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Remark on Poincaré duality for $SU_q(2)$

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Abstract

Let $\mathcal{A}$ be the C^*-algebra associated with $SU_q(2)$, J be the modular conjugation coming from the Haar state and let D be the generic equivariant Dirac operator for $SU_q(2)$. We prove in this article that there is no element in $J\mathcal{A}J$, other than the scalars, that have bounded commutator with D. This shows in particular that $J\mathcal{A}J$ does not contain any Poincaré dual for $SU_q(2)$.

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1 Introduction

In noncommutative geometry, spaces are described by a triple $(\mathcal{A}, \mathcal{H}, D)$, called a spectral triple. In this spectral point of view, one requires D to be nontrivial in the sense that the associated Kasparov module should give a nontrivial element in K-homology. The motivation behind this formalism is, in the context of closed Riemannian spin manifolds, one intuitively thinks that the induced Kasparov module should correspond to the fundamental class via the Chern isomorphism. One characteristic feature of the fundamental class is the following: on taking intersection product with this class we get the Poincaré duality. When tranferred to K-theory/K-homology, this property gives the commuting square

$$\begin{array}{cccccc} \otimes[D] & : & K_*(C^\infty(M))\otimes\mathbb{Q} & \xrightarrow{\cong} & K^*(C^\infty(M))\otimes\mathbb{Q} \\ & & \downarrow & & \downarrow \\ [\cdot]\cap[M](\cdot)=\int_M \cdot\wedge\cdot & : & H^*(M,\mathbb{Q}) & \xrightarrow{\cong} & H_*(M,\mathbb{Q}) \end{array}$$

where the top row is given by the Kasparov product

$$KK(\mathbb{C}, C^\infty(M)) \times KK(C^\infty(M)\otimes C^\infty(M)^{op}, \mathbb{C}) \to KK(C^\infty(M)^{op}, \mathbb{C}) \cong KK(C^\infty(M), \mathbb{C}).$$

Looking at things in this way helps in extending the notion of Poincaré duality in NCG. See [3] for a detailed formulation of Poincaré duality in NCG, and [6] for an interesting application. It says in particular that there should be a subalgebra $\mathcal{B}$ of the commutant $\mathcal{A}'$ with the same K-theory as that of $\mathcal{A}$ such that $(\mathcal{B}, \mathcal{H}, D)$ is also a spectral triple and the cup product with D gives an isomorphism between $K_*(\mathcal{A})$ and $(K_*(\mathcal{B}))^*$.

In what follows, we will be concerned with Poincaré duality for the quantum $SU(2)$ group, the spectral triple under consideration being the canonical equivariant spectral triple constructed by the authors in [1]. We will mostly follow notations used in that paper. In particular, $\mathcal{A}$ will denote the C^*-algebra of continuous functions on $SU_q(2)$, $\mathcal{A}_f$ will be the *-subalgebra of $\mathcal{A}$ generated by α and β. α_r and β_r will stand for α^r and β^r respectively if $r \geq 0$, and for $(\alpha^*)^{-r}$ and $(\beta^*)^{-r}$ if $r < 0$. Recall ([1]) that the operator $D : e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$, where

$$d(n,i) = \begin{cases} 2n+1 & \text{if } n \neq i, \\ -(2n+1) & \text{if } n = i, \end{cases} \tag{1.1}$$

gives rise to an equivariant spectral triple of dimension 3 and with nontrivial K-homology class. Let J be the modular conjugation associated with the Haar state of $SU_q(2)$. The main result in this article says that an element in $J\mathcal{A}J$ will have bounded commutator with D only if it is scalar.

2 Closer look at $L_2(h)$

Recall (cf. [1]) that $L_2(h)$ has a natural orthonormal basis $\{e_{ij}^{(n)} : n \in \frac{1}{2}\mathbb{N}, i, j = -n, -n+1, \ldots, n\}$, and the left multiplication operators in this basis are given by

$$\alpha : e_{ij}^{(n)} \mapsto a_+(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + a_-(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}, \tag{2.1}$$

$$\beta : e_{ij}^{(n)} \mapsto b_+(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + b_-(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}, \tag{2.2}$$

where

$$\begin{aligned}
a_+(n,i,j) &= \Big(q^{2(n+i)+2(n+j)+2}\frac{(1-q^{2n-2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\Big)^{\frac{1}{2}},\\
a_-(n,i,j) &= \Big(\frac{(1-q^{2n+2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\Big)^{\frac{1}{2}},\\
b_+(n,i,j) &= -\Big(q^{2(n+j)}\frac{(1-q^{2n-2j+2})(1-q^{2n+2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\Big)^{\frac{1}{2}},\\
b_-(n,i,j) &= \Big(q^{2(n+i)}\frac{(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n})(1-q^{4n+2})}\Big)^{\frac{1}{2}}.
\end{aligned}$$

We will also need the following operators on $L_2(h)$:

$$\widehat{\alpha} : e_{ij}^{(n)} \mapsto \hat{a}_+(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + \hat{a}_-(n,i,j)e_{i-\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}, \tag{2.3}$$

$$\widehat{\beta} : e_{ij}^{(n)} \mapsto \hat{b}_+(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n+\frac{1}{2})} + \hat{b}_-(n,i,j)e_{i+\frac{1}{2},j-\frac{1}{2}}^{(n-\frac{1}{2})}, \tag{2.4}$$

where

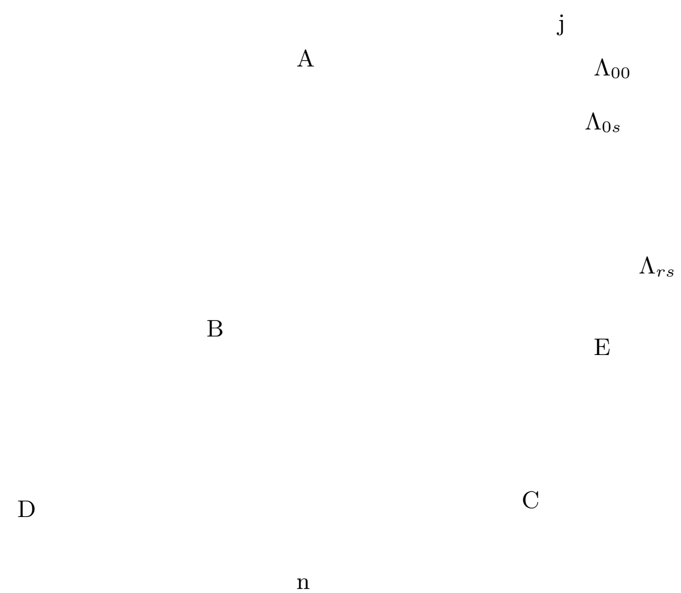
$$\begin{aligned}
\hat{a}_+(n,i,j) &= q^{2n+i+j+1},\\
\hat{a}_-(n,i,j) &= (1-q^{2n+2i})^{\frac{1}{2}}(1-q^{2n+2j})^{\frac{1}{2}},\\
\hat{b}_+(n,i,j) &= -q^{n+j}(1-q^{2n+2i+2})^{\frac{1}{2}},\\
\hat{b}_-(n,i,j) &= q^{n+i}(1-q^{2n+2j})^{\frac{1}{2}}.
\end{aligned}$$

It is easy to see that $\widehat{\alpha}$ and $\widehat{\beta}$ are compact perturbations of α and β respectively.

We will now decompose the space $L_2(h)$ as a direct sum of smaller subspaces, and study the behaviour of the above operators with respect to that decomposition. Note that the set $\Lambda = \{(n,i,j) : n \in \frac{1}{2}\mathbb{N}, i,j = -n,-n+1,\ldots,n\}$ parametrizes the canonical orthonormal basis for $L_2(h)$. For each $n \in \frac{1}{2}\mathbb{Z}$, denote by Λ_n the minimal subset of Λ containing the point $(|n|,-n,-n)$ and closed under the translations

$$(a,b,c) \mapsto (a+\frac{1}{2}, b+\frac{1}{2}, c-\frac{1}{2}), \quad (a,b,c) \mapsto (a+\frac{1}{2}, b-\frac{1}{2}, c+\frac{1}{2}).$$

For $n,k \in \frac{1}{2}\mathbb{Z}$, denote by Λ_{nk} the minimal subset of Λ_n that contains $(|n|+|k|,-n+k,-n-k)$ and is closed under the translation $(a,b,c) \mapsto (a+1,b,c)$. Thus all the Λ_{nk}'s are disjoint, $\Lambda = \cup_n \Lambda_n$, $\Lambda_n = \cup_{n,k} \Lambda_{nk}$. The following diagram will make these definitions clearer. Represent the lattice Λ as a pyramid. Then Λ_n are precisely the vertical cross-sections parallel to the plane ABC. In particular, Λ_0 is the cross-section given by the plane ABC. Λ_{nk} are vertical lines in the plane Λ_n.



Let us also note that the family of maps $\phi_n : \Lambda_n \to \Lambda_0$ given by

$$\phi_n(a,b,c) = (a-|n|, b+n, c+n) \tag{2.5}$$

give bijections between Λ_n and Λ_0 whose restriction to Λ_{nk} yield a bijection from Λ_{nk} to Λ_{0k}.

Let $\mathcal{H}_r$ denote the closed span of $\{e_{ij}^{(n)} : (n,i,j) \in \Lambda_r\}$, $\mathcal{H}_{rs}$ denote the closed span of $\{e_{ij}^{(n)} : (n,i,j) \in \Lambda_{rs}\}$, P_r denote the projection onto $\mathcal{H}_r$ and P_{rs} denote the projection onto $\mathcal{H}_{rs}$. Let U_n denote the unitary operator from $\mathcal{H}_n$ to $\mathcal{H}_0$ induced by the bijection ϕ_n.

Proposition 2.1 *Let A stand for α or $\widehat{\alpha}$, and B stand for β or $\widehat{\beta}$. Then one has*

$$P_{n+\frac{1}{2}}AP_n = AP_n, \qquad BP_n = P_nB, \tag{2.6}$$

$$P_{r+\frac{1}{2},s}AP_{rs} = AP_{rs}, \qquad P_{r,s+\frac{1}{2}}BP_{rs} = BP_{rs}, \tag{2.7}$$

$$P_{rs}B^*B = B^*BP_{rs}, \tag{2.8}$$

where $n, r, s \in \frac{1}{2}\mathbb{Z}$.

Moreover, for all $n \in \frac{1}{2}\mathbb{N}$, the operators $U_n\widehat{\alpha}U_n^*$ and $U_n\widehat{\beta}U_n^*$ are independent of n.

Lemma 2.2 *$\widehat{\alpha}$ and $\widehat{\beta}$ satisfy the following commutation relations:*

$$\widehat{\alpha}^*\widehat{\alpha} + \widehat{\beta}^*\widehat{\beta} = I,\ \widehat{\alpha}\widehat{\alpha}^* + q^2\widehat{\beta}\widehat{\beta}^* = I,\ \widehat{\alpha}\widehat{\beta} - q\widehat{\beta}\widehat{\alpha} = 0,\ \widehat{\alpha}\widehat{\beta}^* - q\widehat{\beta}^*\widehat{\alpha} = 0,\ \widehat{\beta}^*\widehat{\beta} = \widehat{\beta}\widehat{\beta}^*. \tag{2.9}$$

Following is a simple consequence of the above commutation relations.

Corollary 2.3 *Let $\gamma = \beta^*\beta$ and $\widehat{\gamma} = \widehat{\beta}^*\widehat{\beta}$. Then $\sigma(\widehat{\gamma}) = \{q^{2k} : k \in \mathbb{N}\} \cup \{0\} = \sigma(\gamma)$, and $\ker\widehat{\alpha}^* = \{0\} = \ker\alpha^*$.*

Note that the action of $\widehat{\gamma}$ on the basis vectors are given by

$$\widehat{\gamma}e_{ij}^{(n)} = c_+(n,i,j)e_{ij}^{(n+1)} + c_0(n,i,j)e_{ij}^{(n)} + c_-(n,i,j)e_{ij}^{(n-1)}, \tag{2.10}$$

where

$$\begin{aligned}
c_+(n,i,j) &= -q^{2n+i+j+1}(1-q^{2n+2i+2})^{\frac{1}{2}}(1-q^{2n+2j+2})^{\frac{1}{2}},\\
c_0(n,i,j) &= (q^{2n+2j}(1-q^{2n+2i}) + q^{2n+2i}(1-q^{2n+2j+2}),\\
c_-(n,i,j) &= -q^{2n+i+j-1}(1-q^{2n+2i})^{\frac{1}{2}}(1-q^{2n+2j})^{\frac{1}{2}}.
\end{aligned}$$

It is not too difficult to check, using (2.10) and (2.8) that $\ker\widehat{\gamma} = \{0\}$.

Lemma 2.4 *Let $r \in \frac{1}{2}\mathbb{N}$, and $s \in \frac{1}{2}\mathbb{Z}$. The restriction $P_{rs}\widehat{\gamma}P_{rs}$ of $\widehat{\gamma}$ to $\mathcal{H}_{rs}$ is compact for every r, s and $\sigma(P_{rs}\widehat{\gamma}P_{rs}) = \sigma(\widehat{\gamma})$, where $\sigma(\cdot)$ denotes the spectrum.*

Proof: Observe that for $r \in \frac{1}{2}\mathbb{N}$, $U_r(P_{rs}\widehat{\gamma}P_{rs})U_r^* = P_{0s}\widehat{\gamma}P_{0s}$. So it is enough to prove the statement for $r = 0$.

It is easy to see that $P_{0s}\widehat{\gamma}P_{0s}$ is compact by using equation (2.10). This, along with the second equality in (2.7) and the fact that $\widehat{\beta}$ and $\widehat{\gamma}$ commute, tells us that $\sigma(P_{0s}\widehat{\gamma}P_{0s})$ is independent of s, and consequently $\sigma(P_{0s}\widehat{\gamma}P_{0s}) = \sigma(P_0\widehat{\gamma}P_0)$ and in fact, this is same as the essential spectrum $\sigma_{ess}(P_0\widehat{\gamma}P_0)$.

Let us next show that $\sigma(P_0\widehat{\gamma}P_0) = \sigma(\widehat{\gamma})$. Let K be the operator on $\mathcal{H}_0$, given on the basis vectors $e_{i,-i}^{(n)}$ as follows:

$$Ke_{i,-i}^{(n)} = c_+(n,i,-i)e_{i,-i}^{(n+1)} + (q^{2n+2|i|} - q^{4n} - q^{4n+2})e_{i,-i}^{(n)} + c_-(n,i,-i)e_{i,-i}^{(n-1)}. \tag{2.11}$$

It is easy to see that K is compact, the restriction T of $P_0\widehat{\gamma}P_0 - K$ to $\mathcal{H}_{0s}$ is independent of s, and $\sigma(T) = \sigma(\widehat{\gamma})$. Hence $\sigma(P_0\widehat{\gamma}P_0 - K) = \sigma_{ess}(P_0\widehat{\gamma}P_0 - K) = \sigma(\widehat{\gamma})$. Since $\sigma_{ess}(P_0\widehat{\gamma}P_0 - K) = \sigma_{ess}(P_0\widehat{\gamma}P_0)$, the proof follows. $\square$

The following lemma will play a key role in some of the proofs in section 4.

Lemma 2.5 *Let $r \in \frac{1}{2}\mathbb{N}$. If $e^{(r)}_{-r,-r}$ is an eigenvector for $f(\widehat{\gamma})$, then $f(\widehat{\gamma})$ is a scalar. In particular, if $f(\widehat{\gamma})e^{(r)}_{-r,-r} = 0$, then $f(\widehat{\gamma}) = 0$.*

Proof: Suppose the corresponding eigenvalue is λ. Let $E = f^{-1}\{\lambda\}$. If $E \neq \sigma(\widehat{\gamma})$, μ is some nonzero element of $\sigma(\widehat{\gamma})\backslash E$, and v is an eigenvector of $P_{r0}\widehat{\gamma}P_{r0}$ corresponding to the eigenvalue μ, then we must have $\langle e^{(r)}_{-r,-r}, v\rangle = 0$. But it is easy to see that for any eigenvector v of $P_{r0}\widehat{\gamma}P_{r0}$, the above inner product is nonzero. Hence we must have $E = \sigma(\widehat{\gamma})$, which implies that $f(\widehat{\gamma})$ is a scalar. $\square$

3 The modular conjugation

Denote by S the operator $a \mapsto a^*$ in $L_2(h)$. Then $\{e^{(n)}_{ij} : (n,i,j) \in \Lambda\}$ is contained in the domain of S and

$$\begin{aligned} Se^{(n)}_{ij} &= \sum_{m,k,l} \langle e^{(m)}_{kl}, Se^{(n)}_{ij}\rangle e^{(m)}_{kl} \\ &= \sum_{m,k,l} \langle e^{(n)}_{ij} e^{(m)}_{kl}, 1\rangle e^{(m)}_{kl}. \end{aligned}$$

Using properties of Clebsch-Gordon coefficients as can be found for example in [5], one can show that

$$Se^{(n)}_{ij} = (-1)^{2n+i+j} q^{i+j} e^{(n)}_{-i,-j}. \tag{3.1}$$

Let J denote the antilinear operator, given on the basis elements by

$$Je^{(n)}_{ij} = (-1)^{2n+i+j} e^{(n)}_{-i,-j}, \tag{3.2}$$

and let Δ be given by

$$\Delta e^{(n)}_{ij} = q^{2i+2j} e^{(n)}_{ij}. \tag{3.3}$$

Then it follows from (3.1) that $S = J\Delta^{\frac{1}{2}}$, i.e. J is the modular conjugation and Δ is the modular operator associated with the haar state.

Remark 3.1 Let R be the operator given by

$$Re^{(n)}_{ij} = \begin{cases} e^{(n)}_{ij} & \text{if } |i| < n, \\ -e^{(n)}_{ij} & \text{if } i = \pm n. \end{cases}$$

Then we have $DJ = RJD$, where R satisfies $R^2 = I$ and $DR = RD$.

4 Poincaré duality

Let τ be the action of $S^1 \times S^1$ on $\mathcal{A}$ by automorphisms given by

$$\tau_{z,w} : \begin{cases} \alpha \mapsto z\alpha, \\ \beta \mapsto w\beta. \end{cases}$$

For $r, s \in \mathbb{Z}$, denote by π_r and ρ_s the following maps:

$$\pi_r(a) = \int_{S^1} z^{-r}\tau_{z,1}(a)dz, \quad \rho_s(a) = \int_{S^1} w^{-s}\tau_{1,w}(a)dw.$$

For $n \in \mathbb{N}$, write $\widetilde{\alpha}_n(a) = \sum_{|r|\leq n} \pi_r(a)$ and $\widetilde{\beta}_n(a) = \sum_{|s|\leq n} \rho_s(a)$. Both these maps are contractive and for $a \in \mathcal{A}_f$, one has $a = \lim_n \widetilde{\alpha}_n(a), a = \lim_n \widetilde{\beta}_n(a)$. It is easy to see that for any $a \in \mathcal{A}$, $\pi_r(a)$ is of the form $\alpha_r f(\beta)$ and for any $a \in \mathcal{A}$ of the form $f(\beta)$, $\rho_s(a)$ is of the form $\beta_s g(\beta^*\beta)$. Let $T \in \mathcal{A}$. Denote by T_{rs} the element $\rho_s(\pi_r(a))$. Observe that T_{rs} is of the form $\alpha_r\beta_r f(\gamma)$ for some continuous function f on $\sigma(\gamma)$. One can prove the following proposition from the above discussion.

Proposition 4.1 *Let $T \in \mathcal{A}$. Then T can be written as a sum $\sum_{r,s} T_{rs}$, where the sum converges in norm.*

Let D be the operator given by (1.1), and let $F = \operatorname{sign} D$. It is easy to check that F as well as JFJ commute with all the projections P_{rs}. Let $Q = \frac{I-JFJ}{2}$. In the diagram of section 2, Q is the projection onto the span of $e_{ij}^{(n)}$'s for (n, i, j) belonging to the face ADC.

We now come to the main result in this section.

Theorem 4.2 *Let $T \in \mathcal{A}$ and J be the modular conjugation given by (3.2). If $[F, JTJ]$ is compact, then T is a scalar.*

We will divide the proof into several lemmas.

Lemma 4.3 *If $[F, JTJ]$ is compact, then $[Q, T_{rs}]$ is compact for all r, s.*

Proof: Let $V_{z,w} : L_2(h) \to L_2(h)$ be given by

$$V_{z,w}e_{ij}^{(n)} = z^{-i-j}w^{i-j}e_{ij}^{(n)}.$$

Then $\tau_{z,w}(a) = V_{z,w}aV_{z,w}^*$ for all $a \in \mathcal{A}$. Now observe that $[F, JTJ]$ is compact if and only if $[Q, T]$ is compact, and $V_{z,w}[Q, T]V_{z,w}^* = [Q, \tau_{z,w}(T)]$. Since $T_{rs} = \int_{S^1\times S^1} z^{-r}w^{-s}\tau_{z,w}(T)dzdw$, it follows that compactness of $[Q, T]$ forces compactness of $[Q, T_{rs}]$. □

Lemma 4.4 *Suppose $[Q, T_{00}]$ is compact. Then T_{00} must be a scalar.*

Proof: T_{00} is of the form $f(\gamma)$ where f is a continuous function on $\sigma(\gamma)$. Since $\widehat{\beta}$ is a compact perturbation of β and they have the same spectrum, it follows that $f(\widehat{\gamma}) - f(\gamma)$ is compact. Hence if $[Q, T_{00}]$ compact, then $[Q, f(\widehat{\gamma})]$ is compact. Now $[Q, f(\widehat{\gamma})]$ decomposes as $\oplus_{n\in\frac{1}{2}\mathbb{N}}[P_nQP_n, P_nf(\widehat{\gamma})P_n]$, and $U_n[P_nQP_n, P_nf(\widehat{\gamma})P_n]U_n^* = [P_0QP_0, P_0f(\widehat{\gamma})P_0]$ for all $n \in \frac{1}{2}\mathbb{N}$. Hence compactness of $[Q, f(\widehat{\gamma})]$ forces $[P_0QP_0, P_0f(\widehat{\gamma})P_0] = 0$. Since $\widehat{\gamma}$ commutes with all the P_{ni}'s, $f(\widehat{\gamma})$ must commute with QP_{00}. But this means $e_{00}^{(0)}$ is an eigenvector for $f(\widehat{\gamma})$. By lemma 2.5, it follows that $f(\widehat{\gamma})$ is a scalar. Therefore f is a constant function, and $f(\gamma)$ is also a scalar. □

Lemma 4.5 *Suppose $s \neq 0$ and $[Q, T_{0s}]$ is compact. Then $T_{0s} = 0$.*

Proof: Observe that T_{0s} is of the form $\beta_s f(\gamma)$. Hence, as in the proof of the previous lemma, compactness of $[Q, T_{0s}]$ implies compactness of $[Q, \widehat{\beta}_s f(\widehat{\gamma})]$. Since this decomposes as $\oplus_{n\in\frac{1}{2}\mathbb{N}}[P_nQP_n, P_n\widehat{\beta}_s f(\widehat{\gamma})P_n]$, and since $U_n[P_nQP_n, P_n\widehat{\beta}_s f(\widehat{\gamma})P_n]U_n^* = [P_0QP_0, P_0\widehat{\beta}_s f(\widehat{\gamma})P_0]$ for all $n \in \frac{1}{2}\mathbb{N}$, we must have $[P_0QP_0, P_0\widehat{\beta}_s f(\widehat{\gamma})P_0] = 0$. Now, let us consider two cases:
Case I. $s > 0$. Observe that $0 = [P_0QP_0, P_0\widehat{\beta}_s f(\widehat{\gamma})P_0]e_{00}^{(0)} = \widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)}$. Since kernel of $\widehat{\beta}_s$ is $\{0\}$, we have $f(\widehat{\gamma})e_{00}^{(0)} = 0$. By lemma 2.5, we conclude that $f(\widehat{\gamma})$ must be zero.
Case II. $s < 0$. It follows by taking adjoints that $[P_0QP_0, P_0\widehat{\beta}_{-s}\bar{f}(\widehat{\gamma})P_0] = 0$. Using the argument used in the earlier case, we now get $\bar{f}(\widehat{\gamma}) = 0$, which implies $T_{0s} = 0$. □

Lemma 4.6 *Let r be a positive integer. Then $[Q, \widehat{\alpha}_r]$ is not compact.*

Proof: Direct computation gives us $|\langle [JFJ, \widehat{\alpha}_r]e_{n,n-\frac{r}{2}}^{(n)}, e_{n+\frac{r}{2},n}^{(n+\frac{r}{2})}\rangle| = 2|\langle \widehat{\alpha}_r e_{-n,-n+\frac{r}{2}}^{(n)}, e_{-n-\frac{r}{2},-n}^{(n+\frac{r}{2})}\rangle| = 2q^{r(\frac{r}{2}+1)}$. This does not go to zero as n tends to ∞. So clearly $[JFJ, \widehat{\alpha}_r]$ can not be compact. □

Lemma 4.7 *Suppose $r \neq 0$ and $[Q, T_{rs}]$ is compact. Then $T_{rs} = 0$.*

Proof: T_{rs} is of the form $\alpha_r\beta_s f(\gamma)$, where f is a continuous function on $\sigma(\gamma)$. Hence as before, compactness of $[Q, T_{rs}]$ will lead to compactness of $[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]$. Now

$$[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})] = \sum_{n\in\frac{1}{2}\mathbb{Z}} [Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_n = \sum_{n\in\frac{1}{2}\mathbb{Z}} P_{n+\frac{r}{2}}[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_n.$$

Hence in the case $r > 0$, one has

$$U_{n+\frac{r}{2}}[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_nU_n^* = U_{\frac{r}{2}}[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_0 \text{ for all } n \in \frac{1}{2}\mathbb{N},$$

and in the case $r < 0$, one has

$$U_{n+\frac{r}{2}}[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_nU_n^*U_{\frac{|r|}{2}} = [Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_{\frac{|r|}{2}} \text{ for all } n \in \frac{|r|}{2} + \frac{1}{2}\mathbb{N}.$$

Hence if $[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]$ is compact, then we must have

$$[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_0 = 0, \qquad \text{if } r > 0, \tag{4.1}$$

$$[Q, \widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})]P_{\frac{|r|}{2}} = 0, \qquad \text{if } r < 0. \tag{4.2}$$

Let us next look at the cases $s > 0$, $s < 0$ and $s = 0$ separately.

Case I. $s > 0$. Suppose first that $r > 0$. Then we have (4.1). Evaluating both sides at $e_{00}^{(0)}$, we get $\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)} = Q\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)}$. But the right hand side is zero, since $\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)} \in \mathcal{H}_{\frac{r}{2},s}$. Now $\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)} = 0$ implies $(\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma}))^*\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma})e_{00}^{(0)} = 0$. But now the left hand side is of the form $g(\widehat{\gamma})$, so that by lemma 2.5, we get $\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma}) = 0$.

Suppose next that $r < 0$, so that we have (4.2). This time evaluating both sides at $e^{(\frac{|r|}{2})}_{\frac{r}{2},\frac{r}{2}}$, and using the same kind of argument, one gets $\widehat{\alpha}_r\widehat{\beta}_s f(\widehat{\gamma}) = 0$.

Case II. $s < 0$. Taking adjoints in equations (4.1) and (4.2), we get

$$P_0[Q, \bar{f}(\widehat{\gamma})\widehat{\beta}_{-s}\widehat{\alpha}_{-r}] = [Q, \bar{f}(\widehat{\gamma})\widehat{\beta}_{-s}\widehat{\alpha}_{-r}]P_{\frac{r}{2}} = 0, \qquad \text{if } r > 0, \tag{4.3}$$

$$P_{\frac{|r|}{2}}[Q, \bar{f}(\widehat{\gamma})\widehat{\beta}_{-s}\widehat{\alpha}_{-r}] = [Q, \bar{f}(\widehat{\gamma})\widehat{\beta}_{-s}\widehat{\alpha}_{-r}]P_0 = 0, \qquad \text{if } r < 0. \tag{4.4}$$

Since $\bar{f}(\widehat{\gamma})\widehat{\beta}_{-s}\widehat{\alpha}_{-r}$ is of the form $\widehat{\alpha}_{-r}\widehat{\beta}_{-s}g(\widehat{\gamma})$ for some appropriate g, we are through by the argument in case I.

Case III. $s = 0$. Suppose $r > 0$. By applying $U_{\frac{r}{2}}$ on both sides in (4.1), we get $[Q, U_{\frac{r}{2}}\widehat{\alpha}_r f(\widehat{\gamma})]P_0 = 0$. Evaluating both sides on $e^{(0)}_{00}$, $U_{\frac{r}{2}}\widehat{\alpha}_r f(\widehat{\gamma})e^{(0)}_{00} = QU_{\frac{r}{2}}\widehat{\alpha}_r f(\widehat{\gamma})e^{(0)}_{00}$. Now $U_{\frac{r}{2}}\widehat{\alpha}_r f(\widehat{\gamma})$ is of the form $g(\widehat{\gamma})U_{\frac{r}{2}}\widehat{\alpha}_r$. Hence we have $g(\widehat{\gamma})U_{\frac{r}{2}}\widehat{\alpha}_r e^{(0)}_{00} = Qg(\widehat{\gamma})U_{\frac{r}{2}}\widehat{\alpha}_r e^{(0)}_{00}$. It is easy to see that $U_{\frac{r}{2}}\widehat{\alpha}_r e^{(0)}_{00} = \lambda e^{(0)}_{00}$ for some λ. Hence it follows that $e^{(0)}_{00}$ is an eigenvector of $g(\widehat{\gamma})$. By lemma 2.5, $g(\widehat{\gamma})$ is a scalar. By lemma 4.6, $g(\widehat{\gamma})$ has to be zero. Hence $\widehat{\alpha}_r f(\widehat{\gamma}) = 0$.

If $r < 0$, then taking adjoint in (4.2) reduces it to the above case □

Combining lemmas 4.3–4.7, proof of theorem 4.2 is immediate.

Remark 4.8 By the characterization of equivariant spectral triples in [1] (see the discussion preceeding proposition 4.4, [1]), for any equivariant D, $\operatorname{sign} D$ has to be of the form $2P - I$ or $I - 2P$, where P is the projection onto the subspace spanned by $\{e^{(n)}_{ij} : n \in \frac{1}{2}\mathbb{N}, n - i \in E, j = -n, -n+1, \ldots, n\}$, E being some finite subset of $\mathbb{N}$. Essentially the same argument used in the proof of theorem 4.2 will work for the sign of this D also.

Corollary 4.9 *Suppose $a \in J\mathcal{A}J$. If $[D, a]$ is bounded, then a must be a scalar.*

Remark 4.10 Poincaré duality for the equivariant spectral triples would fail to hold if one could prove the above corollary for elements in $\mathcal{A}' = J\mathcal{A}''J$, which amounts to proving lemma 4.5 and 4.7 for $T_{rs} = \alpha_r\beta_s f(\gamma)$, where f is a bounded measurable function on $\sigma(\gamma)$. Thus although the above result gives a strong indication that Poincaré duality might fail to hold, it does not rule out the possibility completely.

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