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Spectral triples and associated Connes-de Rham complex for the quantum $SU(2)$ and the quantum sphere

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Abstract

In this article, we take up the construction of spectral triples and associated calculus in the context of $SU_q(2)$ and S^2_{qc}. In order to construct explicit spectral triples, we begin with the computation of K-groups, and then from explicit generators we construct spectral triples which induce generating elements in K-homology. Using these spectral triples, we compute a modified version of the space of Connes-de Rham forms and the associated calculus. The space of L^2 forms have also been described explicitly.

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1 Introduction

Quantum $SU(2)$-group is one of the most well-known examples of a noncommutative space. This was studied in the topological setting by Woronowicz, and subsequently by many others. In the present article, we study a class of spectral triples on this noncommutative space and compute the exterior complexes corresponding to these. We begin with computation of K-groups, and then from explicit generators we construct spectral triples which induce generating elements in K-homology.

Let us start with a brief description of the C^*-algebra of continuous functions on the quantum $SU(2)$, to be denoted by $C(SU_q(2))$. This is the canonical C^*-algebra generated by two elements α and β satisfying the following relations:

$$\alpha^*\alpha + \beta^*\beta = I, \qquad \alpha\alpha^* + q^2\beta\beta^* = I,$$
$$\alpha\beta - q\beta\alpha = 0, \qquad \alpha\beta^* - q\beta^*\alpha = 0,$$
$$\beta^*\beta = \beta\beta^*.$$

The C^*-algebra $C(SU_q(2))$ can be described more concretely as follows. Let $\{e_i\}_{i\geq 0}$ and $\{e_i\}_{i\in\mathbb{Z}}$ be the canonical orthonormal bases for $L_2(\mathbb{N})$ and $L_2(\mathbb{Z})$ respectively. We denote by

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the same symbol N the operator $e_k \mapsto ke_k$, $k \geq 0$, on $L_2(\mathbb{N})$ and $e_k \mapsto ke_k$, $k \in \mathbb{Z}$, on $L_2(\mathbb{Z})$. Similarly, denote by the same symbol ℓ the operator $e_k \mapsto e_{k-1}$, $k \geq 1$, $e_0 \mapsto 0$ on $L_2(\mathbb{N})$ and the operator $e_k \mapsto e_{k-1}$, $k \in \mathbb{Z}$ on $L_2(\mathbb{Z})$. Now take $\mathcal{H}$ to be the Hilbert space $L_2(\mathbb{N}) \otimes L_2(\mathbb{Z})$, and define π to be the following representation of $C(SU_q(2))$ on $\mathcal{H}$:

$$\pi(\alpha) = \ell\sqrt{I - q^{2N}} \otimes I, \quad \pi(\beta) = q^N \otimes \ell.$$

Then π is a faithful representation of $C(SU_q(2))$, so that one can identify $C(SU_q(2))$ with the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\pi(\alpha)$ and $\pi(\beta)$. Image of π contains $\mathcal{K} \otimes C(S^1)$ as an ideal with $C(S^1)$ as the quotient algebra, that is we have a useful short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} C(S^1) \longrightarrow 0. \tag{1.1}$$

The Haar state h on $C(SU_q(2))$ is given by,

$$h : a \mapsto (1 - q^2) \sum_{i=0}^{\infty} q^{2i} \langle e_{i0}, ae_{i0} \rangle.$$

Remark 1.1 This representation admits a nice interpretation. Let M be a compact topological manifold and E, a Hermitian vector bundle on M. Let $\Gamma(M, E)$ be the space of continuous sections. Then $\Gamma(M, E)$ is a finitely generated projective $C(M)$ module. Define an inner product on $\Gamma(M, E)$ as

$$\langle s_1, s_2 \rangle := \int (s_1(m), s_2(m))_m d\nu(m),$$

where ν is a smooth measure on M and $(\cdot, \cdot)_m$ is the inner product on the fibre on m. Let $\mathcal{H}_E$ be the Hilbert space completion of $\Gamma(M, E)$. Then we have a natural representation of $C(M)$ in $\mathcal{L}(\mathcal{H}_E)$. The same program can be carried out in the noncommutative context also. Let $\mathcal{A}$ be a C^*-algebra and E a Hilbert $\mathcal{A}$-module with its $\mathcal{A}$ valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. Let τ be a state on $\mathcal{A}$. Consider the inner product on E given by $\langle e_1, e_2 \rangle = \tau(\langle e_1, e_2 \rangle_{\mathcal{A}})$. If we denote by $\mathcal{H}_E$ the Hilbert space completion of E, then we get a natural representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H}_E)$. Now in the context of $C(SU_q(2))$, let $p = |e_0\rangle\langle e_0| \otimes I \in C(SU_q(2))$. Then it is easy to verify that $\mathcal{H}_E = l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$ for $E = C(SU_q(2))p$ with its natural left Hilbert $C(SU_q(2))$-module structure. Moreover, the associated representation is nothing but the representation of $C(SU_q(2))$ described above.

2 Generators of K-homology

One way to have some idea about spectral triples is to compute the generators of K-homology. We will write $\mathcal{A}$ for the C^*-algebra $C(SU_q(2))$ and $\mathcal{A}_f$ for the *-subalgebra of $C(SU_q(2))$ generated by the two elements α and β. Restriction of π to $\mathcal{A}_f$ gives a representation of $\mathcal{A}_f$ on $\mathcal{H}$, which we denote by the same symbol π. The short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} C(S^1) \longrightarrow 0 \tag{2.1}$$

gives rise to the folowing six-term exact sequence

$$\begin{array}{ccccc} K^0(C(S^1)) & \xrightarrow{\sigma^0} & K^0(\mathcal{A}) & \xrightarrow{i^0} & K^0(\mathcal{K}\otimes C(S^1)) \\ \uparrow & & & & \downarrow \\ K^1(\mathcal{K}\otimes C(S^1)) & \xleftarrow{i^1} & K^1(\mathcal{A}) & \xleftarrow{\sigma^1} & K^1(C(S^1)). \end{array}$$

It is known that $K_0(\mathcal{A}) = \mathbb{Z} = K_1(\mathcal{A})$. Since these are free abelian groups, It follows from the results of Rosenberg-Schochet ([6]) that $K^0(\mathcal{A}) = \mathbb{Z} = K^1(\mathcal{A})$. Therefore the six term sequence above becomes

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\sigma^0} & \mathbb{Z} & \xrightarrow{i^0} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ K^1(\mathcal{K}\otimes C(S^1)) & \xleftarrow{i^1} & \mathbb{Z} & \xleftarrow{\sigma^1} & K^1(C(S^1)) \end{array}$$

Lemma 2.1 *i^1 and σ^0 are isomorphisms while i^0 and σ^1 are zero morphisms.*

Proof: We know that $K^1(\mathcal{K}\otimes C(S^1)) \cong K^1(C(S^1)) \cong \mathbb{Z}$. Therefore by the exactness of the diagram above it is enough to show that i^1 is onto. For that observe $\mathcal{H} = L_2(\mathbb{N}) \otimes L_2(\mathbb{Z})$ and $F = I \otimes S$, where S denotes the operator

$$S : e_k \mapsto \begin{cases} e_k & \text{if } k \geq 0, \\ -e_k & \text{if } k < 0, \end{cases}$$

is an odd Fredholm module on $\mathcal{A}$ and hence on $i(\mathcal{K}\otimes C(S^1)) \subseteq \mathcal{A}$. Moreover this Fredholm module is a generator of $K^1(\mathcal{K}\otimes C(S^1))$ implying surjectivity of i^1. □

Remark 2.2 Proof of the above lemma also shows that $(\mathcal{H}, F)$ is a generating Fredholm module for $K^1(\mathcal{A})$.

3 Spectral triples

In this section we construct spectral triples with nontrivial Chern character. For $p \in [0, \infty)$, let D_p be the operator $N^p \otimes S + I \otimes N$ on $\mathcal{H} = L_2(\mathbb{N}) \otimes L_2(\mathbb{Z})$ with S as defined above.

Proposition 3.1 *Let $\mu_n(T)$ denote the n^{th} largest singular value of an opeartor T. Then*

$$\mu_n(|D_p|^{-1-\frac{1}{p}}) \sim \frac{1}{n}.$$

Proof: Check that the action of $|D_p|$ on $\mathcal{H}$ is given by $e_i \otimes e_j \mapsto (i^p + |j|)e_i \otimes e_j$. If we denote by λ_r the number of elements in $\{(i,j) : i, j \in \mathbb{N}, i^p + j \leq r\}$, then a simple calculation tells us that $\frac{\lambda_r}{r^{1+\frac{1}{p}}} \to \frac{2p}{1+p}$ as $r \to \infty$. It follows from this that the $\mathrm{n^{th}}$ eigenvalue of $|D_p|$ is of the order of $n^{\frac{1}{1+1/p}}$, which gives us the required result. □

Lemma 3.2 *Define a functional ϕ on $\mathcal{A}_f$ by*

$$\phi(a) := \lim_{t \to 0} t^{1+1/p}\, tr(a \exp(-tD_p^2)).$$

Then $\phi(\alpha_i \beta^j \beta^{*k}) = \delta_{i0}\delta_{j0}\delta_{k0}$. In particular, ϕ does not depend on p.

Proof: Observe that $\exp(-tD_p^2)e_r \otimes e_s = \exp(-t(r^p + |s|)^2)e_r \otimes e_s$, and

$$\alpha_i \beta^j \beta^{*k} e_r \otimes e_s \begin{cases} = q^{2rk} e_r \otimes e_s & \text{if } i = 0,\ j = k \\ \in \mathbb{C}e_{r-i} \otimes e_{s+j-k} & \text{otherwise.} \end{cases}$$

Hence we have

$$\operatorname{tr}(\alpha_i \beta^j \beta^{*k} \exp(-tD_p^2)) = \begin{cases} \sum_{r=0}^{\infty} \sum_{s\in\mathbb{Z}} q^{2rk} \exp(-t(r^p + |s|)^2) & \text{if } i = 0,\ j = k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore it follows that $\phi(\alpha_i \beta^j \beta^{*k}) = 0$ for $i \neq 0$. Now note that

$$\sum_{r=0}^{\infty} \sum_{s\in\mathbb{Z}} q^{2rk} \exp(-t(r^p + |s|)^2) = 2 \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} q^{2rk} \exp(-t(r^p + |s|)^2) + \sum_{r=0}^{\infty} q^{2rk} \exp(-tr^{2p}),$$

and

$$\begin{aligned} \sum_{s=1}^{\infty} q^{2rk} \exp(-t(r^p + |s|)^2) &< \int_0^{\infty} \exp(-t(r^p + x)^2)\, dx \\ &= \frac{1}{\sqrt{2t}} \int_{r^p\sqrt{2t}}^{\infty} \exp(-\frac{1}{2}y^2)\, dy \\ &< \frac{1}{\sqrt{2t}}. \end{aligned}$$

Hence for $i = 0$ and $j = k \neq 0$, $\phi(\alpha_i \beta^j \beta^{*k}) = \lim_{t\to 0} t^{1+1/p} \operatorname{tr}(\alpha_i \beta^j \beta^{*k} \exp(-tD_p^2)) = 0$. Finally, from the previous lemma, it follows that $\phi(I) = \operatorname{tr}_\omega(|D_p|^{-1-1/p}) = 1$. $\square$

Proposition 3.3 *For each $p \in (0, 1]$, $\mathcal{S}_p := (\mathcal{A}_f, \mathcal{H}, D_p)$ defines an odd spectral triple.*

Proof: Self-adjointness of D_p is trivial, and it follows from proposition 3.1 that D_p has compact resolvent. Let $\mathcal{H}_0 = \operatorname{span}\{e_i \otimes e_j : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and is invariant under the actions of D_p and the elements of $\mathcal{A}_f$. In view of this and the self-adjointness of D_p, it is enough to show that $[D_p, \alpha]$ and $[D_p, \beta]$ are bounded. Straightforward calculation now gives

$$\begin{aligned} [D_p, \alpha] &= \alpha(((N - I)^p - N^p) \otimes S), \\ [D_p, \beta] &= q^N N^p \otimes [S, \ell^*] + \beta. \end{aligned} \tag{3.2}$$

Therefore $\mathcal{S}_p$ is a spectral triple. $\square$

Remark 3.4 The circle group S^1 has an action on $\mathcal{A}$ given by $\phi_z : \alpha \mapsto z\alpha, \beta \mapsto \beta$, where $z \in S^1$. D_p is equivariant with respect to this action. Equivariance follows from the fact that D_p commutes with the generator of the action $N \otimes I + I \otimes N$.

Theorem 3.5 *The spectral triple $(\mathcal{A}_f, \mathcal{H}, D_p)$ has nontrivial Chern character.*

Proof: For this one only has to note that the operator F constructed in the proof of 2.1 is nothing but $sign(D_p)$. We give an explicit description of the pairing with $K_1(\mathcal{A})$. Let $E = \frac{1+F}{2} = I(N \geq 0)$ and $u = I_{\{1\}}(\beta^*\beta)(\beta-1)+1$. u gives an element $[u] \in K_1(\mathcal{A})$. By proposition 2 (page 289 of [3]), EuE is a Fredholm operator and $\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = Index(EuE)$. It is easily seen that the last quantity is -1. Since $K_1(\mathcal{A}) = \mathbb{Z}$, this shows $[u]$ generates $K_1(\mathcal{A})$ and describes the pairing with $K_1(\mathcal{A})$ completely. $\square$

Note that the following corollary is immediate from the proof of this theorem.

Corollary 3.6 *Let $u = I_{\{1\}}(\beta^*\beta)(\beta - 1) + 1$. Then $[u]$ generates $K_1(\mathcal{A})$.*

4 Modified Connes-de Rham complex

Let $\Omega^\bullet(\mathcal{A}_f) = \oplus_n \Omega^n(\mathcal{A}_f)$ be the universal graded differential algebra over $\mathcal{A}_f$, i.e. $\Omega^n(\mathcal{A}_f) = \text{span}\{a_0(\delta a_1)\ldots(\delta a_n) : a_i \in \mathcal{A}_f, \delta(ab) = a(\delta b) + (\delta a)b\}$. The universal differential algebra is not very interesting from the cohomological point of view. Interesting cohomologies are obtained from the representations of the algebra. For the spectral triple $(\mathcal{A}_f, \mathcal{H}, D_p)$, one has the standard Connes-de Rham complex of noncommutative exterior forms $\Omega_D^\bullet(\mathcal{A}_f)$, given by

$$\Omega_D^\bullet(\mathcal{A}) := \Omega^\bullet(\mathcal{A})/(K + \delta K) \cong \pi(\Omega^\bullet(\mathcal{A}))/\pi(\delta K).$$

where $K = \oplus_{p \geq 0} K_p$ is the two sided ideal of $\Omega^\bullet(\mathcal{A})$ given by $K_p = \{\omega \in \Omega^p(\mathcal{A}) : \pi(\omega) = 0\}$. But often, the explicit computation of this complex is rather difficult. What we will do is the following. We will compute the complex obtained from the representation $\theta \circ \pi : \Omega^\bullet(\mathcal{A}) \to \mathcal{Q}(\mathcal{H})$ where $\theta : \mathcal{L}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the projection onto the Calkin algebra. More specifically, let $\widetilde{d} : \mathcal{A}_f \to \mathcal{L}(\mathcal{H})$ be given by $\widetilde{d}a = [D_p, \pi(a)]$. Define $\pi_n : \Omega^n(\mathcal{A}_f) \to \mathcal{L}(\mathcal{H})$ by $\pi_n(a_0(\delta a_1)\ldots(\delta a_n)) = \pi(a_0)(\widetilde{d}a_1)\ldots(\widetilde{d}a_n)$. Define $d = \theta \circ \widetilde{d}$, $\psi_n = \theta \circ \pi_n$, and $\psi := \oplus \psi_n : \oplus \Omega^n \to \mathcal{Q}(\mathcal{H})$. Let $J_n = \ker \psi_n$. Define $\Omega_d^n(\mathcal{A}_f) = \Omega^n(\mathcal{A}_f)/(J_n + \delta J_{n-1})$. Then $\Omega_d^n(\mathcal{A}_f) = \psi(\Omega^n(\mathcal{A}_f))/\psi(\delta J_n)$. We will compute these cohomologies $\Omega_d^n(\mathcal{A}_f)$. Before entering the computations, it should be stressed here that by computing these rather than the standard complex, we do not lose much. Because, first, since for a compact operator K one has $\text{Tr}_\omega(K|D_p|^{-1-1/p}) = 0$, proposition 5, page 550, [3] concerning the Yang-Mills functional holds in our present case. Second, in the context of the canonical spectral triple associated with a compact Riemannian spin manifold this prescription also gives back the exterior complex.

4.1 The case $p = 1$

We will write D for D_1 throughout this subsection.

First, we need the following lemma which will be very useful for the computations.

Lemma 4.1 *Assume $a, b \in \mathcal{A}_f$ and $c \in \mathcal{K}(\mathcal{H})$. If $a(I \otimes S) + b = c$, then $a = b = 0$.*

Proof: For a functional ρ on $\mathcal{L}(L_2(\mathbb{N}))$, and $T \in \mathcal{L}(\mathcal{H})$, denote by a_ρ the operator $(\rho \otimes \mathrm{id})T$. Now observe that for any $a \in \mathcal{A}_f$ and any functional ρ,

$$a_\rho \ell = \ell a_\rho. \tag{4.3}$$

Write $P = \frac{1}{2}(I+S)$. It is easy to see that the given condition implies that $(b_\rho - a_\rho) + 2a_\rho P = c_\rho$, which in turn implies that

$$\begin{aligned} (b_\rho - a_\rho)e_i &= c_\rho e_i \quad \forall i < 0, & (4.4)\\ (b_\rho + a_\rho)e_i &= c_\rho e_i \quad \forall i \geq 0. & (4.5) \end{aligned}$$

Now from (4.3) and (4.4), it follows that for any $i, j \in \mathbb{Z}$ and $j < 0$,

$$\begin{aligned} \|(b_\rho - a_\rho)e_i\| &= \|(b_\rho - a_\rho)\ell^{j-i}e_j\| \\ &= \|\ell^{j-i}(b_\rho - a_\rho)e_j\| \\ &= \|(b_\rho - a_\rho)e_j\| \\ &= \|c_\rho e_j\|. \end{aligned}$$

Since c is compact, $\lim_{j\to-\infty} \|c_\rho e_j\| = 0$. Hence $(b_\rho - a_\rho)e_i = 0$ for all i. In other words, $(b_\rho - a_\rho) = 0$. Since this is true for any ρ, we get $a = b$. Using this equality, together with equations (4.3) and (4.5), a similar reasoning yields $a = 0$. □

Lemma 4.2 *Let $\mathcal{I}_\beta$ denote the ideal in $\mathcal{A}_f$ generated by β and β^*. Then for $n \geq 1$, we have*

$$\psi(\Omega^n(\mathcal{A}_f)) = (I \otimes S)^n \mathcal{A}_f + (I \otimes S)^{n+1}\mathcal{I}_\beta. \tag{4.6}$$

Proof: Let us first prove the equality for $n = 1$. Let $Z_k = q^{N+k}(N+k)$, $B_{jk} = \sum_{i=j-k+1}^{j} |e_{i-1}\rangle\langle e_i|$, and

$$C_j = \begin{cases} \sum_{i=0}^{j-1} |e_i\rangle\langle e_{i-1}| & \text{if } j \geq 1, \\ 0 & \text{if } j = 0. \end{cases}$$

It follows from (3.2) that

$$\begin{aligned} [D, \alpha_i\beta^j\beta^{*k}] &= -i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k} + 2(Z_i \otimes C_j)\alpha_i\beta^{j-1}\beta^{*k} \\ &\qquad -2(Z_i \otimes B_{jk})\alpha_i\beta^j\beta^{*k-1}. \end{aligned} \tag{4.7}$$

Hence $d(\alpha_i\beta^j\beta^{*k}) = -i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k}$. Thus for any $a \in \mathcal{A}_f$,

$$da = (I \otimes S)b + c, \qquad \text{where } b \in \mathcal{A}_f, \quad c \in \mathcal{I}_\beta. \tag{4.8}$$

Note that for any $a' \in \mathcal{A}_f$, $\psi(a')(I \otimes S) = (I \otimes S)\psi(a')$ in $\mathcal{Q}(\mathcal{H})$. Hence $\psi(a'(\delta a))$ is again of the form $(I \otimes S)b + c$, where $b \in \mathcal{A}_f$, $c \in \mathcal{I}_\beta$, i.e. is a member of $(I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. Thus $\psi(\Omega^1(\mathcal{A}_f)) \subseteq (I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. For the reverse inclusion, observe that $(I \otimes S) = (1-q^2)^{-1}((d\alpha)\alpha^* + q^2(d\alpha^*)\alpha)$, $\beta = d\beta$ and $\beta^* = -d\beta^*$.

The inductive step follows easily from (4.8). □

Lemma 4.3 *$J_0 = \{0\}$, and for $n \geq 1$, we have*

$$\psi(\delta J_n) = (I \otimes S)^{n+1}\mathcal{A}_f + (I \otimes S)^{n+2}\mathcal{I}_\beta. \tag{4.9}$$

Proof: By lemma 4.1, $\psi : \mathcal{A}_f \to \mathcal{Q}(\mathcal{H})$ is faithful. Hence it follows that $J_0 = \{0\}$.

We will prove here (4.9) by induction. From lemma 4.2, we have $\psi(\delta J_1) \subseteq \psi(\Omega^2(\mathcal{A}_f)) = \mathcal{A}_f + (I \otimes S)\mathcal{I}_\beta$. Let us show that I, $(I \otimes S)\beta$ and $(I \otimes S)\beta^*$ are all members of $\psi(\delta J_1)$.

Choose $\omega \in \Omega^1(\mathcal{A}_f)$ such that $\psi(\omega) = (I \otimes S)$. Let $\omega_k = k\alpha_k\omega - \delta(\alpha_k), k = \pm 1$. Then it follows from (3.2) that $\psi(\omega_k) = k\alpha_k(I \otimes S) - k\alpha_k(I \otimes S) = 0$, so that $\omega_k \in J_1$. $\psi(\delta\omega_k) = \psi(k(\delta\alpha_k)\omega) = k^2\alpha_k = \alpha_k \in \psi(\delta J_1)$, i.e. both α and α^* are in $\psi(\delta J_1)$. It follows from this that $I \in \psi(\delta J_1)$.

Next we show that $(I \otimes S)\beta \in \psi(\delta J_1)$. Take $\omega = \frac{1}{2}(\alpha(\delta\beta) - \delta(\alpha\beta) + q\beta(\delta\alpha))$. Then $\psi(\omega) = 0$ and $\psi(\delta\omega) = (I \otimes S)\alpha\beta$. So $(I \otimes S)\alpha\beta \in \psi(\delta J_1)$. Similarly taking $\omega = \frac{1}{2}(\alpha^*(\delta\beta) - \delta(\alpha^*\beta) + q^{-1}\beta(\delta\alpha^*))$, it follows that $(I \otimes S)\alpha^*\beta \in \psi(\delta J_1)$. These two together imply $(I \otimes S)\beta \in \psi(\delta J_1)$.

A similar argument shows that $(I \otimes S)\beta^*$ is also in $\psi(\delta J_1)$. Thus $\mathcal{A}_f + (I \otimes S)\mathcal{I}_\beta = \psi(\delta J_1)$.

For the inductive step, notice that $\psi(\delta J_n) \subseteq \psi(\Omega^{n+1}(\mathcal{A}_f)) = (I \otimes S)^{n+1}\mathcal{A}_f + (I \otimes S)^{n+2}\mathcal{I}_\beta$. We will show that the following are all elements of $\psi(\delta J_n)$:

$$\begin{array}{lll} (I \otimes S)^{n+1}\alpha, & (I \otimes S)^{n+2}\alpha\beta, & (I \otimes S)^{n+2}\alpha\beta^*, \\ (I \otimes S)^{n+1}\alpha^*, & (I \otimes S)^{n+2}\alpha^*\beta, & (I \otimes S)^{n+2}\alpha^*\beta^*. \end{array}$$

From the right $\mathcal{A}_f$-module structure of $\psi(\delta J_n)$, it will then follow that $(I \otimes S)^{n+1}$, $(I \otimes S)^{n+2}\beta$ and $(I \otimes S)^{n+2}\beta^*$ are in $\psi(\delta J_n)$, giving us the other inclusion.

Choose $\omega \in J_{n-1}$ such that $\psi(\delta\omega) = (I \otimes S)^n$. Take $\omega_k = k\omega(\delta\alpha_k)$, $k = \pm 1$. Then $\omega_k \in J_n$ and $\psi(\delta\omega_k) = (I \otimes S)^{n+1}\alpha_k$. Similarly choosing ω such that $\psi(\delta\omega) = (I \otimes S)^{n+1}\beta$ and ω_k as before, we get $\omega_k \in J_n$ and $\psi(\delta\omega_k) = q^{-k}(I \otimes S)^{n+2}\alpha\beta$. Finally, take ω such that $\psi(\delta\omega) = (I \otimes S)^{n+1}\beta^*$ and ω_k as before to show that $(I \otimes S)^{n+2}\alpha_k\beta^* \in \psi(\delta J_n)$. □

Proposition 4.4

$$\Omega_d^n(\mathcal{A}_f) = \begin{cases} \mathcal{A}_f \oplus \mathcal{I}_\beta & \text{if } n = 1, \\ \{0\} & \text{if } n \geq 2. \end{cases}$$

Proof: Proof follows from lemmas 4.2 and 4.3. □

4.2 The case $0 < p < 1$

Let us first introduce a few notations. Let X_{rs} denote the operator $(N + r)^p - (N + s)^p$, Z_r stand for $q^{N+r}(N + r)^p$ and let B_{rs} and C_r be as in the earlier subsection. We have, then,

$$\begin{aligned} [D_p, \alpha_i\beta^j\beta^{*k}] &= (I \otimes S)(X_{0i} \otimes I)\alpha_i\beta^j\beta^{*k} + (j - k)\alpha_i\beta^j\beta^{*k} + 2(Z_i \otimes C_j)\alpha_i\beta^{j-1}\beta^{*k} \\ &\qquad -2(Z_i \otimes B_{jk})\alpha_i\beta^j\beta^{*k-1}. \\ &= (I \otimes S)(X_{0i} \otimes I)\alpha_i\beta^j\beta^{*k} + (j - k)\alpha_i\beta^j\beta^{*k} + \text{compact} \end{aligned} \tag{4.10}$$

and hence,

$$\begin{aligned}[D_p, (X_{r_1s_1}\dots X_{r_ks_k}\otimes I)\alpha_i\beta^j\beta^{*k}] &= (I\otimes S)(X_{r_1s_1}\dots X_{r_ks_k}X_{0i}\otimes I)\alpha_i\beta^j\beta^{*k}\\ &\quad + (j-k)(X_{r_1s_1}\dots X_{r_ks_k}\otimes I)\alpha_i\beta^j\beta^{*k} + \text{compact} \qquad (4.11)\end{aligned}$$

We will work with the algebra $\widetilde{\mathcal{A}}_f$ generated by $\{X_{rs}\otimes I) : r,s\in\mathbb{Z}\}$ and the elements of $\mathcal{A}_f$. Note that $\widetilde{\mathcal{A}}_f$ is nothing but the span of $\{(X_{0s_1}\dots X_{0s_n}\otimes I)\alpha_i\beta^j\beta^{*k}\}$. Now first of all observe that in the proof of lemma 4.1, the only property of $\mathcal{A}_f$ that has been used is that

$$a(I\otimes\ell) = (I\otimes\ell)a \qquad (4.12)$$

for all $a\in\mathcal{A}_f$, so that one has equation (4.3). Since (4.12) is satisfied by elements of $\widetilde{\mathcal{A}}_f$ also, it follows that lemma 4.1 remains valid even when $\mathcal{A}_f$ is replaced by the bigger algebra $\widetilde{\mathcal{A}}_f$.

Lemma 4.5 *Let $\widetilde{\mathcal{I}}_\beta$ denote the ideal in $\widetilde{\mathcal{A}}_f$ generated by β and β^*. Then for $n\geq 1$, we have*

$$\psi(\Omega^n(\widetilde{\mathcal{A}}_f)) = (I\otimes S)^n\widetilde{\mathcal{A}}_f + (I\otimes S)^{n+1}\widetilde{\mathcal{I}}_\beta. \qquad (4.13)$$

Lemma 4.6 *Let $\widetilde{J}_n$ be the kernel of ψ restricted to $\Omega^n(\widetilde{\mathcal{A}}_f)$. Then $\widetilde{J}_0=\{0\}$, and for $n\geq 1$, we have*

$$\psi(\delta\widetilde{J}_n) = (I\otimes S)^{n+1}\widetilde{\mathcal{A}}_f + (I\otimes S)^{n+2}\widetilde{\mathcal{I}}_\beta. \qquad (4.14)$$

Proof: Arguments used for proving lemma 4.3 goes through. □

Proposition 4.7

$$\Omega^n_d(\widetilde{\mathcal{A}}_f) = \begin{cases}\widetilde{\mathcal{A}}_f\oplus\widetilde{\mathcal{I}}_\beta & \text{if } n=1,\\ \{0\} & \text{if } n\geq 2.\end{cases}$$

Proof: Lemma 4.5, and 4.6 yields this as in proposition 4.4. □

5 L^2-complex of Frohlich et. al.

In this section we will compute the complex of square integrable forms for the spectral triple corresponding to $p=1$. For that we begin with similar computations for the spectral triple $(\mathbb{C}[z,z^{-1}],\mathcal{H}_0=L_2(\mathbb{Z}),D_0=N)$ associated with the algebra $\mathbb{C}[z,z^{-1}]$. Here we consider the embedding $\pi_0:\mathbb{C}[z,z^{-1}]\to\mathcal{L}(\mathcal{H})$ that maps z to ℓ.

Lemma 5.1 (i) $\widetilde{\Omega}^n_{D_0}(\mathbb{C}[z,z^{-1}])=0$, *for $n\geq 2$,*
(ii) $\widetilde{\Omega}^1_{D_0}(\mathbb{C}[z,z^{-1}])=\mathbb{C}[z,z^{-1}]$.

Proof: (i) Let $\omega=\sum\alpha_{n_0,\cdots,n_k}z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k}\in\Omega^k(\mathbb{C}[z,z^{-1}])$, where the sum is a finite one and δ is the universal differential. Then it is easily verified that

$$(\omega,\omega)_{D_0} = \int\big(\sum n_1\cdots n_k\alpha_{n_0,\cdots,n_k}z^{\sum_0^k n_j}\big)^*\big(\sum n_1\cdots n_k\alpha_{n_0,\cdots,n_k}z^{\sum_0^k n_j}\big)dz,$$

where dz is the Lebesgue measure on the circle. Therefore,

$$\begin{aligned} K_k(\mathbb{C}[z,z^{-1}]) &= \{\omega \in \Omega^k(\mathbb{C}[z,z^{-1}]) : (\omega,\omega)_{D_0} = 0\} \\ &= \{\sum \alpha_{n_0,\cdots,n_k} z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} : \sum_{n_0+\cdots+n_k=r} n_1\cdots n_k \alpha_{n_0,\cdots,n_k} = 0, \forall r\}. \end{aligned}$$

Consequently we have,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} - n_1\cdots n_k z^{\sum_0^k n_i - k}\delta z\cdots\delta z \in K_k(\mathbb{C}[z,z^{-1}]), \tag{5.15}$$

$$\delta z^r \delta z\cdots\delta z - r z^r \delta z\cdots\delta z \in K_k(\mathbb{C}[z,z^{-1}]), \tag{5.16}$$

$$z^r\delta z\cdots\delta z - \frac{1}{r+1}\delta z^{r+1}\delta z\cdots\delta z \in K_{k-1}(\mathbb{C}[z,z^{-1}]). \tag{5.17}$$

From (5.17) we get $\delta z^r\delta z\cdots\delta z \in \delta K_{k-1}(\mathbb{C}[z,z^{-1}])$. Combining this with (5.15) and (5.16) we get,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in K_k(\mathbb{C}[z,z^{-1}]) + \delta K_{k-1}(\mathbb{C}[z,z^{-1}]) \text{ for large } n_0.$$

Since $K_k(\mathbb{C}[z,z^{-1}]) + \delta K_{k-1}(\mathbb{C}[z,z^{-1}])$ is a bimodule we have

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in K_k(\mathbb{C}[z,z^{-1}]) + \delta K_{k-1}(\mathbb{C}[z,z^{-1}]) \quad \forall\, n_0,\cdots,n_k.$$

This proves (i).
(ii) It suffices to note that

$$z^{n_0}\delta z^{n_1} - n_1 z^{n_0+n_1-1}\delta z \in K_1(\mathbb{C}[z,z^{-1}]).$$

The induced $d : \widetilde{\Omega}^0_{D_0}(\mathbb{C}[z,z^{-1}]) \to \mathbb{C}[z,z^{-1}]$ is given by $d(z^n) = nz^n$. □

Now we are in a position to compute the complex of square integrable forms for $\mathcal{A}_f$ for the spectral triple associated with $p = 1$.

Theorem 5.2 (i) $\widetilde{\Omega}^n_D(\mathcal{A}_f) = 0$ *for* $n \geq 2$.
(ii) $\widetilde{\Omega}^n_D(\mathcal{A}_f) = \mathbb{C}[z,z^{-1}]$ *for* $n = 0, 1$ *here equality is as an* $\mathcal{A}_f$ *bimodule.*

Proof: Note that the homomorphism σ in (2.1) induces a surjective homomorphism denoted by the same symbol from $\mathcal{A}_f$ to $\mathbb{C}[z,z^{-1}]$. We have the following short exact sequence

$$0 \longrightarrow \mathcal{I}_\beta \longrightarrow \mathcal{A}_f \xrightarrow{\sigma} \mathbb{C}[z,z^{-1}] \longrightarrow 0,$$

Let $\sigma_k : \Omega^k(\mathcal{A}_f) \to \Omega(\mathbb{C}[z,z^{-1}])$ be the induced surjective map. One easily verifies that $(\omega,\omega)_D = (\sigma_k(\omega),\sigma_k(\omega))_{D_0}$. Therefore,

$$K_k(\mathcal{A}_f) = \{\omega \in \Omega^k(\mathcal{A}_f) : (\omega,\omega)_D = 0\} = \sigma_k^{-1}(K_k(\mathbb{C}[z,z^{-1}])).$$

We have the following commutative diagram

$$\begin{array}{ccccccc}
K_0 = I_\beta & \longrightarrow & \mathcal{A}_f & \xrightarrow{\sigma} & \mathbb{C}[z, z^{-1}] & \longrightarrow & \pi_0(\mathbb{C}[z, z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
K_1(\mathcal{A}_f) & \longrightarrow & \Omega^1(\mathcal{A}_f) & \xrightarrow{\sigma_1} & \Omega^1(\mathbb{C}[z, z^{-1}]) & \longrightarrow & \tilde{\Omega}^1_{D_0}(\mathbb{C}[z, z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
K_2(\mathcal{A}_f) & \longrightarrow & \Omega^2(\mathcal{A}_f) & \xrightarrow{\sigma_2} & \Omega^2(\mathbb{C}[z, z^{-1}]) & \longrightarrow & \tilde{\Omega}^2_{D_0}(\mathbb{C}[z, z^{-1}]) \\
 & \cdots & & \cdots & & \cdots & \cdots \\
K_n(\mathcal{A}_f) & \longrightarrow & \Omega^n(\mathcal{A}_f) & \xrightarrow{\sigma_n} & \Omega^n(\mathbb{C}[z, z^{-1}]) & \longrightarrow & \tilde{\Omega}^n_{D_0}(\mathbb{C}[z, z^{-1}]).
\end{array}$$

This along with the previous lemma proves the theorem. We will only illustrate (i).

Let $\omega_n \in \Omega^n(\mathcal{A}_f)$, then by the previous lemma $\sigma_n(\omega_n) = \omega_{1,n} + \delta\omega_{2,n-1}$ where $\omega_{1,n} \in K_n(\mathbb{C}[z, z^{-1}]), \omega_{2,n-1} \in K_{n-1}(\mathbb{C}[z, z^{-1}])$. Let $\omega'_{1,n} = \sigma_n^{-1}(\omega_{1,n}), \omega'_{2,n-1} = \sigma_{n-1}^{-1}(\omega_{2,n-1})$, then $\sigma_n(\omega_n - \omega'_{1,n} - \delta\omega'_{2,n-1}) = 0$ implying $\omega_n \in K_n + \delta K_{n-1}$. □

6 Computations for the quantum sphere

In this section we will do similar computations for quantum spheres. At times we will be sketchy because some of the arguments are very similar to the earlier one. Quantum sphere was introduced by Podles in [5]. This is the universal C*-algebra, denoted by $C(S^2_{qc})$, generated by two elements A and B subject to the following relations:

$$A^* = A, \qquad B^*B = A - A^2 + cI,$$
$$BA = q^2AB, \qquad BB^* = q^2A - q^4 + cI.$$

Here the deformation parameters q and c satisfy $|q| < 1, c > 0$. For later purpose we also note down two irreducible representations whose direct sum is faithful. Let $\mathcal{H}_+ = l^2(\mathbb{N}), \mathcal{H}_- = \mathcal{H}_+$. Define $\pi_\pm(A), \pi_\pm(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$ by

$$\begin{array}{lcl}
\pi_\pm(A)(e_n) = \lambda_\pm q^{2n} e_n & \text{where} & \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{1/2} \\
\pi_\pm(B)(e_n) = c_\pm(n)^{1/2} e_{n-1} & \text{where} & c_\pm(n) = \lambda_\pm q^{2n} - (\lambda_\pm q^{2n})^2 + c.
\end{array}$$

Since $\pi = \pi_+ \oplus \pi_-$ is a faithful representation, an immediate corollary follows.

Theorem 6.1 (Sheu) (i) $C(S^2_{qc}) \cong C^*(\mathscr{T}) \oplus_\sigma C^*(\mathscr{T}) := \{(x, y) : \ x, y \in C^*(\mathscr{T}), \sigma(x) = \sigma(y)\}$ *where* $C^*(\mathscr{T})$ *is the Toeplitz algebra and* $\sigma : C^*(\mathscr{T}) \to C(S^1)$ *is the symbol homomorphism.*
(ii) *We have a short exact sequence*

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} C^*(\mathscr{T}) \longrightarrow 0 \qquad (6.18)$$

Proof: (i) An explicit isomorphism is given by $x \mapsto (\pi_+(x), \pi_-(x))$.
(ii) Define $\alpha((x, y)) = x$ then $\ker \alpha = \mathcal{K}$. □

Corollary 6.2 (i) $K_0(C(S^2_{qc})) = K^0(C(S^2_{qc})) = \mathbb{Z} \oplus \mathbb{Z}$.
(ii) $K_1(C(S^2_{qc})) = K^1(C(S^2_{qc})) = 0$.

Proof: The six term exact sequence associated with (6.18) along with the KK-equivalence of $\mathcal{K}, C^*(\mathscr{T})$ with $\mathbb{C}$ proves the result $\square$

Proposition 6.3 *Let $\mathcal{A}_{fin}$ be the *-subalgebra of $C(S^2_{qc})$ generated by A and B. Then*

$$\left(\mathcal{A}_{fin},\ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,\ D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix},\ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

is an even spectral triple.

Proof: We only have to show that $[D, a]$ is bounded for $a \in \mathcal{A}_{fin}$. For that it is enough to note that,
(i) $N\pi_\pm(A),\ \pi_\pm(A)N$ are bounded.
(ii) $n(c_\pm(n)^{1/2} - \sqrt{c})$ is bounded as n becomes large.
(iii) $[N, l] = l$. $\square$

Remark 6.4 This spectral triple has nontrivial Chern character. This can be seen as follows: let $P_0 = i(|e_0\rangle\langle e_0|) \in C(S^2_{qc})$, then applying proposition 4, page 296, [3],we get the index pairing $\langle [P_0], [(\mathcal{A}_{fin}, \mathcal{H}, D, \gamma)]\rangle = -1$, implying nontriviality of the spectral triple.

Now we will briefly indicate the computations of the complex $(\Omega^\bullet_d(\mathcal{A}_{fin}), d)$ introduced at the beginning of section 4.

Proposition 6.5 (i) $\Omega^n_d(\mathcal{A}_{fin}) = 0$ *for* $n \geq 2$.
(ii) $\Omega^1_d(\mathcal{A}_{fin}) = \mathbb{C}[z, z^{-1}]$, *here also equality is as an $\mathcal{A}_{fin}$ bimodule.*

Proof: Let π be the associated representation of $\Omega^\bullet(\mathcal{A}_{fin})$ in $\mathcal{L}(\mathcal{H})$. Then straightforward verification gives (i) $[D, A]$ is compact, (ii) $[D, B] = l \otimes \kappa +$ compact, and (iii) $[D, B^*] = -l^* \otimes \kappa +$ compact, where $\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, modulo compacts

$$\begin{aligned} \pi(\Omega^{2k+1}(\mathcal{A}_{fin})) &= C^*_{fin}(\mathscr{T}) \otimes \kappa \\ \pi(\Omega^{2k}(\mathcal{A}_{fin})) &= C^*_{fin}(\mathscr{T}) \otimes I_2, \end{aligned}$$

where $C^*_{fin}(\mathscr{T})$ is the *-algebra generated by $\mathscr{T}$. Now for (i), note that

$$\omega_n = B\delta B^* \underbrace{\delta B \cdots \delta B}_{n-2\ times} + B^*\delta B \underbrace{\delta B \cdots \delta B}_{n-2\ times}$$

satisfies (a) $\pi(\omega_n)$ is compact and (b)$\pi(\delta\omega_n) = 2I$ is invertible, hence (i) follows.
For (ii), observe that if $a \in \mathcal{A}_{fin}$ and $\pi(a)$ is compact then Na and aN both compact. Hence, $\Omega^1_d(\mathcal{A}_{fin}) = \pi(\Omega^1(\mathcal{A}_{fin})) = \mathbb{C}[z, z^{-1}]$ because modulo compacts $\mathbb{C}[z, z^{-1}]$ is $C^*(\mathscr{T})$. $\square$

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