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May 10, 2002
http://www.isid.ac.in/~statmath/eprints

# Equivariant spectral triples on the quantum $SU(2)$ group

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# Equivariant spectral triples on the quantum $SU(2)$ group

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**Abstract**

We characterize all equivariant odd spectral triples on the quantum $SU(2)$ group having a nontrivial Chern character. It is shown that the dimension of an equivariant spectral triple is at least three, and there does exist a 3-summable equivariant spectral triple. We also show that given any odd spectral triple, there is an odd equivariant spectral triple that induces the same element in $K^1$.

**AMS Subject Classification No.:** 58B34, 46L87, 19K33
**Keywords.** Spectral triples, quantum group.

## 1 Introduction

Study of quantum groups originated in the early eighties in the work of Fadeev, Sklyanin & Takhtajan in the context of quantum inverse scattering theory. It picked up momentum during the mid eighties, and connections were established with various other areas in mathematics. They were first studied in the topological setting by Woronowicz, who treated the $q$-deformation of the $SU(2)$ group and then went on to characterize the family of compact quantum groups and studied their representation theory.

Noncommutative geometry was introduced around the same time by Alain Connes, drawing inspiration mainly from the work of Atiyah and Kasparov. While their classical counterparts are very intimately connected to each other, there has so far been very little work on the connection and relationship between the two notions of quantum groups and noncommutative geometry. One of the first questions that one would like to settle, for example, is that given a compact quantum group, does it admit a Dirac operator that is equivariant under its own (co-)action. There has been some work on this theme ([1], [5]), but none of these resolve the question satisfactorily. Present article is a modest attempt towards answering this.

The most well-known example of a compact quantum group is the $q$-deformation of the $SU(2)$-group, which has been studied very thoroughly for real values of the deformation parameter $q$ by Woronowicz in [10]. It has a natural (co-)action on itself, so that it can be thaught of as an $SU_q(2)$-homogeneous space. We investigate geometries on this homogeneous

*The first author would like to acknowledge support from the National Board of Higher Mathematics, India.

space equivariant under the (co-)action of $SU_q(2)$. The earliest work on noncommutative geometry on $SU_q(2)$ is probably the paper by Masuda and Watanabe ([7]), but their treatment was more from the point of view of noncommutative topology; they do not talk about the Dirac operator which is of fundamental importance in Connes' theory, and which captures topological as well as geometric information about the space concerned. More recently, Bibikov & Kulish ([1]) and Goswami ([5]) made attempts to get an equivariant 'Dirac' operator on $SU_q(2)$, but none of them could accomplish it satisfactorily within the framework of Connes' theory, which is what we plan to do in the present article. We restrict ourselves to odd spectral triples. We characterize all equivariant odd spectral triples on the $L_2$ space of the haar state. In particular, we show the existence of a 3-summable equivariant spectral triple. It is also shown that an equivariant spectral triple can not be $p$-summable for $p < 3$. Then we go on to prove that the associated Chern character is nontrivial by computing the pairing between the induced Fredholm module and a generator for $K_1(C(SU_q(2)))$, which is $\mathbb{Z}$. Computation of the pairing along with the results of Rosenberg & Schochet ([9]) shows that the associated Fredholm module is a generator of $K^1$. One immediate corollary is the universality of equivariant odd spectral triples in the sense that given any odd spectral triple, there is an equivariant one that induces the same element in $K^1$.

We will assume throughout that $q$ *is a real parameter lying in the interval* $(0, 1)$.

## 2 Preliminaries

To fix notation, let us give here a very brief description of the quantum $SU(2)$ group. The $C^*$-algebra of continuous functions on $SU_q(2)$, to be denoted by $\mathcal{A}$, is the $C^*$-algebra generated by two elements $\alpha$ and $\beta$ satisfying the following relations:

$$\alpha^*\alpha + \beta^*\beta = I, \qquad \alpha\alpha^* + q^2\beta\beta^* = I,$$
$$\alpha\beta - q\beta\alpha = 0, \qquad \alpha\beta^* - q\beta^*\alpha = 0,$$
$$\beta^*\beta = \beta\beta^*.$$

We will denote by $\mathcal{A}_f$ the dense $*$-subalgebra of $\mathcal{A}$ generated by $\alpha$ and $\beta$. Group structure is given by the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\beta^* \otimes \beta,$$
$$\Delta(\beta) = \beta \otimes \alpha + \alpha^* \otimes \beta.$$

For two continuous linear functionals $\rho_1$ and $\rho_2$ on $\mathcal{A}$, one defines their convolution product by: $\rho_1 * \rho_2(a) = (\rho_1 \otimes \rho_2)\Delta(a)$. It is known ([10]) that $\mathcal{A}$ admits a faithful state $h$, called the Haar state, that satisfies

$$h * \rho(a) = h(a)\rho(I) = \rho * h(a)$$

for all continuous linear functionals $\rho$ and all $a \in \mathcal{A}$. We will denote by $\mathcal{H}$ the GNS space associated with this state.

The representation theory of $SU_q(2)$ is strikingly similar to its classical counterpart. In particular, for each $n \in \{0, \frac{1}{2}, 1, \ldots\}$, there is a unique irreducible unitary representation $t^{(n)}$ of dimension $2n+1$. Denote by $t_{ij}^{(n)}$ the $ij^{\text{th}}$ entry of $t^{(n)}$. These are all elements of $\mathcal{A}_f$ and they form an orthogonal basis for $\mathcal{H}$. Denote by $e_{ij}^{(n)}$ the normalized $t_{ij}^{(n)}$'s, so that $\{e_{ij}^{(n)} : n = 0, \frac{1}{2}, 1, \ldots, i, j = -n, -n+1, \ldots, n\}$ is an orthonormal basis.

**Remark 2.1** One has to be a little careful here, because, unlike in the classical case, the choice of matrix entries does affect the orthogonality relations. Therefore one has to specify the matrix entries he/she is working with. In our case, $t_{ij}^{(n)}$'s are the same as in Klimyk & Schmuedgen (page 74, [6]).

*We will use the symbol* $\nu$ *to denote the number* $1/2$ throughout this article, just to make some expressions occupy less space. Using formulas for Clebsch-Gordon coefficients, and the orthogonality relations (page 80–81 and equation (57), page 115 in [6]), one can write down the actions of $\alpha$, $\beta$ and $\beta^*$ on $\mathcal{H}$ explicitly as follows:

$$\alpha : e_{ij}^{(n)} \mapsto a_+(n,i,j) e_{i-\nu,j-\nu}^{(n+\nu)} + a_-(n,i,j) e_{i-\nu,j-\nu}^{(n-\nu)}, \tag{2.1}$$

$$\beta : e_{ij}^{(n)} \mapsto b_+(n,i,j) e_{i+\nu,j-\nu}^{(n+\nu)} + b_-(n,i,j) e_{i+\nu,j-\nu}^{(n-\nu)}, \tag{2.2}$$

$$\beta^* : e_{ij}^{(n)} \mapsto b_+^+(n,i,j) e_{i-\nu,j+\nu}^{(n+\nu)} + b_-^+(n,i,j) e_{i-\nu,j+\nu}^{(n-\nu)}, \tag{2.3}$$

where

$$a_+(n,i,j) = \left(q^{2(n+i)+2(n+j)+2} \frac{(1-q^{2n-2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^{\nu}, \tag{2.4}$$

$$a_-(n,i,j) = \left(\frac{(1-q^{2n+2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^{\nu}, \tag{2.5}$$

$$b_+(n,i,j) = -\left(q^{2(n+j)} \frac{(1-q^{2n-2j+2})(1-q^{2n+2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^{\nu}, \tag{2.6}$$

$$b_-(n,i,j) = \left(q^{2(n+i)} \frac{(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^{\nu}, \tag{2.7}$$

$$b_+^+(n,i,j) = \left(q^{2(n+i)} \frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^{\nu}, \tag{2.8}$$

$$b_-^+(n,i,j) = -\left(q^{2(n+j)} \frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^{\nu}. \tag{2.9}$$

# 3 Equivariant spectral triples

In this section, we will formulate the notion of equivariance, and investigate the behaviour of $D$, where $D$ is the Dirac operator of an equivariant spectral triple.

In the classical context of a compact Lie group $G$, a left invariant differential operator is one that commutes with the left regular representation of $G$. Now in the case of abelian $G$, the $C^*$-algebra generated by the left regular representation is nothing but $C(\widehat{G})$. Therefore we can rephrase the left invariance condition as a commutation condition with $C(\widehat{G})$. For $C(SU_q(2))$, Woronowicz has explicitly described the generators for $C(\widehat{G})$. Therefore, a proper analog of a left invariant Dirac operator would be a Dirac operator commuting with these generators.

Let $A_0$ and $A_1$ be the following operators on $\mathcal{H}$:

$$\begin{aligned} A_0 &: e_{ij}^{(n)} \mapsto q^j e_{ij}^{(n)}, \\ A_1 &: e_{ij}^{(n)} \mapsto \begin{cases} 0 & \text{if } j = n, \\ (q^{-2n} + q^{2n+2} - q^{-2j} - q^{2j+2})^\nu e_{ij+1}^{(n)} & \text{if } j < n. \end{cases} \end{aligned}$$

The operators $A_0$ and $A_1$ generate the $C^*$-algebra of continuous functions on the dual of $SU_q(2)$ and thus are the 'generators' of the regular representation of $SU_q(2)$ (For more details, see [8]; $A_0$ and $A_1$ are the operators **a** and **n** there). We say that an operator $T$ on $\mathcal{H}$ is **equivariant** if it commutes with $A_0$, $A_1$ and $A_1^*$. It is clear that any equivariant self-adjoint operator with discrete spectrum must be of the form

$$D : e_{ij}^{(n)} \mapsto d(n,i) e_{ij}^{(n)}, \tag{3.1}$$

where $d(n,i)$'s are real. Assume then that $D$ is such an operator. Let us first write down the commutators of $D$ with $\alpha$ and $\beta$.

$$\begin{aligned} [D,\alpha] e_{ij}^{(n)} &= a_+(n,i,j)(d(n+\nu,i-\nu) - d(n,i)) e_{i-\nu,j-\nu}^{(n+\nu)} \\ &\quad + a_-(n,i,j)(d(n-\nu,i-\nu) - d(n,i)) e_{i-\nu,j-\nu}^{(n-\nu)}, \qquad (3.2) \\ [D,\beta] e_{ij}^{(n)} &= b_+(n,i,j)(d(n+\nu,i+\nu) - d(n,i)) e_{i+\nu,j-\nu}^{(n+\nu)} \\ &\quad + b_-(n,i,j)(d(n-\nu,i+\nu) - d(n,i)) e_{i+\nu,j-\nu}^{(n-\nu)}. \qquad (3.3) \end{aligned}$$

We are now in a position to prove the following.

**Proposition 3.1** *Let $D$ be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i) e_{ij}^{(n)}$. Then $[D,a]$ is bounded for all $a \in \mathcal{A}_f$ if and only if $d(n,i)$'s satisfy the following two conditions:*

$$d(n+\nu,i+\nu) - d(n,i) = O(1), \tag{3.4}$$

$$d(n+\nu,i-\nu) - d(n,i) = O(n+i+1). \tag{3.5}$$

*Proof*: Assume that $[D,a]$ is bounded for all $a \in \mathcal{A}_f$. Then, in particular, $[D,\alpha]$ and $[D,\beta]$ are bounded, so that there is a positive constant $C$ such that

$$\|[D,\alpha]\| \le C, \quad \|[D,\beta]\| \le C.$$

It follows from equations (3.2) and (3.3) that

$$|a_+(n,i,j)(d(n+\nu,i-\nu) - d(n,i))|^2 + |a_-(n,i,j)(d(n-\nu,i-\nu) - d(n,i))|^2 \le C^2, \tag{3.6}$$

$$|b_+(n,i,j)(d(n+\nu,i+\nu) - d(n,i))|^2 + |b_-(n,i,j)(d(n-\nu,i+\nu) - d(n,i))|^2 \le C^2 \tag{3.7}$$

for all $n$, $i$ and $j$. From the second inequality above, we get

$$|b_+(n,i,j)(d(n+\nu,i+\nu) - d(n,i))| \le C \quad \forall n,i,j.$$

Now

$$|b_+(n,i,j)| = \left(\frac{q^{2n+2j} - q^{4n+2}}{1 - q^{4n+2}}\right)^{\nu} \left(\frac{1 - q^{2n+2i+2}}{1 - q^{4n+4}}\right)^{\nu}.$$

Hence

$$1 - q^2 \leq \frac{1 - q^2}{1 - q^{4n+4}} \leq \max_j |b_+(n,i,j)| = \frac{1 - q^{2n+2i+2}}{1 - q^{4n+4}} \leq \frac{1}{1 - q^4}.$$

Hence $|d(n+\nu, i+\nu) - d(n,i)| \leq \frac{C}{1-q^2}$ for all $n$, $i$, i. e. we have (3.4). We also have from equation (3.6), $|a_+(n,i,j)(d(n+\nu, i-\nu) - d(n,i))| \leq C$. But

$$a_+(n,i,j) = q\left(\frac{q^{2n+2j} - q^{4n+2}}{1 - q^{4n+2}}\right)^{\nu} \left(\frac{q^{2n+2i} - q^{4n+2}}{1 - q^{4n+4}}\right)^{\nu}.$$

Hence

$$\max_j |a_+(n,i,j)| = q\left(\frac{q^{2n+2i} - q^{4n+2}}{1 - q^{4n+4}}\right)^{\nu}.$$

Therefore

$$q\left(\frac{q^{2n+2i} - q^{4n+2}}{1 - q^{4n+4}}\right)^{\nu} |d(n+\nu, i-\nu) - d(n,i)| \leq C \quad \forall n, i.$$

Consequently, $q^{n+i}|d(n+\nu, i-\nu) - d(n,i)| \leq q^{-1}C\frac{1}{(1-q^2)^{\nu}}$, i. e.

$$|d(n+\nu, i-\nu) - d(n,i)| = O(q^{-n-i}). \tag{3.8}$$

Let us next write the difference $d(n+\nu, i-\nu) - d(n,i)$ as follows:

$$\begin{aligned}
&\sum_{r=0}^{n+i-1} \left(d(n+\nu-r\nu, i-\nu-r\nu) - d(n+\nu-(r+1)\nu, i-\nu-(r+1)\nu)\right) \\
&\quad - \sum_{r=0}^{n+i-1} \left(d(n-r\nu, i-r\nu) - d(n-(r+1)\nu, i-(r+1)\nu)\right) \\
&\quad + d(n+\nu-(n+i)\nu, i-\nu-(n+i)\nu) - d(n-(n+i)\nu, i-(n+i)\nu)
\end{aligned}$$

Using this expression together with (3.8) for the case $n+i=0$ and (3.4), we get (3.5).

Next assume that the $d(n,i)$'s satisfy the conditions (3.4) and (3.5). We will show that $[D,\alpha]$ and $[D,\beta]$ are bounded, which in turn will ensure that $[D,a]$ is bounded for all $a \in \mathcal{A}_f$. It follows from (3.4) and (3.5) that there is a positive constant $C > 0$ such that

$$|d(n+\nu, i+\nu) - d(n,i)| \leq C, \quad q^{n+i}|d(n+\nu, i-\nu) - d(n,i)| \leq C.$$

It follows from the above two inequalities that

$$\begin{aligned}
|a_+(n,i,j)(d(n+\nu, i-\nu) - d(n,i))| &\leq C(1-q^4)^{-1/2}, \\
|a_-(n,i,j)(d(n-\nu, i-\nu) - d(n,i))| &\leq Cq^{-1}(1-q^2)^{-1/2}.
\end{aligned}$$

We now conclude from (3.2) that $[D,\alpha]$ is bounded. Proof of boundedness of $[D,\beta]$ is similar.

□

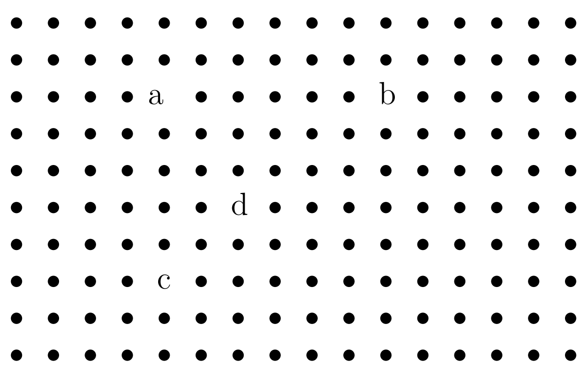
Next, we exploit the condition that $D$ must have compact resolvent. It is straightforward to see that a necessary and sufficient condition for an operator $D$ of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ to have compact resolvent is that if we write the $d(n,i)$'s in a single sequence, it should not have any limit point other than $\infty$ or $-\infty$. As we shall see below, in presence of (3.4) and (3.5), we can say much more about the $d(n,i)$'s. In particular, we can extract information about the sign of $D$ also.

**Proposition 3.2** *Let $D$ be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ such that $d(n,i)$'s satisfy conditions (3.4) and (3.5) and $D$ has compact resolvent. Then*

1. *For each $k \in \mathbb{N}$, there exists an $r_k \in \mathbb{N}$, $r_k \geq k$ such that $d(n, n-k)$ are of the same sign for all $n \geq r_k$.*
2. *There exists an $r \in \mathbb{N}$ such that for all $k \geq r$ and for all $n$, $d(n, n-k)$ are of the same sign.*

(3.9)

*Proof*: In the following diagram, each dot stands for a $d(n,i)$, the dot at the $i^{\text{th}}$ row and $j^{\text{th}}$ column representing $d(\frac{i+j}{2}, \frac{j-i}{2})$ (here $i$ and $j$ range from 0 onwards).




There are two restrictions imposed on these numbers, given by equations (3.4) and (3.5). Equation (3.4) says that: (i) the difference of two consecutive numbers along any row is bounded by a fixed constant, and (3.5) says that: (ii) the difference of two consecutive numbers along the $j^{\text{th}}$ column is $O(j+1)$. Suppose $C$ is a big enough constant which works for both (i) and (ii).

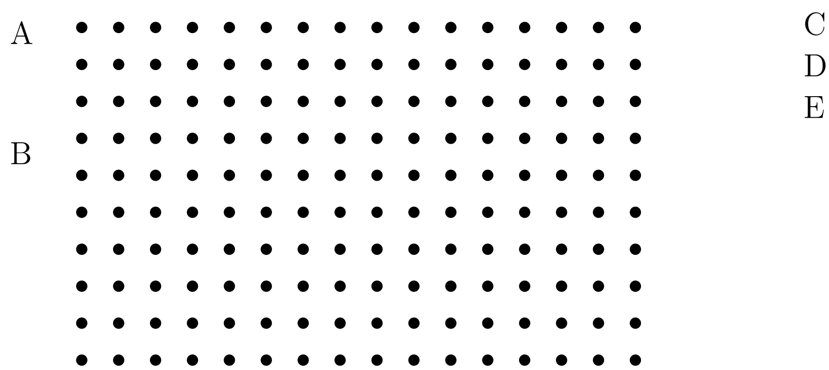
Now suppose $a$ and $b$ are two elements in the same row. Connect them with a path as in the diagram. If $a$ and $b$ are of opposite sign, then because of restriction (i) above, there has to be some dot between $a$ and $b$ for which the corresponding $d(n,i)$ lies in $[-C, C]$. Therefore, if the signs of the $d(n,i)$'s change infinitely often along a row, one can produce infinitely many $d(n,i)$'s in the interval $[-C, C]$. But this will prevent $D$ from having a compact resolvent. This proves part 1.

For part 2, employ a similar argument, this time connecting two dots, say $c$ and $d$, by a path as shown in the diagram, and observing that the difference between any two consecutive numbers along the path is bounded by $C$. $\square$

Let $m$ and $n$ be two nonnegative integers. Let

$$\begin{aligned} F(m,n) &= \left\{ d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : 0 \le i \le m, 0 \le j \le n \right\}, \\ S(m,n,r) &= \left\{ d\left(\frac{j+r}{2}, \frac{j-r}{2}\right) : j > n \right\}, \quad 0 \le r \le m, \\ T(m) &= \left\{ d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : i > m, j \ge 0 \right\}. \end{aligned}$$

In the following diagram, for example, $A$ is $F(2,4)$, $B$ is $T(2)$, and $C$, $D$ and $E$ are $S(2,4,0)$, $S(2,4,1)$ and $S(2,4,2)$ respectively.




What the last proposition says is the following. There exist big enough integers $m$ and $n$ such that in each of the sets $T(m)$, $S(m,n,0), \ldots, S(m,n,m)$, all elements are of the same sign, i. e. each of the sets $T(m)$, $S(m,n,0), \ldots, S(m,n,m)$ is contained in either $\mathbb{R}_+$ or $-\mathbb{R}_+$.

**Remark 3.3** One can extend the argument in the proof of the last proposition a little further and prove that if $D$ is as in the previous proposition, then

$$\left.\begin{array}{l} \textit{given any nonnegative real } N, \textit{ there exist positive integers } m \textit{ and } n \textit{ such} \\ \textit{that each of the sets } T(m),\ S(m,n,0),\ldots,S(m,n,m) \textit{ is contained in ei-} \\ \textit{ther } \{x \in \mathbb{R} : x > N\} \textit{ or } \{x \in \mathbb{R} : x < -N\}. \end{array}\right\} \qquad (3.10)$$

**Theorem 3.4** *An operator $D$ on $L_2(h)$ gives rise to an equivariant spectral triple if and only if it is of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$, where $d(n,i)$'s are real and satisfy conditions (3.4), (3.5) and (3.10).*

*Proof*: It is enough to prove that if the $d(n,i)$'s obey condition (3.10), then $D$ has compact resolvent. But this is clear, because (3.10) implies that for any real number $N > 0$, the interval $[-N, N]$ contains only a finite number of the $d(n,i)$'s. □

It is clear then that up to a compact perturbation, $D$ will have nontrivial sign if and only if the following condition holds:

$$\left.\begin{array}{l}\text{1. there exist positive integers } m \text{ and } n \text{ such that in each of the sets}\\ \quad T(m),\ S(m,n,0),\ldots,S(m,n,m), \text{ all elements are of the same sign,}\\ \quad \text{and}\\ \text{2. there are two sets in this collection whose elements are of opposite}\\ \quad \text{sign.}\end{array}\right\} \tag{3.11}$$

A natural question to ask now is whether there does indeed exist any $D$ with nontrivial sign satisfying (3.4) and (3.5). It is easy to see that the operator $D$ determined by the family $d(n,i)$, where

$$d(n,i) = \begin{cases} 2n+1 & \text{if } n \neq i, \\ -(2n+1) & \text{if } n = i, \end{cases} \tag{3.12}$$

satisfy all the requirements in propositions 3.1 and 3.2. In fact, one can easily see that $D^{-3} \in \mathcal{L}^{(1,\infty)}$, where $\mathcal{L}^{(1,\infty)}$ stands for the ideal of Dixmier traceable operators. Thus we have the following.

**Theorem 3.5** *$SU_q(2)$ admits an equivariant odd 3-summable spectral triple.*

The classical $SU(2)$ has (both topological as well as metric) dimension 3. For $SU_q(2)$, however, the topological dimension turns out to be 1, as can be seen from the following short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \longrightarrow \mathcal{A} \longrightarrow C(S^1) \longrightarrow 0,$$

where $\mathcal{K}$ denotes the algebra of compact operators. The next theorem tell us that as far as metric dimension is concerned, it behaves more like its classical counterpart; in fact along with the previous theorem, it says that the metric dimension of $SU_q(2)$ is 3.

**Theorem 3.6** *Let $(\mathcal{A}, \mathcal{H}, D)$ be an equivariant odd spectral triple. Then $D$ can not be $p$-summable for $p < 3$.*

*Proof*: Conditions (3.4) and (3.5) impose the following growth restriction on the $d(n,i)$'s:

$$\max_i |d(n,i)| = O(n). \tag{3.13}$$

The conclusion of the theorem follows easily from this. □

**Remark 3.7** 1. Goswami ([5]) has also obtained an equivariant Dirac operator $D$, but in his case the commutators with algebra elements are not bounded, and in order to circumvent that, one has to pass on to the absolute value $|D|$, thus making the sign trivial. This implies that the associated Chern character vanishes, which means in some sense the geometry given by $D$ does not capture the whole picture. In our case, we will show in the next section that the associated Chern character does not vanish.

2. For the $D$ described by (3.12), $|D|$ is essentially same as the $|D|$ in ([5]). Since only $|D|$ is used in capturing the haar state there (theorem 3.3, part (4) in [5]), same can be done with our $D$ as well.

The next proposition says that the derivative of any nonconstant function is nonzero.

**Proposition 3.8** *Let $D$ be given by (3.12). Then for $a \in \mathcal{A}_f$, $[D, a] = 0$ if and only if $a$ is a scalar.*

*Proof*: Take $a = \sum_{(i,j,k)\in F} c_{ijk}\alpha_i\beta^j\beta^{*k}$, where $F$ is a finite subset of $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ and all the $c_{ijk}$'s are nonzero. We will show that $[D, a] \neq 0$.

Let $m = \max\{|i|+j+k : (i,j,k) \in F\}$. Let $(r, s, t)$ be a point of $F$ such that $|r|+s+t = m$. Write $p = \frac{1}{2}(s-t-r)$, $p' = \frac{1}{2}(t-s-r)$. Then it is easy to see that

$$\begin{aligned}
&\langle e^{(n+m/2)}_{pp'}, [D, a]e^{(n)}_{00}\rangle \\
&= \langle e^{(n+m/2)}_{pp'}, [D, c_{rst}\alpha_r\beta^s\beta^{*t}]e^{(n)}_{00}\rangle \\
&= c_{rst}\prod_{i=1}^{t} b_+^+\left(n+\frac{i-1}{2}, -\frac{i-1}{2}, \frac{i-1}{2}\right)\prod_{i=t+1}^{t+s} b_+\left(n+\frac{i-1}{2}, -t+\frac{i-1}{2}, t-\frac{i-1}{2}\right) \\
&\quad\times\prod_{i=s+t+1}^{m} a_+^{\#}\left(n+\frac{i-1}{2}, p+\operatorname{sign}(r)\frac{m-i}{2}, p'+\operatorname{sign}(r)\frac{m-i}{2}\right) \\
&\qquad\times\Big(d(n+m/2,p)-d(n,0)\Big),
\end{aligned}$$

where $a_+^{\#}$ stands for $a_+$ or $a_+^+$ depending on the sign of $r$. The right hand side above is clearly nonzero because of our choice of $D$. □

The above proposition says, in particular, that the Dirac operator given by (3.12) is really a Dirac operator for the full tangent bundle rather than that of some lower dimensional subbundle.

## 4 Nontriviality of the Chern character

In this section we will examine the $D$ given by the family (3.12) in more detail and see that the nontriviality in sign does indeed result in nontriviality at the Fredholm level. For this, we will compute the pairing between $\operatorname{sign} D$ and a generator of $K_1(\mathcal{A})$. Let $u$ denote the element $I_{\{1\}}(\beta^*\beta)(\beta - I) + I$ of $\mathcal{A}$. It can be shown that this is a generator of $K_1(\mathcal{A})$. What we will do is the following. We will choose an invertible element $\gamma$ in $\mathcal{A}_f$ that is close enough to $u$ so that $\gamma$ and $u$ are the same in $K_1(\mathcal{A})$. We then compute the pairing between $\operatorname{sign} D$ and this $\gamma$.

**Theorem 4.1** *The Chern character of the spectral triple $(\mathcal{A}_f, \mathcal{H}, D)$ is nontrivial.*

Before we begin the proof, let us observe from equations (2.2) and (2.3) that the action of $\beta\beta^*$ on $\mathcal{H}$ is given by

$$(\beta\beta^*)(e_{ij}^{(n)}) = \sum_{\epsilon=-1}^{1} k_\epsilon(n,i,j) e_{ij}^{(n+\epsilon)}, \tag{4.1}$$

where

$$k_1(n,i,j) = -\Big(q^{4n+2i+2j+2}\frac{1-q^{2n+2j+2}}{1-q^{4n+2}}\frac{1-q^{2n-2i+2}}{1-q^{4n+4}}\frac{1-q^{2n-2j+2}}{1-q^{4n+4}}\frac{1-q^{2n+2i+2}}{1-q^{4n+6}}\Big)^\nu, \tag{4.2}$$

$$k_0(n,i,j) = q^{2(n+j)}\frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})} + q^{2(n+i)}\frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}, \tag{4.3}$$

$$k_{-1}(n,i,j) = -\Big(q^{4n+2i+2j-2}\frac{(1-q^{2n-2j})(1-q^{2n+2i})(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n-2})(1-q^{4n})(1-q^{4n})(1-q^{4n+2})}\Big)^\nu. \tag{4.4}$$

**Proof of theorem 4.1** : Choose $r \in \mathbb{N}$ such that $q^{2r} < \frac{1}{2} < q^{2r-2}$. Define $\gamma_r = (\beta^*\beta)^r(\beta - I) + I$. By our choice of $r$, we have

$$\begin{aligned}\|\gamma_r - u\| &\leq \|(\beta^*\beta)^r - I_{\{1\}}(\beta^*\beta)\| \cdot \|\beta - I\| \\ &\leq 2q^{2r} < 1.\end{aligned}$$

Hence $\gamma_r$ and $u$ are the same in $K_1(\mathcal{A})$. Therefore it is enough for our purpose if we can show that the pairing between $\operatorname{sign} D$ and $\gamma_r$ is nontrivial. Denote by $P_k$ the projection onto the space spanned by $\{e_{n-k,j}^{(n)} : n, j\}$. Then $\operatorname{sign} D = I - 2P_0$. Therefore we now want to compute the index of the operator $P_0\gamma_r P_0$ thaught of as an operator on $P_0\mathcal{H}$.

It follows from (4.1) that

$$(\beta\beta^*)^r(e_{ij}^{(n)}) = \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1}\epsilon_s, i, j)\Big) e_{ij}^{(n+\sum_1^r \epsilon_s)}. \tag{4.5}$$

Since $\beta$ is normal, we have

$$\begin{aligned}\gamma_r e_{ij}^{(n)} = &\sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)\Big(b_+(n + \sum_1^r \epsilon_s, i, j) e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} \\ &\qquad + b_-(n + \sum_1^r \epsilon_s, i, j) e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \\ &- \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1}\epsilon_s, i, j)\Big) e_{ij}^{(n+\sum_1^r \epsilon_s)} + e_{ij}^{(n)}.\end{aligned} \tag{4.6}$$

Consequently,

$$\begin{aligned}\gamma_r e^{(n)}_{nj} &= \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s,n,j)\Big)\Big(b_+(n+\sum_1^r\epsilon_s,n,j)e^{(n+\sum_1^r\epsilon_s+\nu)}_{n+\nu,j-\nu}\\ &\qquad + b_-(n+\sum_1^r\epsilon_s,n,j)e^{(n+\sum_1^r\epsilon_s-\nu)}_{n+\nu,j-\nu}\Big)\\ &\quad - \sum_{\epsilon_t\in\{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s,n,j)\Big)e^{(n+\sum_1^r\epsilon_s)}_{nj} + e^{(n)}_{nj}.\end{aligned}$$

When we cut this off by $P_0$, we get

$$\begin{aligned}P_0\gamma_r e^{(n)}_{nj} &= \sum_{\sum\epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s,n,j)\Big)b_+(n,n,j)e^{(n+\nu)}_{n+\nu,j-\nu}\\ &\quad + \sum_{\sum\epsilon_t=1} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s,n,j)\Big)b_-(n+1,n,j)e^{(n+\nu)}_{n+\nu,j-\nu}\\ &\quad - \sum_{\sum\epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s,n,j)\Big)e^{(n)}_{nj} + e^{(n)}_{nj}.\end{aligned}$$

A closer look at the quantities $k_\epsilon$ and $b_\pm$ tells us that if we do the calculations modulo compact operators, which we can because we want to compute the index, we find that there is no contribution from the second term, while in the case of the first and the third term, contribution comes from only the coefficient where the product $\prod_{t=1}^r k_{\epsilon_t}(n+\epsilon_1+\ldots+\epsilon_{t-1},n,j)$ consists solely of $k_0$'s, i. e. when each $\epsilon_t=0$. A further examination of the terms $k_0$ and $b_+$ then yield the following:

$$\begin{aligned}P_0\gamma_r P_0 e^{(n)}_{nj} &= k_0(n,n,j)^r b_+(n,n,j)e^{(n+\nu)}_{n+\nu,j-\nu} + (1-k_0(n,n,j)^r)e^{(n)}_{nj}\\ &= -q^{2rn+2rj}(1-q^{2n-2j})^r q^{n+j}(1-q^{2n-2j+2})^{1/2}e^{(n+\nu)}_{n+\nu,j-\nu}\\ &\qquad + \big(1-q^{2rn+2rj}(1-q^{2n-2j})^r\big)\,e^{(n)}_{nj},\end{aligned}$$

and

$$\begin{aligned}P_0\gamma_r^* P_0 e^{(n)}_{nj} &= -q^{2rn+2rj}(1-q^{2n-2j-2})^r q^{n+j}(1-q^{2n-2j})^{1/2}e^{(n-\nu)}_{n-\nu,j+\nu}\\ &\qquad + \big(1-q^{2rn+2rj}(1-q^{2n-2j})^r\big)\,e^{(n)}_{nj}\end{aligned}$$

From these, one can easily show that the index of $P_0\gamma_r P_0$ is $-1$. Since $P_0$ is the eigenspace corresponding to the eigenvalue $-1$ of $\operatorname{sign} D$, the value of the $K$-homology–$K$-theory pairing $\langle[u],[(\mathcal{A},\mathcal{H},D)]\rangle$ coming from Kasparov product of $K_1$ and $K_1$ is $-\operatorname{index} P_0\gamma_r P_0$, which is nonzero. □

**Remark 4.2** Strictly speaking, it is not essential to introduce the element $u$ as a generator for $K_1(\mathcal{A})$. It is enough if one computes the pairing between $\operatorname{sign} D$ and a suitable $\gamma_r$ and show that it is nontrivial. But the introduction of $u$ makes the choice of $\gamma_r$'s and hence the proof above more transparent.

It follows from proposition 3.2 that for the purposes of computing the index pairing, sign of any equivariant $D$ must be of the form $I-2P$ where $P=\sum_{k\in F} P_k$, $F$ being a finite subset of $\mathbb{N}$ ( the actual $P$ would be a compact perturbation of this). Conversely, given a $P$ of this form, it is easy to produce a $D$ satisfying the conditions in proposition 3.2 for which $\operatorname{sign} D = I-2P$. One could, for example, take the $D$ given by $d(n,i)$'s, where

$$d(n,i)=\begin{cases}-(2n+1) & \text{if } n-i\in F,\\ 2n+1 & \text{otherwise.}\end{cases}$$

We are now in a position to prove the following.

**Proposition 4.3** *Given any $m\in\mathbb{Z}$, there exists an equivariant spectral triple $D$ acting on $\mathcal{H}$ such that $\langle\gamma_r,[(\mathcal{A},\mathcal{H},D)]\rangle = m$, where $\langle,\cdot,\cdot\rangle : K_1(\mathcal{A})\times K^1(\mathcal{A})\to\mathbb{Z}$ denotes the map coming from the Kasparov product.*

*Proof*: It is enough to prove the statement for $m$ positive. Let $D$ be an equivariant Dirac operator whose sign is $I-2P$ where $P=\sum_{k\in F} P_k$, $F$ being a subset of size $m$ of $\mathbb{N}$. In order to compute the pairing $\langle\gamma_r,[(\mathcal{A},\mathcal{H},D)]\rangle$, we must first have a look at $P_{k+l}\gamma_r P_k$.

We get from equation (4.6)

$$\begin{aligned}\gamma_r e^{(n)}_{n-k,j} &= \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big)\\ &\quad\times\Big(b_+(n+\sum_1^r\epsilon_s, n-k, j)e^{(n+\sum_1^r\epsilon_s+\nu)}_{n-k+\nu,j-\nu} + b_-(n+\sum_1^r\epsilon_s, n-k, j)e^{(n+\sum_1^r\epsilon_s-\nu)}_{n-k+\nu,j-\nu}\Big)\\ &\quad - \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big)e^{(n+\sum_1^r\epsilon_s)}_{n-k,j} + e^{(n)}_{n-k,j}.\end{aligned}$$

and consequently,

$$\begin{aligned}P_{k+l}\gamma_r e^{(n)}_{n-k,j} &= \sum_{\sum\epsilon_t=l}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big)b_+(n+l, n-k, j)e^{(n+l+\nu)}_{n-k+\nu,j-\nu}\\ &\quad+\sum_{\sum\epsilon_t=l+1}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big)b_-(n+l+1, n-k, j)e^{(n+l+\nu)}_{n-k+\nu,j-\nu}\\ &\quad-\sum_{\sum\epsilon_t=l}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n-k, j)\Big)e^{(n+l)}_{n-k,j} + \delta_{l0}e^{(n)}_{n-k,j}.\end{aligned}$$

Now because of the nature of the quantities $k_\epsilon$ and $b_\pm$, we see that for index calculations, none of the terms contribute anything for $l\neq 0$, while for $l=0$, the first, third and the fourth term survive, with coefficient in the first term being $k_0(n, n-k, j)^r b_+(n, n-k, j)$ and that in the third being $(1 - k_0(n, n-k, j)^r)$. It follows from here that

$$\operatorname{index} P_k \gamma_r P_k = -1,$$

and $P_{k+l}\gamma_r P_k$ is compact for $l \neq 0$. Therefore the pairing between $\operatorname{sign} D$ and $\gamma_r$ produces $m$. □

An immediate corollary of the above proposition and theorem 1.17 in [9] is the following universality property of equivariant spectral triples.

**Corollary 4.4** *Given any odd spectral triple $(\mathcal{A}, \mathcal{K}, D)$, there is an equivariant triple $(\mathcal{A}, \mathcal{H}, D')$ inducing the same element in $K^1(\mathcal{A})$.*

Finally, we have the following characterization theorem for equivariant Dirac operators.

**Theorem 4.5** *$(\mathcal{A}, \mathcal{H}, D)$ is an equivariant odd spectral triple with nontrivial Chern character if and only if $D$ is given by (3.1) and the $d(n, i)$'s obey conditions (3.4), (3.5), (3.10) and (3.11).*

*Proof*: If $D$ is of the form $e_{ij}^{(n)} \mapsto d(n, i)e_{ij}^{(n)}$, where $d(n, i)$'s are real and satisfy conditions (3.4), (3.5), (3.10) and (3.11), then proposition 3.1 says $[D, a]$ is bounded and nontriviality of Chern character follows from arguments of proposition 4.3. Conversely, if $D$ is equivariant, then by propositions 3.1, 3.2 and remark 3.3, we have (3.4), (3.5) and (3.10). Since $D$ has nontrivial Chern character, it has nontrivial sign so that we have (3.11). □

## References


[1] Bibikov, P. N. & Kulish, P. P. : Dirac operators on the quantum group $\mathrm{SU}_q(2)$ and the quantum sphere. (Russian. English, Russian summary) *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (POMI) 245 (1997), *Vopr. Kvant. Teor. Polya i Stat. Fiz.* 14, 49–65, 283; translation in *J. Math. Sci.* (New York) 100 (2000), no. 2, 2039–2050.

[2] Blackadar, Bruce : *K-theory for operator algebras*, Second edition. MSRI Publications, 5. Cambridge University Press, Cambridge, 1998.

[3] Connes, A. : *Noncommutative Geometry*, Academic Press, 1994.

[4] Connes, Alain & Landi, Giovanni : Noncommutative manifolds, the instanton algebra and isospectral deformations. *Comm. Math. Phys.* 221 (2001), no. 1, 141–159.

[5] Goswami, D. : Some Noncommutative Geometric Aspects of $SU_q(2)$, math-ph/0108003.

[6] Klimyk, A. & Schmuedgen, K. : *Quantum Groups and their Representations*, Springer, New York, 1998.

[7] Masuda, T. & Watanabe, J. : Quantum groups as objects of noncommutative geometry. *Current topics in operator algebras* (Nara, 1990), 359–370, World Sci. Publishing, River Edge, NJ, 1991.

[8] Podleś, P. & Woronowicz, S. L. : Quantum deformation of Lorentz group, *Comm. Math. Phys.*, 130(1990), 381–431.

[9] Rosenberg, J. & Schochet, C. : The Künneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor, *Duke Math. J.* 55 (1987), no. 2, 431–474.

[10] Woronowicz, S. L. : Twisted $SU(2)$ group. An example of a noncommutative differential calculus, *Publ. RIMS*, Kyoto University, 23(1987), 117–181.

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