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High density asymptotics of the Poisson random connection model

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ABSTRACT Consider a sequence of independent Poisson point processes $X_1, X_2, \ldots$ with densities $\lambda_1, \lambda_2, \ldots$ respectively and connection functions $g_1, g_2, \ldots$ defined by $g_n(r) = g(nr)$, for $r > 0$ and for some integrable function g. The Poisson random connection model (X_n, λ_n, g_n) is a random graph with vertex set X_n and, for any two points x_i and x_j in X_n, the edge $< x_i, x_j >$ is included in the random graph with a probability $g_n(|x_i - x_j|)$ independent of the point process as well as other pairs of points. We show that if $\lambda_n/n^d \to \lambda$, $(0 < \lambda < \infty)$ as $n \to \infty$ then for the number $I_{(n)}(K)$ of isolated vertices of X_n in a compact set K with non-empty interior, we have $(\mathrm{Var}(I_{(n)}(K)))^{-1/2}(I_{(n)}(K) - E(I_{(n)}(K)))$ converges in distribution to a standard normal random variable. Similar results may be obtained for clusters of finite size. The importance of this result is in the statistical simulation of such random graphs.

Keywords Random connection model, Continuum percolation, Poisson point process, central limit theorem

1 Introduction

The random connection model is an example of random graphs used in describing small world networks. It consists of a countably infinite set of vertices distributed according to a well defined mechanism (random or otherwise), with pairs of vertices connected by edges according to some random mechanism. More formally, let $\mathcal{G}$ be a complete graph of a vertex set $X = \{x_1, x_2, \ldots\} \subset \mathbb{R}^d$ and containing edges $< x_i, x_j >$ for all possible pairs of vertices x_i and x_j of X. The random connection model (RCM) is a random sub-graph of $\mathcal{G}$ consisting of the vertex set X and an edge set $\mathcal{E}(X)$ formed according to a random mechanism determined by a *connection function* $g : [0, \infty) \to [0, 1]$. The edge $< x_i, x_j >$ connecting the vertices x_i and x_j is included in the edge set $\mathcal{E}(X)$ with a probability $g(|x_i - x_j|)$, where $|\cdot|$ denotes the d–dimensional Euclidean distance. The inclusion or non-inclusion of an edge in $\mathcal{E}(X)$ is independent of the inclusion or non-inclusion of all other edges of $\mathcal{G}$.

It is immediate that for $X = \mathbb{Z}^d$ and $g(|x|) = p1_{\{|x| \leq 1\}}$, the random graph above coincides with the nearest neighbour independent bond percolation model. Also for X arising as points of a Poisson point process and $g(|x|) = 1_{\{|x| \leq 2r\}}$, for some $r > 0$, we obtain the Poisson Boolean model of continuum percolation with balls of a fixed radius r.

In this paper the object of study is the Poisson RCM where the vertex set X is the points of a Poisson point process of density λ on $\mathbb{R}^d$, i.e., for disjoint Borel sets $A, B \subseteq \mathbb{R}^d$,

(a) $\#(X \cap A)$ and $\#(X \cap B)$ are independent random variables.

(b) Given $\#(X \cap A) = k$, these k points are uniformly distributed in A.

The connection function g is arbitrary and the probabilistic mechanism forming the edge set is independent of the Poisson point process generating X. We shall denote this RCM by (X, λ, g). For a more formal account of the mathematical set-up of this model we refer the reader to Meester and Roy [1996].

Clearly the graph (X, λ, g) can be decomposed as a disjoint union of *maximal* connected sub-graphs. For a vertex $x \in X$, let $C(x)$ denote the maximal connected sub-graph containing x. The vertex x is said to admit a *cluster of order* k if $C(x)$ contains exactly k vertices. The vertex x is *isolated* if it has a cluster of order 1. In case $C(x)$ contains infinitely many vertices, the vertex x is said to admit an *unbounded cluster*. Burton and Meester [1993] have shown that the Poisson RCM admits at most one unbounded cluster almost surely.

It can be easily seen that in (X, λ, g), for a fixed vertex $u \in X$, the vertices $x \in X$ such that the edges $< u, x >$ are in the graph (X, λ, g) form an inhomogeneous Poisson point process with density $\lambda g(|x-u|)$. Thus, for any k,

$$\begin{aligned} & P\{u \text{ is connected to } k \text{ distinct points}\} \\ = \quad & \exp\left(-\lambda \int_{\mathbb{R}^d} g(|x-u|)dx\right) \frac{(\lambda \int_{\mathbb{R}^d} g(|x-u|)dx)^k}{k!}. \end{aligned} \tag{1}$$

Hence, if

$$\int_{\mathbb{R}^d} g(|x|)dx = \infty, \tag{2}$$

then, for any $k \geq 1$, $P\{u \text{ has a cluster of order } k\} = 0$, i.e., u admits an unbounded cluster almost surely. Combining this with the result of Burton and Meester we see that, for g satisfying (2), (X, λ, g) is an unbounded connected graph almost surely.

Similarly, if $\int_{\mathbb{R}^d} g(|x|)dx = 0$, then (X, λ, g) is just a collection of isolated vertices. Thus the Poisson RCM worth considering is for g satisfying

$$0 < \int_{\mathbb{R}^d} g(|x|)dx < \infty. \tag{3}$$

Penrose [1991] has shown that for g satisfying (3), and for a fixed vertex $u \in X$, there exists $\lambda_c(g) > 0$ such that

$$\theta_g(\lambda) := P\{u \text{ admits an unbounded cluster}\} \begin{cases} = 0 & \text{for } \lambda < \lambda_c(g) \\ > 0 & \text{for } \lambda > \lambda_c(g). \end{cases}$$

Moreover,

$$\lim_{\lambda \to \infty} \frac{-\log(1-\theta_g(\lambda))}{\lambda \int_{\mathbb{R}^d} g(|x|)dx} = 1, \tag{4}$$

which implies that the rate at which $\theta_g(\lambda)$ tends to 1 as λ tends to ∞ corresponds to the rate at which the probability of u being isolated tends to 0. Thus as the density increases to ∞, the Poisson RCM tends to becoming a connected graph.

This however need not occur if the connection function were to also change with λ. Indeed, consider a sequence of independent Poisson point processes $X_1, X_2, \dots$ with densities $\lambda_1, \lambda_2, \dots$ respectively, where $\lambda_n \to \infty$ as $n \to \infty$; and connection functions $g_1, g_2, \dots$ defined by

$$g_n(r) = g(nr), \text{ for } r > 0, \tag{5}$$

for some g satisfying (3). We shall obtain asymptotic distributions of the number of isolated vertices in a bounded region K admitted by (X_n, λ_n, g_n) as $n \to \infty$.

The mosaic structure of Poisson Boolean model of continuum percolation is a similar structure where one considers Boolean models whose intensities λ_n increase to infinity while the radii of the balls in each of the models decrease to 0. Hall [1988] has obtained asymptotic distribution of the number of 'clumps' of finite order according to the rate at which the radius goes to 0. We shall obtain results similar to that obtained by Hall, however, in our case, because of the arbitrary nature of g and since the dependence structure of a general RCM is significantly different from that of the Boolean model the calculations involved are more intricate.

For simplicity with the calculations we assume here that

$$g \text{ is a non-increasing function.} \tag{6}$$

Fix a compact subset K of $\mathbb{R}^d$ with non-empty interior and let $I_{(n)}(K)$ be the number of isolated vertices in K of (X_n, λ_n, g_n). Our main result is

Theorem 1 *For Poisson RCM's (X_n, λ_n, g_n) as above, if $\lambda_n / n^d \to \lambda, (0 < \lambda < \infty)$ as $n \to \infty$ then*

$$\frac{I_{(n)}(K) - E(I_{(n)}(K))}{\sqrt{Var(I_{(n)}(K))}} \Longrightarrow Z$$

where Z is a standard normal random variable.
Here $\Longrightarrow$ denotes convergence in distribution.

One expects similar results for clusters of finite sizes; however the computations involved are quite forbidding.

2 Moment computations

In preparation for our subsequent analysis, let $U \in \mathbb{R}^d$ be a point uniformly distributed on K and independent of all other underlying processes. For $a_1, a_2, \dots, a_k \in \mathbb{R}^d$, $(X \cup \{a_1, a_2, \dots, a_k\}, \lambda, g)$ denotes the RCM obtained with vertex set $X \cup \{a_1, a_2, \dots, a_k\}$ and connection function g, where X is a Poisson point process of intensity λ.

Define

$$p(\lambda, g) := P\Big\{\mathbf{0} \text{ is isolated in } (X \cup \{\mathbf{0}\}, \lambda, g)\Big\},$$

where $\mathbf{0}$ denotes the origin. By the translation invariance of the process

$$p(\lambda, g) = P\Big\{U \text{ is isolated in } (X \cup \{U\}, \lambda, g)\Big\}.$$

Moreover, given that the process has a point at the origin, the conditional distribution of $(X \cup \{\mathbf{0}\}, \lambda, g)$ is the same as that of a process (X, λ, g) with a point of the process taken to be the origin, thus from (1) we have

$$p(\lambda.g) = \exp\Big(-\lambda \int_{\mathbb{R}^d} g(|x|)dx\Big). \tag{7}$$

Clearly, for λ_n such that $\lambda_n/n^d \to \lambda$ as $n \to \infty$, for some $\lambda \in (0, \infty)$, we have from (7),

$$\begin{aligned} p(\lambda_n, g_n) &= P\Big\{U \text{ is isolated in } (X_n \cup \{U\}, \lambda_n, g_n)\Big\} \\ &= \exp\Big(-\lambda_n \int_{\mathbb{R}^d} g_n(|x|)dx\Big) \\ &= \exp\Big(-\frac{\lambda_n}{n^d} \int_{\mathbb{R}^d} g(|x|)dx\Big) \\ &= p\Big(\frac{\lambda_n}{n^d}, g\Big) \\ &\to p(\lambda, g) \text{ as } n \to \infty. \end{aligned} \tag{8}$$

Now we introduce some notation. For a Borel set $A \subseteq \mathbb{R}^d$, we denote the points of the Poisson points in A by $X(A) := X \cap A$ and its cardinality by $N(A) := \#X(A)$. Also, for $t > 0$, the fattening of A by t is denoted by $A^t := \{y \in \mathbb{R}^d : |x - y| \leq t \text{ for some } x \in A\}$. Finally, let $I(A)$ be the number of isolated vertices of X in A, i.e., $I(A) := \sum_{\xi \in X} 1_{\{\xi \in A \text{ and } \xi \text{ is isolated}\}}$ where $\{\xi \text{ is isolated}\}$ is the event that ξ is not connected to any $\xi' \in X$. Also, denote by $I_t(A) := \sum_{\xi \in X} 1_{\{\xi \in A \text{ and } \xi \text{ is isolated in } A^t\}}$ where $\{\xi \text{ is isolated in } A^t\}$ is the event that ξ is not connected to any $\xi' \in X(A^t)$. Note, both $I(A)$ and $I_t(A)$ are 0 if $X(A) = \emptyset$.

For K compact with non-empty interior we have $I_t(K) \downarrow I(K)$ almost surely as $t \uparrow \infty$, and for any $t > 0$, $E(I_t(K))^2 \leq E(N(K))^2 < \infty$; thus applying the dominated convergence theorem we have

$$E(I_t(K)) \downarrow E(I(K)) \text{ and } \mathrm{Var}(I_t(K)) \to \mathrm{Var}(I(K)) \text{ as } t \to \infty. \tag{9}$$

Now we will compute $E(I_t(K))$ and $E(I_t(K))^2$. First note that the points $x \in X$ such that either $< x, u_1 >$ or $< x, u_2 >$ is an edge in $(X \cup \{u_1, u_2\}, \lambda, g)$ form an inhomogeneous Poisson point process. A simple inclusion–exclusion calculation shows that the intensity function of this inhomogeneous process is $\lambda h(x; u_1, u_2)$ where $h(x; u_1, u_2) := g(|x - u_1|) + g(|x - u_2|) - g(|x - u_1|)g(|x - u_2|)$.

Lemma 2 *For any $t > 0$, we have*

$$E(I_t(K)) = \lambda \int_K dx \exp(-\lambda \int_{K^t} dy g(|x - y|))$$

and

$$\begin{aligned} E(I_t(K))^2 &= E(I_t(K)) \\ &\quad +\lambda^2 \int_K \int_K dx_1 dx_2 (1 - g(|x_1 - x_2|)) \exp(-\lambda \int_{K^t} dy h(y; x_1, x_2)). \end{aligned}$$

Proof: The proof is by direct computation.

$$\begin{aligned}
&E(I_t(K)) \\
&= \sum_{m=1}^{\infty} E\left(\sum_{\xi\in X} 1_{\{\xi\in K \text{ and } \xi \text{ is isolated in } K^t\}}|N(K^t)=m\right) P(N(K^t)=m) \\
&= \sum_{m=1}^{\infty} \frac{\exp(-\lambda\ell(K^t))[\lambda\ell(K^t)]^m}{m!} \\
&\qquad mP\left(\xi_1\in K \text{ and } \xi_1 \text{ is isolated in } K^t|X(K^t)=\{\xi_1,\ldots,\xi_m\}\right) \\
&= \sum_{m=1}^{\infty} \frac{\exp(-\lambda\ell(K^t))\lambda^m}{(m-1)!}\int_K dx_1 \int_{K^t}\cdots\int_{K^t} dx_m\ldots dx_2 \prod_{i=2}^{m}(1-g(|x_1-x_i|)) \\
&= \lambda\sum_{m=1}^{\infty} \frac{\exp(-\lambda\ell(K^t))\lambda^{m-1}}{(m-1)!}\int_K dx_1\left[\int_{K^t} dy\,(1-g(|x_1-y|))\right]^{m-1} \\
&= \lambda\exp(-\lambda\ell(K^t))\int_K dx_1 \sum_{m=0}^{\infty}\frac{1}{m!}\left[\lambda\int_{K^t} dy\,(1-g(|x_1-y|))\right]^{m} \\
&= \lambda\int_K dx\exp\left(-\lambda\int_{K^t} dy g(|x-y|)\right). \qquad (10)
\end{aligned}$$

For the second moment, note that

$$\begin{aligned}
(I_t(K))^2 &= \left(\sum_{\xi\in X} 1_{\{\xi\in K \text{ and } \xi \text{ is isolated in } K^t\}}\right)^2 \\
&= \sum_{\xi\in X} 1_{\{\xi\in K \text{ and } \xi \text{ is isolated in } K^t\}} \\
&\qquad \sum_{\xi\neq\xi'\in X} 1_{\{\xi,\xi'\in K \text{ and both are isolated in } K^t\}} \\
&= I_t(K)+V_t(K) \ \text{(say)}. \qquad (11)
\end{aligned}$$

We now calculate $E(V_t(K))$.

$$\begin{aligned}
&E(V_t(K)) \\
&= \sum_{m=2}^{\infty} E\left(\sum_{\xi\neq\xi'\in X} 1_{\{\xi,\xi'\in K \text{ and both are isolated in } K^t\}}|(N(K^t)=m\right) P(N(K^t)=m) \\
&= \sum_{m=2}^{\infty} \frac{\exp(-\lambda\ell(K^t))[\lambda\ell(K^t)]^m}{m!} m(m-1) \\
&\qquad P\left(\xi_1\neq\xi_2\in K \text{ are isolated in } K^t|X(K^t)=\{\xi_1,\ldots,\xi_m\}\right) \\
&= \sum_{m=2}^{\infty} \frac{\exp(-\lambda\ell(K^t))\lambda^m}{(m-2)!}\int_K\int_K dx_1dx_2\,(1-g(|x_1-x_2|)) \\
&\qquad \left(\int_{K^t}\cdots\int_{K^t} dx_m\ldots dx_3\prod_{i=3}^{m}(1-g(|x_1-x_i|))(1-g(|x_2-x_i|))\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 \sum_{m=2}^{\infty} \frac{\exp(-\lambda\ell(K^t))\lambda^{m-2}}{(m-2)!} \int_K \int_K dx_1 dx_2\, (1-g(|x_1-x_2|)) \\
&\qquad \left[\int_{K^t} dy(1-g(|y-x_1|))(1-g(|y-x_2|))\right]^{m-2} \\
&= \lambda^2 \exp(-\lambda\ell(K^t)) \int_K \int_K dx_1 dx_2\, (1-g(|x_1-x_2|)) \\
&\qquad \left(\sum_{m=0}^{\infty} \frac{1}{m!}\left[\lambda \int_{K^t} dy(1-g(|y-x_1|))(1-g(|y-x_2|))\right]^{m}\right) \\
&= \lambda^2 \exp(-\lambda\ell(K^t)) \int_K \int_K dx_1 dx_2\, (1-g(|x_1-x_2|)) \\
&\qquad \exp\left(\lambda \int_{K^t} dy(1-g(|x_1-y|))(1-g(|x_2-y|))\right) \\
&= \lambda^2 \int_K \int_K dx_1 dx_2 (1-g(|x_1-x_2|)) \exp\left(-\lambda \int_{K^t} dy h(y;x_1,x_2)\right). \qquad (12)
\end{aligned}
$$

This proves the lemma. ■

Letting $t \to \infty$ and using the dominated convergence theorem as in (9), we obtain from (10),

$$
\begin{aligned}
EI(K) &= \lambda \int_K dx \exp\left(-\lambda \int_{\mathbb{R}^d} g(|x-y|)dy\right) \\
&= \lambda \int_K dx \exp\left(-\lambda \int_{\mathbb{R}^d} g(|y|)dy\right) \\
&= \lambda\ell(K)p(\lambda, g) \qquad (13)
\end{aligned}
$$

and, from (11) and (12), we have

$$
\begin{aligned}
&\mathrm{Var}(I(K)) \\
&\quad = -[\lambda\ell(K)]^2 \exp(-2\lambda \int_{\mathbb{R}^d} dy g(|y|)) + \lambda\ell(K)\exp(-\lambda \int_{\mathbb{R}^d} g(|y|)dy) + \\
&\qquad \lambda^2 \int_K \int_K dx_1 dx_2 \left[(1-g(|x_1-x_2|))\exp(-\lambda \int_{\mathbb{R}^d} dy h(y;x_1,x_2))\right] \\
&\quad = \lambda\ell(K)p(\lambda,g) + (\lambda\ell(K))^2 \int_K \int_K dx_1 dx_2 \frac{1}{\ell(K)^2} p^2(\lambda, g) \\
&\qquad \left[(1-g(|x_1-x_2|))\exp(\lambda \int_{\mathbb{R}^d} dy g(|x_1-y|)g(|x_2-y|)) - 1\right]. \qquad (14)
\end{aligned}
$$

Now let $I_{(n)}(K)$ be the number of isolated Poisson points in K for the model (X_n, λ_n, g_n) (i.e. the equivalent of $I(K)$ for the process X_n).

Lemma 3 *As $n \to \infty$, we have*

$$
\left(\lambda_n \ell(K)\right)^{-1} E(I_{(n)}(K)) \to p(\lambda, g)
$$

and

$$\left(\lambda_n \ell(K)\right)^{-1} Var(I_{(n)}(K)) \to p(\lambda,g) + \lambda p^2(\lambda,g) \int_{\mathbb{R}^d} dv \\ \Big[(1-g(|v|)) \exp(\lambda \int_{\mathbb{R}^d} g(|z|)g(|z-v|)dz) - 1\Big]. \quad (15)$$

Proof: First, from (13) and (8) we have

$$\begin{aligned} \left(\lambda_n \ell(K)\right)^{-1} E(I_{(n)}(K)) &= p(\lambda_n, g_n) \\ &\to p(\lambda, g) \text{ as } n \to \infty. \end{aligned} \quad (16)$$

Also,

$$\begin{aligned} &\int_K \int_K dx_1 dx_2 \Big[(1-g_n(|x_1-x_2|)) \exp(\lambda_n \int_{\mathbb{R}^d} g_n(|x_1-y|)g_n(|x_2-y|)dy) - 1\Big] \\ =&\int_K \int_K dx_1 dx_2 \Big[(1-g(n|x_1-x_2|)) \exp(\lambda_n \int_{\mathbb{R}^d} g(n|x_1-y|)g(n|x_2-y|)dy) - 1\Big] \\ =&\int_K dx_1 \int_{K-x_1} dv_1 \Big[(1-g(n|v_1|)) \exp(\lambda_n \int_{\mathbb{R}^d} g(n|z|)g(n|z-v_1|)dy) - 1\Big] \\ =&\frac{1}{n^d} \int_K dx_1 \int_{\mathbb{R}^d} dv 1_{n(K-x_1)} \left[(1-g(|v|)) \exp(\frac{\lambda_n}{n^d} \int_{\mathbb{R}^d} g(|z|)g(|z-v|)dz) - 1\right]. \end{aligned} \quad (17)$$

The inner integrand $1_{n(K-x_1)}\Big[(1-g(|v|)) \exp\big((\lambda_n/n^d) \int_{\mathbb{R}^d} dz g(|z|)g(|z-v|)\big) - 1\Big]$ of the expression in (17) tends to $\Big[(1-g(|v|)) \exp(\lambda \int_{\mathbb{R}^d} dz g(|z|)g(|z-v|)) - 1\Big]$ as $n \to \infty$ for almost all (Lebesgue) $x_1 \in K$.

Note that for any $y \in \mathbb{R}^d$, we must have either $|y| \geq |v|/2$ or $|v-y| \geq |v|/2$, thus

$$\begin{aligned} \int_{\mathbb{R}^d} dy g(|y|)g(|v-y|) &\leq \int_{|y|<|v|/2} dy g(|y|)g(|v-y|) + \int_{|y|\geq|v|/2} dy g(|y|)g(|v-y|) \\ &\leq g(|v|/2) \int_{|y|<|v|/2} dy g(|y|) + g(|v|/2) \int_{|y|\geq|v|/2} dy g(|v-y|) \\ &\leq 2g(|v|/2) \int_{\mathbb{R}^d} dy g(|y|). \end{aligned}$$

Since $\lambda_n/n^d \to \lambda$ as $n \to \infty$, $\int_{\mathbb{R}^d} g(|y|)dy < \infty$ and g is non-increasing, we can choose M and N so large that $(2\lambda_n/n^d)g(M/2) \int_{\mathbb{R}^d} g(|y|)dy \leq 1$ for all $n \geq N$. Hence, for $|v| > M$

$$\begin{aligned} \exp\left(\frac{\lambda_n}{n^d} \int_{\mathbb{R}^d} g(|y|)g(|v-y|)dy\right) &\leq \exp\left(2\frac{\lambda_n}{n^d} g(|v|/2) \int_{\mathbb{R}^d} g(|y|)dy\right) \\ &\leq 1 + 2e\frac{\lambda_n}{n^d} g(|v|/2) \int_{\mathbb{R}^d} g(|y|)dy, \end{aligned} \quad (18)$$

where we have used the inequality $e^x \leq 1 + ex$ for $x \leq 1$.

Combining the above inequalities we have for $n \geq N$,

$$\begin{aligned} &(1-g(|v|)) \exp\left(\frac{\lambda_n}{n^d} \int_{\mathbb{R}^d} g(|y|)g(|v-y|)dy\right) - 1 \\ \leq& \ (1-g(|v|))(1 + C(g,d)g(|v|/2)) - 1 \end{aligned}$$

$$\leq \quad (2C(g,d)+1)g(|v|/2),$$

where $C(g,d)$ is a constant depending only on g and d. Thus, applying the dominated convergence theorem to (17) and combining it with (14) and (16) we have, as $n \to \infty$,

$$\begin{aligned}\big(\lambda_n \ell(K)\big)^{-1}\mathrm{Var}(I_{(n)}(K)) \;\to\; & p(\lambda,g) \\ & + \lambda p^2(\lambda,g)\int_{\mathbb{R}^d} dv\Big[(1-g(|v|))\exp(\lambda\int_{\mathbb{R}^d} g(|z|)g(|z-v|)dz)-1\Big].\end{aligned}$$

This completes the proof of the lemma. ■

3 Asymptotic normality

We shall apply Lyapunov's central limit theorem for arrays to obtain the required asymptotic normality. For this we first need to truncate the connection function g. Let $R>0$ be fixed and consider the functions $g_R, g^R : \mathbb{R} \to [0,1]$ defined by $g_R(x) = 1_{\{x \leq R\}} g(x)$ and $g^R(x) = 1_{\{x>R\}} g(x)$. Let $J_R(K)$ be the number of isolated vertices of (X,λ,g_R) in K and let $L_R(K)$ be the number of isolated vertices of (X,λ,g_R) in K which are not isolated in (X,λ,g). Clearly $L_R(K) = J_R(K) - I(K)$. Similarly define $g_{n,R}, g_n^R : \mathbb{R}^d \to [0,1]$ by $g_{n,R}(x) = 1_{\{x \leq R\}} g_n(x)$ and $g_n^R(x) = 1_{\{x>R\}} g_n(x)$. Let $J_{n,R}(K)$ and $L_{n,R}(K)$ be the corresponding quantities for the model (X_n,λ_n,g_n).

We first compute the mean and variance of $L_R(K)$. For this we need some more notations. Given two functions $f_1, f_2 : \mathbb{R} \to [0,1], x_1, x_2 \in \mathbb{R}^d$ and any Borel set $A \subseteq \mathbb{R}^d$ and $\lambda > 0$, we set

$$p_{A,\lambda}^{f_1,f_2}(x_1,x_2) = \exp\Big(\lambda \int_A f_1(|y-x_1|) f_2(|y-x_2|)dy\Big).$$

By a change of variables, it is clear that $p_{\mathbb{R}^d,\lambda}^{f_1,f_2}(x_1,x_2) = p_{\mathbb{R}^d,\lambda}^{f_1,f_2}(0,x_2-x_1) = p_{\mathbb{R}^d,\lambda}^{f_1,f_2}(x_1-x_2,0)$.

Lemma 4 *For $R > diam(K)$,*

$$E(L_R(K)) = \lambda \ell(K) p(\lambda,g_R)\big(1-p(\lambda,g^R)\big) \tag{19}$$

and

$$\begin{aligned} Var(L_R(K)) = {} & \lambda\ell(K)p(\lambda,g_R)(1-p(\lambda,g^R)) \\ & +\lambda^2 \int_K\int_K dx_1dx_2\Bigg[(1-g(|x_1-x_2|))\Big[p^2(\lambda,g_R)p_{\mathbb{R}^d,\lambda}^{g_R,g_R}(x_1,x_2) \\ & \quad -2p^2(\lambda,g_R)p(\lambda,g^R)p_{\mathbb{R}^d,\lambda}^{g_R,g}(x_1,x_2) + p^2(\lambda,g)p_{\mathbb{R}^d,\lambda}^{g,g}(x_1,x_2)\Big] \\ & \quad -p^2(\lambda,g_R)\big(1-p(\lambda,g^R)\big)^2\Bigg]. \end{aligned} \tag{20}$$

Proof : We employ the same technique as in Lemma 2 to compute the moments. Define, as earlier, $L_{t,R}(K) := \sum_{i=1}^{N(K^t)} 1_{F_i}$ where $F_i := \{\xi_i \in K$ is not connected to any $\xi_j \in X(K^t)$ at a distance R or less from ξ_i, but is connected to some $\xi_j \in X(K^t)$ at a distance more than R from $\xi_i\}$.

Note that $P(F_1 \mid N(K^t) = m) = P(F_i \mid N(K^t) = m)$ for $i = 2, 3, \ldots, m$, and

$$\begin{aligned}
&P(F_1 \mid N(K^t) = m) \\
&= \frac{1}{\big(\ell(K^t)\big)^m} \int_K dx_1 \Bigg[\int_{K^t} \cdots \int_{K^t} dx_m \cdots dx_2 \\
&\qquad \times \Big[1 - \prod_{i=2}^{m} (1 - g^R(|x_1 - x_i|))\Big] \prod_{i=2}^{m} (1 - g_R(|x_1 - x_i|)) \Bigg] \\
&= \frac{1}{\big(\ell(K^t)\big)^m} \int_K dx_1 \Bigg[\Big(\int_{K^t} dy (1 - g_R(|y - x_1|)) dy \Big)^{m-1} \\
&\qquad - \Big(\int_{K^t} dy (1 - g(|y - x_1|)) dy \Big)^{m-1} \Bigg]
\end{aligned}$$

where we have used the fact that g_R and g^R have disjoint supports and $g = g_R + g^R$. Now, using the above, we have

$$\begin{aligned}
&E(L_{t,R}(K)) \\
&= \sum_{m=1}^{\infty} \frac{\exp(-\lambda \ell(K^t)) \big(\lambda \ell(K^t)\big)^m}{m!} m P(F_1 \mid N(K^t) = m) \\
&= \lambda \int_K dx_1 \Bigg[\sum_{m=0}^{\infty} \frac{\exp(-\lambda \ell(K^t)) \lambda^m}{m!} \Big[\int_{K^t} dy (1 - g_R(|y - x_1|)) \Big]^m \\
&\qquad - \sum_{m=0}^{\infty} \frac{\exp(-\lambda \ell(K^t)) \lambda^m}{m!} \Big[\int_{K^t} dy (1 - g(|y - x_1|)) \Big]^m \Bigg] \\
&= \lambda \int_K dx \exp(-\lambda \int_{K^t} dy\, g_R(|y - x|)) \Big[1 - \exp(-\lambda \int_{K^t} dy g^R(|y - x|)) \Big].
\end{aligned}$$

The integrand above converges to $p(\lambda, g_R)(1 - p(\lambda, g^R))$ as $t \to \infty$ and it is bounded. Using the dominated convergence theorem, we have (19).

Next, note that $L_{t,R}^2(K) = L_{t,R}(K)) + \sum_{1 \le i \ne j \le N(K^t)} 1_{F_i} 1_{F_j}$.

$$\begin{aligned}
&P(F_1 \cap F_2 \mid N(K^t) = m) \\
&= \frac{1}{\big(\ell(K^t)\big)^m} \int_K \int_K dx_1 dx_2 (1 - g(|x_1 - x_2|)) \Bigg[\int_{K^t} \cdots \int_{K^t} dx_m \cdots dx_3 \\
&\qquad \times \Big(1 - \prod_{i=3}^{m} (1 - g^R(|x_1 - x_i|))\Big) \prod_{i=3}^{m} (1 - g_R(|x_1 - x_i|)) \\
&\qquad \times \Big(1 - \prod_{i=3}^{m} (1 - g^R(|x_2 - x_i|))\Big) \prod_{i=3}^{m} (1 - g_R(|x_2 - x_i|)) \Bigg].
\end{aligned} \tag{21}$$

Here we have used the fact that $\mathrm{diam}(K) < R$ and $g_R(x) = g(x)$ for $x \le R$. Writing $\bar{a}_i = g^R(|x_1 - x_i|)$, etc., we simplify the integrand in the inner integral as

$$
\begin{aligned}
&(1 - \prod(1-\bar{a}_i)) \prod(1-\underline{b_i})(1 - \prod(1-\bar{c}_i)) \prod(1-\underline{d_i}) \\
&= \prod(1 - \underline{b_i} - \underline{d_i} + \underline{b_i d_i}) \big(1 - \prod(1-\bar{a}_i) - \prod(1-\bar{c}_i) + \prod(1-\bar{a}_i)(1-\bar{c}_i)\big) \\
&= \prod(1 - \underline{b_i} - \underline{d_i} + \underline{b_i d_i}) + \prod(1 - \underline{b_i} - \underline{d_i} + \underline{b_i d_i})(1 - \bar{a}_i - \bar{c}_i + \bar{a}_i\bar{c}_i) \\
&\quad - \prod(1 - \underline{b_i} - \underline{d_i} + \underline{b_i d_i})(1-\bar{a}_i) - \prod(1 - \underline{b_i} - \underline{d_i} + \underline{b_i d_i})(1-\bar{c}_i). \qquad (22)
\end{aligned}
$$

Here $\bar{a}_i\underline{b_i} = \bar{c}_i\underline{d_i} = 0$.

We introduce some more notations to write the inner integrand in the square brackets of (21) . For any two functions $f_1, f_2 : \mathbb{R} \to [0,1]$ and $x_1, x_2 \in \mathbb{R}^d$, $\alpha_{x_1}(f_1) = \int_{K^t} f_1(|y - x_1|)dy$ and $\beta(f_1, f_2) = \int_{K^t} f_1(|y-x_1|) f_2(|y-x_2|)dy$. Now, using these notations and expanding as in (22), the inner integrand in the square brackets in (21), can be written as

$$
\begin{aligned}
\Bigg[&\Big[\ell(K^t) - \alpha_{x_1}(g_R) - \alpha_{x_2}(g_R) + \beta(g_R, g_R)\Big]^{m-1} \\
&- \Big[\ell(K^t) - \alpha_{x_1}(g) - \alpha_{x_2}(g_R) + \beta(g_R, g_R) + \beta(g^R, g_R)\Big]^{m-1} \\
&- \Big[\ell(K^t) - \alpha_{x_1}(g_R) - \alpha_{x_2}(g) + \beta(g_R, g_R) + \beta(g_R, g^R)\Big]^{m-1} \\
&+ \Big[\ell(K^t) - \alpha_{x_1}(g) - \alpha_{x_2}(g) + \beta(g, g)\Big]^{m-1} \Bigg].
\end{aligned}
$$

Now, using this expression in (21), we have

$$
\begin{aligned}
&E(L^2_{t,R}(K)) - E(L_{t,R}(K)) \\
&= \sum_{m=2}^{\infty} \frac{\exp(-\lambda\ell(K^t))\big(\lambda\ell(K^t)\big)^m}{m!} m(m-1) P(F_1 \cap F_2 \mid N(K^t) = m) \\
&= \lambda^2 \int_K \int_K dx_1 dx_2 (1 - g(|x_1 - x_2|)) \Bigg[\sum_{m=0}^{\infty} \frac{\exp(-\lambda\ell(K^t))\lambda^m}{m!} \\
&\quad \times \Bigg[\Big[\ell(K^t) - \alpha_{x_1}(g_R) - \alpha_{x_2}(g_R) + \beta(g_R, g_R)\Big]^{m} \\
&\quad - \Big[\ell(K^t) - \alpha_{x_1}(g) - \alpha_{x_2}(g_R) + \beta(g_R, g_R) + \beta(g^R, g_R)\Big]^{m} \\
&\quad - \Big[\ell(K^t) - \alpha_{x_1}(g_R) - \alpha_{x_2}(g) + \beta(g_R, g_R) + \beta(g_R, g^R)\Big]^{m} \\
&\quad + \Big[\ell(K^t) - \alpha_{x_1}(g) - \alpha_{x_2}(g) + \beta(g, g)\Big]^{m} \Bigg] \\
&= \lambda^2 \int_K \int_K dx_1 dx_2 (1 - g(|x_1 - x_2|)) \Big[\exp(-\alpha_{x_1}(g) - \alpha_{x_2}(g_R) + \beta(g_R, g_R)) \\
&\quad - \exp(-\alpha_{x_1}(g) - \alpha_{x_2}(g_R) + \beta(g_R, g_R) + \beta(g^R, g_R))
\end{aligned}
$$

$$
\begin{aligned}
&- \exp(-\alpha_{x_1}(g_R) - \alpha_{x_2}(g) + \beta(g_R, g_R) + \beta(g_R, g^R)) \\
&+ \exp(-\alpha_{x_1}(g) - \alpha_{x_2}(g) + \beta(g, g))\Big].
\end{aligned}
$$

Finally, using the dominated convergence theorem, we conclude (20). ■

Next, we turn our attention to the model (X_n, λ_n, g_n). Similar computations as in Lemma 3 can be carried out here to conclude

Lemma 5 *As $n \to \infty$, we have*

$$
\big(\lambda_n \ell(K)\big)^{-1} E(L_{n,R}(K)) \to p(\lambda, g_R)(1 - p(\lambda, g^R)) \tag{23}
$$

and

$$
\begin{aligned}
&\big(\lambda_n \ell(K)\big)^{-1} Var(L_{n,R}(K)) \\
&\to \quad p(\lambda, g_R)(1 - p(\lambda, g^R)) + \lambda \int_{\mathbb{R}^d} dx \Bigg[(1 - g(|x|)) \\
&\qquad \Big[p^2(\lambda, g_R) p^{g_R, g_R}_{\mathbb{R}^d, \lambda}(x, 0) - 2p^2(\lambda, g_R) p(\lambda, g^R) p^{g_R, g}_{\mathbb{R}^d, \lambda}(x, 0) \\
&\qquad + p^2(\lambda, g) p^{g,g}_{\mathbb{R}^d, \lambda}(x, 0) \Big] - p^2(\lambda, g_R) \big(1 - p(\lambda, g^R)\big)^2 \Bigg].
\end{aligned} \tag{24}
$$

As a corollary to Lemma 5, we obtain

Corollary 6 *As $R \to \infty$, we have*

$$
\lim_{R \to \infty} \lim_{n \to \infty} \big(\lambda_n \ell(K)\big)^{-1} E(L_{n,R}(K)) \to 0
$$

and

$$
\lim_{R \to \infty} \lim_{n \to \infty} \big(\lambda_n \ell(K)\big)^{-1} Var(L_{n,R}(K)) \to 0.
$$

Proof : Since $\int_{\mathbb{R}^d} g(|x|)dx < \infty$, we have $\int_{\mathbb{R}^d} g^R(|x|)dx \to 0$ as $R \to \infty$. Hence $p(\lambda, g^R) \to 1$ as $R \to \infty$, proving the first part.

For the second part, we note that the integrand in (24) converges to 0 as $R \to \infty$. To use the dominated convergence theorem, we need to show that the integrand is bounded by an integrable function. As in the second half of the proof of Lemma 3, we argue that the absolute value of the integrand is bounded by,

$$
\begin{aligned}
& g(|x|) \Big[p^2(\lambda, g_R) p^{g_R, g_R}_{\mathbb{R}^d, \lambda}(x, 0) + 2p^2(\lambda, g_R) p(\lambda, g^R) p^{g_R, g}_{\mathbb{R}^d, \lambda}(x, 0) \\
& + p^2(\lambda, g) p^{g,g}_{\mathbb{R}^d, \lambda}(x, 0) \Big] + p^2(\lambda, g_R) \Big(p^{g_R, g_R}_{\mathbb{R}^d, \lambda}(x, 0) - 1 \Big) \\
& + 2p^2(\lambda, g_R) p(\lambda, g^R) \Big(p^{g_R, g}_{\mathbb{R}^d, \lambda}(x, 0) - 1 \Big) + p^2(\lambda, g) \Big(p^{g,g}_{\mathbb{R}^d, \lambda}(x, 0) - 1 \Big) \\
\le\ & 4g(|x|) + 4 \Big(p^{g,g}_{\mathbb{R}^d, \lambda}(x, 0) - 1 \Big) \\
\le\ & 4(1 + C) g(|x/2|).
\end{aligned}
$$

This proves the result. ■

We now show that it suffices to obtain a central limit theorem for $J_{n,R}(K)$. Indeed, for any $x \in \mathbb{R}$, $\epsilon > 0$ and any fixed $R > 0$, we have

$$P\Big(\frac{I_{(n)}(K) - E(I_{(n)}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x\Big) \le P\Big(\Big|\frac{L_{n,R}(K) - E(L_{n,R})}{\sqrt{\mathrm{Var}(I_{(n)}(K))}}\Big| \ge \epsilon\Big)$$
$$+ P\Big(\frac{J_{n,R}(K) - E(J_{n,R}(K))}{\sqrt{\mathrm{Var}(J_{n,R}(K))}\sqrt{\frac{\mathrm{Var}(J_{n,R}(K))}{\mathrm{Var}(I_{(n)}(K))}}} \le x + \epsilon\Big).$$

Now, note that both the values $\mathrm{Var}(J_{n,R}(K))$ and $\mathrm{Var}(I_{(n)}(K))$ may be computed from (15) to show that $\mathrm{Var}(J_{n,R}(K))/\mathrm{Var}(I_{(n)}(K))$ converges to some constant δ_R as $n \to \infty$ where δ_R is such that $\delta_R \to 1$ as $R \to \infty$. Thus, we have,

$$\begin{aligned}
&\limsup_{n\to\infty} P\Big(\frac{I_{(n)}(K) - E(I_{(n)}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x\Big) \\
\le\ & \lim_{R\to\infty} \limsup_{n\to\infty} P\Big(\frac{J_{n,R}(K) - E(J_{n,R}(K))}{\sqrt{\mathrm{Var}(J_{n,R}(K))}} \sqrt{\frac{\mathrm{Var}(J_{n,R}(K))}{\mathrm{Var}(I_{(n)}(K))}} \le x + \epsilon\Big) \\
& \quad + P\Big(\Big|\frac{L_{n,R}(K) - E(L_{n,R})}{\sqrt{\mathrm{Var}(I_{(n)}(K))}}\Big| \ge \epsilon\Big) \\
\le\ & \lim_{R\to\infty} \Phi((x+\epsilon)/\delta_R) + \lim_{R\to\infty} \lim_{n\to\infty} \frac{\mathrm{Var}(L_{n,R}(K))}{\epsilon^2 \mathrm{Var}(I_{(n)}(K))} \\
=\ & \Phi(x+\epsilon)
\end{aligned}$$

where $\Phi(x)$ is the distribution function of the standard normal random variable. Letting $\epsilon \to 0$, we get $\limsup_{n\to\infty} P\Big(\frac{I_{(n)}(K)-E(I_{(n)}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x\Big) \le \Phi(x)$. Arguing similarly, we may show, using

$$\begin{aligned}
&P\Big(\frac{J_{n,R}(K) - E(J_{n,R}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x - \epsilon\Big) \\
\le\ & P\Big(\Big|\frac{L_{n,R}(K) - E(L_{n,R})}{\sqrt{\mathrm{Var}(I_{(n)}(K))}}\Big| \ge \epsilon\Big) + P\Big(\frac{I_{(n)}(K) - E(I_{(n)}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x\Big)
\end{aligned}$$

that $\liminf_{n\to\infty} P\Big(\frac{I_{(n)}(K)-E(I_{(n)}(K))}{\sqrt{\mathrm{Var}(I_{(n)}(K))}} \le x\Big) \ge \Phi(x)$.

Now, we will prove the central limit theorem for $J_{n,R}(K)$. Thus we assume henceforth that

$$g(x) = 0 \text{ for } |x| \ge R.$$

Fix an integer m and define $B(n) = [-(m+2)R/n, (m+1)R/n]^d$ and $B_0(n) = [-mR/n, mR/n]^d$. Divide $\mathbb{R}^d$ into cubes each being isomorphic to $B(n)$ and centred at the sites of the lattice $L(n) := \{\frac{2(m+1)R}{n}(z_1, z_2, \ldots, z_d) : z_1, z_2, \ldots, z_d \in \mathbb{N}, i = 1, \ldots, d\}$. For a site $x = \frac{2(m+1)R}{n}(z_1, z_2, \ldots, z_d)$, define $B(n,x) = x + B(n)$, $B_0(n,x) = x + B_0(n)$ and $S(x,n) = x + (B(n) \setminus B_0(n))$.

Let $\alpha_1, \dots, \alpha_{k_n}$ be an enumeration all sites x in $L(n)$ for which $B(n,x) \subseteq K$. Define $D_1 = \cup_{i=1}^{k_n} B_0(n,\alpha_i)$ and $D_2 = \cup_{i=1}^{k_n} S(n,\alpha_i)$ and $D_3 = K \backslash (D_1 \cup D_2)$.

Clearly $I_{(n)}(B_0(n,\alpha_i))$ depends only on the region $B(n,\alpha_i)$, and since, for $i \neq j$, $B(n,\alpha_i) \cap B(n,\alpha_j) = \emptyset$, we have $\{I_{(n)}(B_0(n,\alpha_i)) : i = 1, \dots, k_n\}$ is a collection of i.i.d. random variables. Thus, $\mathrm{Var}(I_{(n)}(D_1)) = k_n \mathrm{Var}(I_{(n)}(B_0(n)))$. Now, from equation (14), we have

$$
\begin{aligned}
& \mathrm{Var}(I_{(n)}(B_0(n))) \\
= \quad & \lambda_n \ell(B_0(n)) p(\lambda_n, g_n) + \lambda_n^2 \int_{B_0(n)} \int_{B_0(n)} dx_1 \, dx_2 \, p^2(\lambda_n, g_n) \\
& \left[(1 - g_n(|x_1 - x_2|)) \exp\left(\lambda_n \int_{\mathbb{R}^d} g_n(|y - x_1|) g_n(|y - x_2|) dy \right) - 1 \right] \\
\to \quad & \lambda (mR)^d p(\lambda, g) + \lambda^2 p^2(\lambda, g) \int_{[-mR,mR]^d} \int_{[-mR,mR]^d} dx_1 dx_2 \\
& \left[(1 - g(|x_1 - x_2|)) \exp\left(-\lambda \int_{\mathbb{R}^d} g(|y - x_1|) g(|y - x_2|) dy \right) - 1 \right] \qquad (25)
\end{aligned}
$$

as $n \to \infty$. Here we have used the fact that $nB_0(n) = [-mR, mR]^d$ and then applied the dominated convergence theorem.

We will first show that the array $\{I_{(n)}(B_0(n,\alpha_i)) : i = 1, \dots, k_n\}_{n \geq 1}$ satisfies the conditions for Lyapunov's central limit theorem. For this, besides the independence properties which we obtained earlier, we need to show

$$
\sum_{i=1}^{k_n} \frac{E|(I_{(n)}(B_0(n,\alpha_i))) - E(I_{(n)}(B_0(n,\alpha_i)))|^3}{(\mathrm{Var}(I_{(n)}(D_1)))^{3/2}} \to 0 \quad \text{as } n \to \infty.
$$

Clearly, $I_{(n)}(B_0(n,\alpha_i)) \leq N_n(B_0(n))$ where $N_n(B_0(n))$ is the number of points of the process X_n in $B_0(n)$. Thus,

$$
\begin{aligned}
& E\left(|I_{(n)}(B_0(n,\alpha_i)) - EI_{(n)}(B_0(n,\alpha_i))| \right)^3 \\
\leq \quad & E\left(I_{(n)}(B_0(n,\alpha_i)) + EI_{(n)}(B_0(n,\alpha_i)) \right)^3 \\
= \quad & EI_{(n)}(B_0(n,\alpha_i))^3 + 3E(I_{(n)}(B_0(n,\alpha_i)))^2 EI_{(n)}(B_0(n,\alpha_i)) \\
& \quad +4(E(I_{(n)}(B_0(n,\alpha_i))))^3 \\
\leq \quad & E(I_{(n)}(B_0(n,\alpha_i)))^3 + 3(E(I_{(n)}(B_0(n,\alpha_i)))^3)^{2/3} (E(I_{(n)}(B_0(n,\alpha_i)))^3)^{1/3} \\
& \quad +4(E(I_{(n)}(B_0(n,\alpha_i))))^3 \\
= \quad & 4E(I_{(n)}(B_0(n,\alpha_i)))^3 + 4(E(I_{(n)}(B_0(n,\alpha_i))))^3 \\
\leq \quad & 4E(N_n(B_0(n)))^3 + 4(E(N_n(B_0(n))))^3. \qquad (26)
\end{aligned}
$$

Now $N_n(B_0(n))$ has a Poisson distribution with mean $\lambda_n \ell(B_0(n)) = O(\lambda_n / n^d)$. Since $\lambda_n / n^d \to \lambda$ as $n \to \infty$, we have $4E(N_n(B_0(n)))^3 + 4(E(N_n(B_0(n)))^3 \leq C$, where C is a constant not depending on n or d.

Note first that $k_n = O(n^d)$ as $n \to \infty$. Thus, we have, as $n \to \infty$,

$$
\sum_{i=1}^{k_n} \frac{E(|I_{(n)}(B_0(n,\alpha_i)) - E(I_{(n)}(B_0(n,\alpha_i)))|^3)}{(\mathrm{Var}(I_{(n)}(D_1)))^{3/2}} \leq \frac{k_n C}{k_n^{3/2} \mathrm{Var}(I_{(n)}(B_0(n)))^{3/2}} \to 0.
$$

Now applying Lyapunov's theorem to the array $\{I_{(n)}(B_0(n,\alpha_i)) : i = 1,\dots k_n\}_{n\geq 1}$ we have that $\dfrac{I_{(n)}(B_0(n,\alpha_i)) - E(I_{(n)}(B_0(n,\alpha_i)))}{\sqrt{\mathrm{Var}(I_{(n)}(B_0(n,\alpha_i)))}}$ converges in distribution to a standard normal random variable.

Finally to complete the proof of the theorem, we need to show

$$\lim_{m\to\infty}\limsup_{n\to\infty}\Big(\mathrm{Var}(I_{(n)}(D_1))\Big)^{-1}\mathrm{Var}(I_{(n)}(D_i)) = 0 \text{ for both } i = 2 \text{ and } 3.$$

The above will also imply that $\lim_{m\to\infty}\limsup_{n\to\infty}\mathrm{Var}(I_{(n)}(D_1))/\mathrm{Var}(I_{(n)}(K)) \to 1$ as $n\to\infty$. For $I_{(n)}(D_2)$, we divide the annular region $S(x,n)$ into cubes S_i of size R/n. Thus, we have, $I_{(n)}(D_2) = \sum_{S_i} I_{(n)}(S_i)$. Note that whenever two cubes are separated by two or more cubes between them, they are independent. Now the random variables $I_{(n)}(S_1), I_{(n)}(S_2),\dots$ can be rearranged as $I_{(n)}(S_{i,1}), I_{(n)}(S_{i,2}),\dots$ for $i = 1,2,\dots,2^d$ such that all the random variables $\sum_j I_{(n)}(S_{i,j})$ are identically distributed and, for any fixed i, the random variables $\{I_{(n)}(S_{i,j}) : j \geq 1\}$ are i.i.d. Therefore,

$$\begin{aligned}
&\mathrm{Var}(I_{(n)}(D_2))\\
&= \sum_{i=1}^{2^d}\mathrm{Var}\Big(\sum_j I_{(n)}(S_{i,j})\Big) + \sum_{1\leq i_1\neq i_2\leq 2^d}\mathrm{Cov}\Big(\sum_j I_{(n)}(S_{i_1,j}),\sum_j I_{(n)}(S_{i_2,j})\Big)\\
&\leq 2^d\mathrm{Var}\Big(\sum_j I_{(n)}(S_{1,j})) + 4^d\mathrm{Var}\Big(\sum_j I_{(n)}(S_{1,j})\Big)\\
&\leq 2^{2d+1}\mathrm{Var}\Big(\sum_j I_{(n)}(S_{1,j})\Big)\\
&= 2^{2d+1}\sum_j \mathrm{Var}(I_{(n)}(S_{1,1})). \qquad (27)
\end{aligned}$$

Now, similar calculations as in (25) may be carried out to obtain

$$\mathrm{Var}(I_{(n)}(S_{1,1})) \to \lambda R^d p(\lambda,g) + \lambda^2 p^2(\lambda,g)\int_{[-R,R]^d}\int_{[-R,R]^d} dx_1 dx_2 \Big[(1-g(|x_1-x_2|))\exp(\lambda\int_{\mathbb{R}^d} g(|y-x_1|)g(|y-x_2|)dy) - 1\Big].$$

Now, the number of terms in the sum in (27) is bounded by $C(d)k_n m^{d-1}$ where $C(d)$ is a constant independent of m and n. Therefore,

$$\begin{aligned}
&\lim_{m\to\infty}\limsup_{n\to\infty}\frac{\mathrm{Var}(I_{(n)}(D_2))}{\mathrm{Var}(I_{(n)}(D_1))}\\
&\leq \lim_{m\to\infty}\limsup_{n\to\infty}\frac{2^{2d+1}C(d)k_n m^{d-1}\mathrm{Var}(I_{(n)}(S_{1,1}))}{k_n(2mR)^d p(\lambda_n,g_n)}\\
&= \frac{2^{d+1}C(d)}{R^d\lambda p(\lambda,g)}\lim_{m\to\infty} m^{-1}\lim_{n\to\infty}\mathrm{Var}(I_{(n)}(S_{1,1}))\\
&= 0.
\end{aligned}$$

For $I_{(n)}(D_3)$, from equation (14) and using bounds as in (18), we have,

$$\begin{aligned}
&\lambda_n^{-1}\mathrm{Var}(I_{(n)}(D_3)) \\
&= \ell(D_3)p(\lambda_n, g_n) + \lambda_n p^2(\lambda_n, g_n)\int_{D_3}\int_{D_3} dx_1 dx_2 \\
&\qquad \times\Big[(1 - g_n(|x_1 - x_2|))\exp\Big(\lambda_n \int_{\mathbb{R}^d} g_n(|y - x_1|)g_n(|y - x_1|)dy\Big) - 1\Big] \\
&\leq \ell(D_3)p(\lambda_n, g_n) + \frac{\lambda_n}{n^d}p^2(\lambda_n, g_n)\int_{D_3} dx_1 \int_{n(D_3 - x_1)} du \\
&\qquad \times\Big[(1 - g(|u|))\exp\Big(\frac{\lambda_n}{n^d}\int_{\mathbb{R}^d} g(|y|)g(|y - u|)dy\Big) - 1\Big] \\
&\leq \ell(D_3)p(\lambda_n, g_n) + \frac{\lambda_n}{n^d}p^2(\lambda_n, g_n)\int_{D_3} dx_1 \int_{\mathbb{R}^d} duCg(|u|/2) \\
&= \ell(D_3)\Big[p(\lambda_n, g_n) + \frac{\lambda_n}{n^d}p^2(\lambda_n, g_n)\int_{\mathbb{R}^d} duCg(|u|/2)\Big]
\end{aligned}$$

where C is a constant independent of n. Now noting that $\ell(D_3) \to 0$ as $n \to \infty$, and that $k_n = O(n^d)$, we have, for some other constant C'

$$\frac{\mathrm{Var}(I_{(n)}(D_3))}{\mathrm{Var}(I_{(n)}(D_1))} \leq \frac{C'\lambda_n \ell(D_3)}{k_n(2mR)^d p(\lambda_n g_n)} \to 0 \text{ as } n \to \infty.$$

This completes the proof of the theorem.

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