and Shanthikumar (1988, 1991), Barlow and Mendel (1993), Barlow and Spizzichino (1993) and Bassan and Spizzichino (1999), among others. In this paper we consider the case when the lifetimes of the components have exchangeable joint probability distribution. We introduce some new notions of multivariate aging and also further discuss the properties of the ones recently introduced in the literature. We fix our attention only on the bivariate case, though these ideas can be easily extended to the multivariate case. We shall be assuming that the random variables under consideration are nonnegative. Absolute continuity will be tacitly assumed whenever needed.

Both from the technical and conceptual point of view, the interest in the paper is concentrated on the conditional distributions of residual lifetimes, given different types of survival data. In fact, these objects will play a central role both in the formulation of the different definitions and in the interpretation of related results.

Before we go into the details, let us quickly review some common notions of univariate positive aging and stochastic orderings of various kinds. We shall denote the density function, the survival function and the hazard rate function of a univariate random variable X by f_X, $\overline{F}_X$ and r_{F_X}, respectively.

A random variable X is said to be *stochastically* larger than another random variable Y (denoted by $X \geq_{st} Y$) if $\overline{F}_X(x) \geq \overline{F}_Y(x)$ for all x. A stronger notion of stochastic dominance is that of *hazard rate* ordering. X is said to be larger than Y in *hazard rate* ordering (denoted by $X \geq_{hr} Y$) if $\overline{F}_X(x)/\overline{F}_Y(x)$ is non-decreasing in x. In the continuous case, this is equivalent to $r_{F_X}(x) \leq r_{F_Y}(x)\ \forall x$. Finally, X is said to be larger than Y in *likelihood ratio* ordering (denoted by $X \geq_{lr} Y$) if $f_X(x)/f_Y(x)$ is non-decreasing in x. In case X and Y have a common left end-point of their supports, we have the following chain of implications among the above stochastic orders: $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{st} Y$.

Let X denote the lifetime of a unit with survival function $\overline{F}$ and let us denote by $\overline{F}_t$ the survival function of the residual life of a unit surviving at time t, that is,

$$\begin{aligned} \overline{F}_t(x) &= P[X - t > x | X > t] \\ &= \frac{\overline{F}(x+t)}{\overline{F}(t)}. \end{aligned}$$

Let us denote by X_t, a random variable with survival function $\overline{F}_t$. A unit with survival function $\overline{F}$ ages positively with time if the random variable X_t decreases with time t in *some stochastic sense*. That is if,

$$X_{t_1} \geq_* X_{t_2},$$

for all (t_1, t_2) in some region $A \subset \{(t_1, t_2)|\ 0 \leq t_1 < t_2\}$ where $\geq_*$ can be any of the above one-dimensional stochastic orderings $\geq_{st}$, $\geq_{hr}$, $\geq_{lr}$ or the orderings of the means. Different concepts of positive aging correspond to different regions A and different stochastic orders between X_{t_1} and X_{t_2} for $(t_1, t_2) \in A$ (see the discussion in Spizzichino (2001)). For example, a random variable X is NBU if and only if $X \geq_{st} X_t$, $\forall 0 < t$. Furthermore, a random variable X has IFR distribution if and only if $X_{t_1} \geq_{st} X_{t_2}$, $\forall\ t_1 < t_2$ and also if and only if $X_{t_1} \geq_{hr} X_{t_2}$, $\forall\ t_1 < t_2$.

The mean residual life (MRL) of a unit or a subject at age t is the average remaining life among those population members who have survived until time t. The mean residual life function (MRLF) is defined as

$$\begin{aligned} \mu_F(t) &= \mathbb{E}[X_t] \\ &= \mathbb{E}[X - t|X > t] \\ &= \int_0^\infty \overline{F}_t(x)\, dx \\ &= \frac{\int_t^\infty \overline{F}(x)\, dx}{\overline{F}(t)}. \end{aligned}$$

Like the failure rate function, the MRLF can be used to describe conditional concepts of aging; however, the MRLF is more intuitive, especially in the health sciences. The review article by Guess and Proschan (1988) gives a nice summary of the theory of MRL and an extensive bibliography.

In many applications it is reasonable to assume that the life system is monotonically degenerating. This concept has been modelled several ways, of which increasing failure rate (IFR) is probably the most studied. The somewhat weaker version of decreasing mean residual life (DMRL), which is implied by IFR, is perhaps more clear conceptually and is easier to explain to the user.

A random variable X is said to have a $DMRL$ (decreasing mean residual life) distribution if $\mu_F(t) = \mathbb{E}[X_t]$ is decreasing in t. Note that X is $DMRL$ if and only if the function $\int_t^\infty \overline{F}(x)\, dx$ is log-concave in t. Kochar, Mukerjee and Samaniego (2000) consider the problem of estimating the MRLF of a DMRL distribution.

A weaker concept of positive aging is that of new better than used in expectation (NBUE). A random variable X is said to have NBUE distribution if

$$\mu_F(t) \leq \mu_F(0) \quad \forall\ t > 0. \tag{1.1}$$

See Barlow and Proschan (1981) and Deshpande, Kochar and Singh (1986) for the definitions of other concepts of aging.

Let $\mu_{F_X}(t)$ and $\mu_{F_Y}(t)$ denote the mean residual life functions of X and Y, respectively.

Definition 1.1. X is said to be greater than Y according to mean residual life (MRL) ordering (written as $X \geq_{mrl} Y$) if

$$\mu_{F_X}(t) \geq \mu_{F_Y}(t) \quad \forall t \geq 0$$

It is immediate to check that $X \geq_{mrl} Y$ if and only if

$$\frac{\int_t^\infty \overline{F}(x)\, dx}{\int_t^\infty \overline{G}(x)\, dx} \text{ is nondecreasing in } t.$$

Note that hazard rate ordering implies mean residual life ordering, but the converse is not true. For more details on stochastic orderings, see Chapter 1 of Shaked and Shanthikumar (1994).

The organization of the paper is as follows. In Section 2 we recall the concept of bivariate IFR aging as introduced by Bassan and Spizzichino (1999) and study some of its new properties. In Section 3, we introduce the concept of bivariate DMRL distributions for exchangeable random variables and study its properties. In the Appendix, we review some concepts of dependence that are used in this paper and the readers, who are not familiar with this area, are advised to read the Appendix before reading the next two sections.

2 Bivariate IFR property

Recalling that a random variable X has IFR distribution if and only if $\mathcal{L}(X - t_1|X > t_1) \geq_{st} \mathcal{L}(X - t_2|X > t_2),\ \forall\, t_1 < t_2$ and also if and only if $\mathcal{L}(X - t_1|X > t_1) \geq_{hr} \mathcal{L}(X - t_2|X > t_2)\ \ \forall\, t_1 < t_2$, Bassan and Spizzichino (1999) considered the following two extensions of the concept of univariate IFR aging to the bivariate case.

Definition 2.1. An exchangeable random vector $\mathbf{T} = (T_1, T_2)$ is bivariate IFR (BIFR) if

$$\mathcal{L}(T_1 - t_1|T_1 > t_1, T_2 > t_2) \geq_{st} \mathcal{L}(T_2 - t_2|T_1 > t_1, T_2 > t_2) \quad \text{for } t_1 \leq t_2. \tag{2.1}$$

Definition 2.2. An exchangeable random vector $\mathbf{T} = (T_1, T_2)$ is said to be bivariate IFR in the strong sense (s-BIFR) if

$$\mathcal{L}(T_1 - t_1|T_1 > t_1, T_2 > t_2) \geq_{hr} \mathcal{L}(T_2 - t_2|T_1 > t_1, T_2 > t_2) \quad \text{for } t_1 \leq t_2. \tag{2.2}$$

Notice that (2.1) holds if and only if the joint survival function $\overline{F}(t_1, t_2) = \mathbb{P}[T_1 > t_1, T_2 > t_2]$ of (T_1, T_2) is Schur-concave in (t_1, t_2). For details on Schur-concavity, see Marshall and Olkin (1979); see also the discussion in Spizzichino (2001) and references therein for the use of Schur-concavity in multivariate aging.

On the other hand, (T_1, T_2) is s-BIFR if and only if

$$R(t) = \frac{\overline{F}(x + t, y)}{\overline{F}(y + t, x)} \text{ is increasing in } t \text{ for } x < y. \tag{2.3}$$

Taking logs and then differentiating both sides, we see that this is equivalent to

$$r_{T_1|T_2}(y+t|T_2>x) \geq r_{T_1|T_2}(x+t|T_2>y) \text{ for } t>0 \text{ and } 0 \leq x < y, \tag{2.4}$$

where $r_{T_1|T_2}(.|T_2 > t_2)$ denotes the hazard rate of the conditional distribution of T_1 given $\{T_2 > t_2\}$.

Unless the random variables T_1 and T_2 are independent, the two concepts of multivariate aging as given by Definitions 2.1 and 2.2 are not equivalent. The latter implies the former, but the converse is not true; see the example presented in Remark 3.6 of Bassan and Spizzichino (1999), where some properties of these bivariate notions of aging have been studied in detail.

The first aspect that we focus on concerns the (unidimensional) aging properties of the marginal distributions of the considered bivariate distributions. As it was observed in Bassan and Spizzichino (1999), the marginals of a BIFR random vector need not be IFR unless the component lifetimes are independent. They show that, however, under some negative dependence between the components, the marginal distributions are NBU. In the next theorem we prove that under a suitable negative dependence condition between T_1 and T_2, the marginals of s-BIFR random vectors are IFR.

Theorem 2.1. *Let (T_1, T_2) be s-BIFR and let T_1 be right tail decreasing in T_2. Then the marginal distribution of T_1 is IFR.*

Proof. Since (T_1, T_2) being s-BIFR is equivalent to (2.4), taking $x = 0$ in it, we get for $y > 0\,, t > 0$,

$$\begin{aligned} r_{T_1}(y+t) &\geq r_{T_1|T_2}(t|T_2>y) \\ &\geq r_{T_1}(t) \;\text{ by the RTD property.} \end{aligned}$$

This proves the required result. ∎

Now we give an example of an s-BIFR distribution which satisfies the conditions of the above theorem.

Example 2.1. Let the joint distribution of (T_1, T_2) be

$$\overline{F}(x,y) = exp\{1 - exp(x^2+y^2)\} \; x \geq 0,\, y \geq 0. \tag{2.5}$$

It is easy to see that

$$\begin{aligned} R(t) &= \frac{\overline{F}(x+t,y)}{\overline{F}(y+t,x)} \\ &= exp\{e^{x^2+(y+t)^2} - e^{y^2+(x+t)^2}\} \end{aligned}$$

is increasing in t for $x \leq y$. Hence (T_1, T_2) is s-BIFR.

Also T_1 is right tail decreasing in T_2 since

$$\frac{\overline{F}(t,y)}{\overline{F}(t,0)} = exp[e^{t^2} - e^{t^2+y^2}] \tag{2.6}$$

is clearly decreasing in t for $y > 0$.

Hence this bivariate distribution satisfies the conditions of the above theorem. It is s-BIFR and its marginals are also IFR.

An interesting fact in the literature and in applications of unidimensional aging notions (see e.g. the review by paper Shaked and Spizzichino (2001) and references therein; for a recent contribution, see Finkelstein and Esaulova (2001)) is that mixtures of distributions with some property of positive aging do not necessarily maintain the same property. Then one can be generally interested in finding conditions on the mixtures under which one-dimensional or multivariate aging properties are preserved.

We consider here conditionally i.i.d pairs of lifetimes. In this case, both the joint law and the marginal law can be written as mixtures. It may be interesting to notice that bivariate aging properties may be preserved under mixtures, whereas the opposite may hold for one-dimensional properties. For example, consider two i.i.d. IFR lifetimes. All the conditional joint laws are Schur-concave (BIFR), and hence so is the unconditional joint law, since Schur-concavity is preserved under mixtures (see Barlow and Mendel (1992) for a discussion on this issue). On the other hand, the marginal law is a mixture of IFR laws, and hence it need not be IFR.

In the next theorem we give sufficient conditions under which the joint law of conditionally i.i.d. IFR random variables is s-BIFR. First, since we shall deal repeatedly with conditionally i.i.d. random variables, we single out the relevant assumptions and notations.

Hypotheses 2.1. Θ is a random variable taking values in a set L, Π is its (prior) distribution and, when existing, π is its (prior) density. Given Θ, T_1 and T_2 are conditionally i.i.d. random variables with a common conditional survival function $\overline{G}(\cdot|\theta)$. The joint survival function is then given by

$$\overline{F}(t_1, t_2) = \int_L \overline{G}(t_1|\theta)\overline{G}(t_2|\theta)d\Pi\,(\theta)\,. \tag{2.7}$$

Theorem 2.2. *Let Hypotheses 2.1 be satisfied, with $L \subset \mathbb{R}$. Assume also the following:*

1. *the mapping $(t, \theta) \mapsto \bar{G}(t|\theta)$ is log-concave (this implies, in particular, that all the conditional laws are IFR) and TP_2.*

2. *the mixing distribution Π has a log-concave density π*

Then, for $t_1 < t_2$, and letting $D = \{T_1 > t_1, T_2 > t_2\}$, we have

$$\mathcal{L}\,(T_1 - t_1|D) \geq_{hr} \mathcal{L}\,(T_2 - t_2|D)\,.$$

For the proof, we prefer to single out two simple lemmas which will be helpful in the next section as well.

Lemma 2.1. *Let*

- $(x,z) \mapsto \phi(x,z)$ *be (jointly) log-concave and* TP_2
- $(x,y) \mapsto \psi(x,y)$ *be* TP_2
- $z \mapsto \psi(y,z)$ *be log-concave,* $\forall\, y$
- $z \mapsto h(z)$ *be log-concave*

Then

1. $(x,y) \mapsto \psi(x,y)\, h(y)$ *is* TP_2
2. $(x,z) \mapsto \phi(x,z)\, \psi(y,z)\, h(z)$ *is log-concave,* $\forall\, y$
3. $(x,y) \mapsto \int \phi(x,z)\, \psi(y,z)\, h(z)\, dz$ *is* TP_2
4. $x \mapsto \int \phi(x,z)\, \psi(y,z)\, h(z)\, dz$ *is log-concave,* $\forall\, y$.

Proof. It is a simple matter to check that the first and the second claim hold. The third claim follows from the basic composition formula (Karlin (1968)), and the fourth one from Prékopa's Theorem (Prékopa (1973)). ■

Lemma 2.2. *Let* $(x,y) \mapsto H(x,y)$ *be* TP_2*, and let* $x \mapsto H(x,y)$ *be log-concave,* $\forall\, y$*. Then, for* $x < y$*, the mapping*

$$t \mapsto R(t;x,y) := \frac{H(x+t,y)}{H(y+t,x)}$$

is increasing.

Proof. Write $R(t,x,y) = R_1(t,x,y)R_2(t,x,y)$, with

$$R_1(t,x,y) = \frac{H(x+t,y)}{H(y+t,y)}, \qquad R_2(t,x,y) = \frac{H(y+t,y)}{H(y+t,x)}.$$

We will show that R_1 and R_2 are increasing in t. In fact, R_1 is increasing because $z \mapsto H(z,y)$ is log-concave for every y. Furthermore, from the TP_2 property of H we get

$$H(y+t+a,x)\, H(y+t,y) \le H(y+t,x)\, H(y+t+a,y),$$

and it follows immediately than R_2 is increasing. ■

Proof of Theorem 2.2. Let $\phi(x,\theta) = \psi(x,\theta) = \bar{G}(x|\theta)$, let $h = \pi$ and let H be the joint survival function:

$$H(t_1,t_2) := \bar{F}(t_1,t_2) = \int \bar{G}(t_1|\theta)\bar{G}(t_2|\theta)\pi(\theta)d\theta.$$

We see that ϕ, ψ and h satisfy the conditions of Lemma 2.1, and hence H satisfies the conditions of Lemma 2.2. It follows that

$$t \mapsto \frac{\bar{F}(t_1+t,t_2)}{\bar{F}(t_2+t,t_1)}$$

is increasing, and the claim follows. ■

Remark 2.1. The conditions of Theorem 2.2 guarantee that the marginals are IFR, as one can easily check using Prékopa's Theorem. These conditions are similar to those of Lynch (1999) for a mixture of IFR distributions to be IFR. In this respect we also notice the following:

In the case when T_1, T_2 are conditionally i.i.d. given Θ, the condition (2.2) can be rewritten in the form

$$\int_L \frac{\overline{G}(t_1+r+\tau|\theta)}{\overline{G}(t_1+r|\theta)} \pi(\theta|T_1 > t_1+r, T_2 > t_2) d\theta \geq$$
$$\int_L \frac{\overline{G}(t_2+r+\tau|\theta)}{\overline{G}(t_2+r|\theta)} \pi(\theta|T_1 > t_1, T_2 > t_2+r) d\theta, \forall\ 0 < t_1 < t_2, \text{ and } r, \tau > 0$$

where we let

$$\pi(\theta|T_1 > t_1+r, T_2 > t_2) \propto \pi(\theta)\,\overline{G}(t_1+r|\theta)\overline{G}(t_2|\theta),$$

which, by Bayes Formula, can be interpreted as the conditional distribution of Θ, given the observation of the survival data

$$\{T_1 > t_1+r, T_2 > t_2\}.$$

Analogously for $\pi(\theta|T_1 > t_1, T_2 > t_2+r)$.

Notice, on the other hand, that the condition that the marginal distribution of T_1, T_2 be IFR can be rewritten in the form

$$\int_L \frac{\overline{G}(t_1+\tau|\theta)}{\overline{G}(t_1|\theta)} \pi(\theta|T_1 > t_1) d\theta \geq \int_L \frac{\overline{G}(t_2+\tau|\theta)}{\overline{G}(t_2|\theta)} \pi(\theta|T_1 > t_2) d\theta.$$

3 Bivariate DMRL property

In this section we introduce a weak and a strong notion of bivariate DMRL aging, in analogy with Definitions 2.1 and 2.2. A natural way to do so is to take into account the conditional mean residual life function given an observed bivariate survival data. A first, natural, definition in this vein is as follows.

Definition 3.1. An exchangeable random vector $\mathbf{T} = (T_1, T_2)$ is said to have bivariate DMRL (BDMRL) distribution if for $t_1 < t_2$,

$$\mathbb{E}[T_1 - t_1|T_1 > t_1, T_2 > t_2] \geq \mathbb{E}[T_2 - t_2|T_1 > t_1, T_2 > t_2] \tag{3.1}$$

Note that (T_1, T_2) is BDMRL if and only if one (and hence all) of the following equivalent conditions holds:

$$\int_{t_1}^{\infty} \overline{F}(x, t_2)\, dx \geq \int_{t_2}^{\infty} \overline{F}(x, t_1)\, dx \text{ for } t_1 < t_2$$
$$\int_{t_1}^{\infty}\int_{t_2}^{\infty} \overline{F}(x, y)\, dx\, dy \text{ is Schur-concave in } (t_1, t_2),$$
$$\mu_{T_1|T_2}(t_1|T_2 > t_2) \geq \mu_{T_1|T_2}(t_2|T_2 > t_1) \text{ for } t_1 < t_2,$$

where $\mu_{T_1|T_2}(\cdot|T_2 > t_2)$ denotes the mean residual life function of the conditional distribution of T_1 given $\{T_2 > t_2\}$.

Since stochastic ordering implies the ordering of the means, it follows immediately that bivariate IFR property implies bivariate DMRL property.

Note that for a univariate random variable X, the mean residual life $\mu_t(x)$ of X_t at time x satisfies the relation $\mu_t(x) = \mu_F(t + x)$.

Therefore, X is DMRL

$$\begin{aligned}
&\Leftrightarrow \mu_F(x) \text{ is decreasing in } x \\
&\Leftrightarrow \mu_t(x) \text{ is decreasing in } x, \forall\ t \geq 0, \\
&\Leftrightarrow \mathcal{L}(X - t_1|X > t_1) \geq_{mrl} \mathcal{L}(X - t_2|X > t_2)\ \forall\ t_1 < t_2.
\end{aligned}$$

Then, in the case of i.i.d variables T_1, T_2, the condition that they are DMRL is equivalent to

$$\mathcal{L}(T_1 - t_1|T_1 > t_1, T_2 > t_2) \geq_{mrl} \mathcal{L}(T_2 - t_2|T_1 > t_1, T_2 > t_2)\ \forall\ t_1 < t_2. \tag{3.2}$$

Similarly to what we argued for the bivariate extension of the notion of IFR, notice now that the equivalence between (3.1) and (3.2) is not anymore valid in the case when T_1, T_2 are exchangeable, but not independent variables.

This encourages us to propose the following stronger notion of bivariate DMRL property.

Definition 3.2. We say that an exchangeable random vector (T_1, T_2) is bivariate DMRL in the strong sense (s-BDMRL) if for all $t_1 < t_2$, the inequality (3.2) holds.

Note that (T_1, T_2) is s-BDMRL if and only if one (and hence both) of the following equivalent conditions hold:

$$\begin{aligned}
&\mu_{T_1|T_2}(x + t_1|T_2 > t_2) \geq \mu_{T_1|T_2}(x + t_2|T_2 > t_1) \text{ for } t_1 < t_2 \text{ and for } x > 0 \\
&\frac{\int_{x+t_1}^{\infty} \overline{F}(u, t_2)\, du}{\int_{x+t_2}^{\in fty} \overline{F}(u, t_1)\, du} \text{ is increasing in } x > 0.
\end{aligned}$$

In the following example we show a bivariate law which is BDMRL but not s-BDMRL.

Example 3.1. Let

$$\overline{F}_{T_1,T_2}(u, v) \propto [1 + u^3 + v^3]^{-2}$$

It is easy to check that this survival function is Schur- concave. Thus the joint law is BIFR, and hence it is BDMRL. In order to see whether it is s-BDMRL, we have to examine whether

$$\frac{\int_{x+t_1}^{\infty} \overline{F}(u, t_2)\, du}{\int_{x+t_2}^{y} \overline{F}(u, t_1)\, du} = \frac{\int_{x+1}^{\infty} [65 + u^3]^{-2}\, du}{\int_{x+4}^{\infty} [2 + u^3]^{-2}\, du}$$

is increasing in x. On examining the plot of this function, we see that it is not monotone. Hence this distribution is not s-BDMRL. Since $\geq_{hr}$ implies $\geq_{mrl}$, this distribution is not s-BIFR either.

In the next theorem we give sufficient conditions for bivariate s-DMRL property.

Theorem 3.1. *Let*

(a)$\mu_{T_1|T_2}(x|T_2 > y)$ *be increasing in* y *for every* $x > 0$*, and;*

(b) $\mu_{T_1|T_2}(x|T_2 > y)$ *be decreasing in* x *for every* $y > 0$
(that is, T_1 *is conditionally DMRL for every given* $T_2 > y$*).*

Then (T_1, T_2) *is s-DMRL.*

Proof. For $t_1 < t_2$ and $x > 0$,

$$\begin{aligned} \mu_{T_1|T_2}(x + t_1|T_2 > t_2) &\geq \mu_{T_1|T_2}(x + t_2|T_2 > t_2) \text{ by (b)} \\ &\geq \mu_{T_1|T_2}(x + t_2|T_2 > t_1) \text{ by (a).} \end{aligned}$$

This proves the required result. ■

In the next theorem we show that under a certain type of negative dependence between the variables, the marginal distributions of a bivariate DMRL vector are NBUE.

Theorem 3.2. *Let* (T_1, T_2) *be bivariate DMRL and be negatively dependent in the sense that*

$$\mathbb{E}(T_1|T_2 > t) \leq \mathbb{E}(T_1) \; \forall \, t > 0. \tag{3.3}$$

Then the marginal distribution of T_1 *is NBUE.*

Proof. (T_1, T_2) being DMRL is equivalent to

$$\mu_{T_1|T_2}(t_2|T_2 > t_1) \leq \mu_{T_1|T_2}(t_1|T_2 > t_2) \text{ for } t_1 < t_2. \tag{3.4}$$

Taking $t_1 = 0$ in it, we get, for $t_2 > 0$,

$$\begin{aligned} \mu_{T_1}(t_2) &\leq \mu_{T_1|T_2}(0|T_2 > t_2) \\ &= \mathbb{E}(T_1|T_2 > t_2) \\ &\leq \mathbb{E}(T_1) \quad \text{by (3.3)} . \end{aligned}$$

Hence T_1 is NBUE. ■

Remark 3.1. The condition (3.3) is implied by negative quadrant dependence between T_1 and T_2.

In the next theorem we give sufficient conditions under which the marginals of a multivariate s-DMRL distribution are DMRL.

Theorem 3.3. *Let (T_1, T_2) be s-DMRL and let*

$$\mu_{T_1|T_2}(x|T_2 > y) \leq \mu_{T_1}(x) \quad \forall\, x, y > 0. \tag{3.5}$$

Then the marginal distribution of T_1 is DMRL.

Proof. (T_1, T_2) being s-DMRL is equivalent to

$$\mu_{T_1|T_2}(x + t_2|T_2 > t_1) \leq \mu_{T_1|T_2}(x + t_1|T_2 > t_2) \text{ for } t_1 < t_2 \text{ and for } x > 0$$

Taking $t_1 = 0$ in it, we get,

$$\mu_{T_1|T_2}(x + t_2|T_2 > 0) \leq \mu_{T_1|T_2}(x|T_2 > t_2) \text{ for } t_1 < t_2 \text{ and for } x > 0$$

That is, for $x \geq 0$, $t_2 \geq 0$,

$$\begin{aligned} \mu_{T_1}(x + t_2) &\leq \mu_{T_1|T_2}(x|T_2 > t_2) \\ &\leq \mu_{T_1}(x) \text{ by (3.5)}, \end{aligned}$$

proving thereby that the distribution of T_1 is DMRL. ■

We now turn to consider the BDMRL and s-BDMRL properties for the case of conditionally i.i.d variables. First, we show that conditionally i.i.d. DMRL random variables are bivariate DMRL.

Theorem 3.4. *Let T_1 and T_2 be conditionally i.i.d., as specified in Hypotheses 2.1. Let the conditional law of T_1 (and T_2) given $\Theta = \theta$ be DMRL, for all θ. Then the joint distribution of (T_1, T_2) is bivariate DMRL.*

Proof. As seen earlier, conditionally on $\Theta = \theta$, T_1 is $DMRL$ iff

$$\int_t^\infty \overline{G}(x|\theta)\, dx \text{ is log- concave in } t.$$

Recalling (2.7), we have

$$\begin{aligned} \int_{t_1}^\infty \int_{t_2}^\infty \overline{F}(x, y)\, dx\, dy &= \int_{t_1}^\infty \int_{t_2}^\infty \left[\int_{\theta \in L} \overline{G}(x|\theta)\overline{G}(y|\theta)\, d\Pi(\theta) \right] dx\, dy \\ &= \int_{\theta \in L} \left[\int_{t_1}^\infty \int_{t_2}^\infty \overline{G}(x|\theta)\overline{G}(y|\theta) dx\, dy \right] d\Pi(\theta). \end{aligned}$$

Since this expression is a mixture of log-concave functions, it is Schur-concave. Hence the result. ■

In the next theorem we establish conditions under which conditionally i.i.d. DMRL random variables are bivariate DMRL in the strong sense.

Theorem 3.5. *Let T_1 and T_2 be conditionally i.i.d., as specified in Hypotheses 2.1, with $L \subset \mathbb{R}$ and Π admitting a density π. We assume the following:*

- $\int_x^\infty \overline{G}(\xi|\theta)d\xi$ *is jointly log-concave in* θ *and* x *(and hence, in particular,* $\overline{G}(\cdot|\theta)$ *is DMRL* $\forall\ \theta$*).*

- $\overline{G}(v|\theta)$ *is* TP_2 *and log- concave as a function of* θ

- $\pi(\theta)$ *is log-concave.*

Then the joint distribution of (T_1, T_2) *is bivariate s-DMRL.*

Proof. Let $\bar{F}$ be the joint survival function and let

$$H(t_1, t_2) := \int_{t_1}^{\infty} \bar{F}(u, t_2)du.$$

Clearly,

$$\begin{aligned} H(t_1, t_2) &= \int_{t_1}^{\infty} \left[\int_L \bar{G}(u|\theta)\bar{G}(t_2|\theta)\pi(\theta)d\theta \right] du \\ &= \int_L \left[\int_{t_1}^{\infty} \bar{G}(u|\theta)du \right] \bar{G}(t_2|\theta)\pi(\theta)d\theta \end{aligned}$$

We need to show that, for $t_1 < t_2$,

$$t \mapsto \frac{H(t_1 + t, t_2)}{H(t_2 + t, t_1)}$$

is increasing.

Let

$$\phi(x, \theta) := \int_x^{\infty} \bar{G}(u|\theta)du; \qquad \psi(x, \theta) = \bar{G}(x|\theta); \qquad h = \pi.$$

Observe that $(t_1, \xi) \mapsto \mathbf{1}_{(t_1,\infty)}(\xi)$ is TP_2, and hence by the basic composition formula ϕ is also TP_2. Thus ϕ, ψ and h satisfy the conditions of Lemma 2.1, and hence H satisfies the conditions of Lemma 2.2 and the conclusion follows. ■

4 Appendix

Some notions of dependence

There are several notions of positive and negative dependence between random variables and these have been discussed in detail in Barlow and Proschan (1981) and Shaked (1977). For a brief introduction, see Khaledi and Kochar (2000). The following concepts are used in this paper.

Let (T_1, T_2) be a bivariate random variable with joint density function $f(t_1, t_2)$ and joint survival function $\overline{F}(t_1, t_2) = P[T_1 > t_1, T_2 > t_2]$.

Definition 4.1. (T_1, T_2) is said to be **positively quadrant dependent** (PQD) if

$$\{T_1 | T_2 > t_2\} \geq_{st} T_1 \quad \forall\, t_2 > 0, \tag{4.1}$$

and **negatively quadrant dependent** (NQD) if the inequality is reversed in (4.1).

Definition 4.2. (T_1, T_2) is said to be **right corner set increasing** (RCSI) if

$$\begin{aligned} &\{T_1 | T_2 > t_2\} \geq_{hr} \{T_1 | T_2 > t_1\} \quad \forall\, t_1 < t_2 \\ &\quad \Leftrightarrow \overline{F}(t_1, t_2) \text{is } TP_2 \text{ in } (t_1, t_2). \\ &\quad \Leftrightarrow r_{T_1|T_2}(t_1 | T_2 > t_2) \text{ decreasing in } t_2\, \forall\, t_1 > 0. \end{aligned} \tag{4.2}$$

Shaked (1977) calls this dependence as $DTP(1,1)$ also. If the inequality is reversed in (4.2), we say that (T_1, T_2) is **right corner set decreasing** (RCSD).

Definition 4.3. T_2 is said to be **right tail increasing** (RTI) in T_1 if

$$\begin{aligned} &\{T_1 | T_2 > t_2\} \geq_{hr} T_1 \quad \forall\, t_2 > 0 \\ \Leftrightarrow \quad & \frac{\overline{F}(t_1, t_2)}{\overline{F}(t_1)} \quad \text{is increasing in } t_1 \forall\, t_2 > 0 \\ \Leftrightarrow \quad & P[T_2 > t_2 | T_1 > t_1] \quad \text{is increasing in } t_1 \forall\, t_2 > 0. \end{aligned} \tag{4.3}$$

If the inequality is reversed in (4.3), we say that T_2 is **right tail decreasing** (RTD) in T_1.

Definition 4.4. (T_1, T_2) are **dependent by total positivity of order** $(2,1)$ ($DTP(2,1)$) if

$$\begin{aligned} &\int_t^\infty \overline{F}(x, y)\, dx \text{ is } TP_2 \text{ in } (t, y) \\ &\quad \Leftrightarrow \mu_{T_1|T_2}(t_1 | T_2 > t_2) \text{ increasing in } t_2\, \forall\, t_1 > 0. \end{aligned} \tag{4.4}$$

If $\mu_{T_1|T_2}(t_1 | T_2 > t_2)$is decreasing in $t_2\, \forall\, t_1 > 0$ the variables are negatively dependent and (T_1, T_2) are said to be $DRR(2,1)$.

We have the following implications among the above concepts of positive dependence.

$$DTP(2,1) \Leftarrow RCSI \Rightarrow RTI \Rightarrow PQD$$

Similar implications hold among the above concepts of negative dependence.

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