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On a bivariate lack of memory property under binary associative operation

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ON A BIVARIATE LACK OF MEMORY PROPERTY UNDER BINARY ASSOCIATIVE OPERATION

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Abstract

A binary operation * over real numbers is said to be associative if $(x * y) * z = x * (y * z)$ and it is said to be reducible if $x * y = x * z$ or $y * w = z * w$ if and only if $z = y$. The operation * is said to have an identity element $\tilde{e}$ if $x * \tilde{e} = x$. Roy (2002) introduced a new definition for bivariate lack of memory property and characterized the bivariate exponential distribution introduced by Gumbel (1960) under the condition that the each of the conditional distributions should have the univariate lack of memory property. We generalize this definition and characterize different classes of bivariate probability distributions under binary associative operations between random variables.

Keywords and phrases: Binary associative operation; Bivariate lack of memory property; Bivariate exponential distribution; Bivariate Weibull distribution; Bivariate Pareto distribution; Multivariate exponential distribution; Characterization.

1 Introduction

Different types of bivariate exponential distributions have been investigated for stochastic modelling purposes. Some of these distributions have been developed via characterizing properties such as the lack of memory property (LMP) of the exponential distribution. Marshall and Olkin (1967) proposed a bivariate lack of memory property ($BLMP$) and studied a class of bivariate as well as multivariate exponential distributions. Roy (2002) introduced a new definition for bivariate lack of memory property ($BLMP_1$) and characterized the bivariate exponential distribution introduced by Gumbel (1960) under the condition that the each of the conditional distributions should have the univariate lack of memory property. We generalize this definition and characterize different classes of bivariate probability distributions under binary associative operations between random variables. These include bivariate exponential, bivariate Weibull and bivariate Pareto distributions.

2 Preliminaries

A binary operation * over real numbers is said to be associative if

$$(x * y) * z = x * (y * z) \tag{2. 1}$$

for all real numbers x, y, z. The binary operation * is said to be reducible if $x * y = x * z$ if and only if $y = z$ and if $y * w = z * w$ if and only if $y = z$. It is known that the general reducible continuous solution of the functional equation (2.1) is

$$x * y = g^{-1}(g(x) + g(y)) \tag{2. 2}$$

where $g(.)$ is a continuous and strictly monotone function provided $x, y, x * y$ belong to a fixed (possibly infinite) interval A (cf. Aczel (1966,1987)).The function $g(.)$ in (2.2) is determined up to a multiplicative constant, that is,

$$g_1^{-1}(g_1(x) + g_1(y)) = g_2^{-1}(g_2(x) + g_2(y))$$

for all x, y in a fixed interval A implies $g_2(x) = \alpha g_1(x)$ for all x in that interval for some $\alpha \neq 0$. We assume here after that the binary operation is reducible and associative with the function $g(.)$ continuous and strictly increasing. Further assume that there exists an identity element $\tilde{e} \in \bar{R}$ such that

$$x * \tilde{e} = x, x \in A.$$

It is also known that every continuous, reducible and associative operation defined on an interval A in the real line is commutative (cf. Aczel (1966), p.267). Let X be a random variable with the distribution function $F(x)$ having support A. Define

$$\phi_X^*(s) = \int_A \exp\{isg(x)\} \;\; F(dx), -\infty < s < \infty. \tag{2. 3}$$

Note that the function $\phi_X^*(s)$ is the characteristic function of the random variable $g(X)$ and hence determines the distribution function of the random variable $g(X)$ uniquely.

Examples of such binary operations are given in Castagnoli (1974, 1978, 1982), Muliere (1984) and Castagnoli and Muliere (1984, 1986, 1988). For instance (i) if $A = (-\infty, \infty)$ and $x * y = x + y$, then $g(x) = x$, (ii) if $A = (0, \infty)$ and $x * y = xy, x > 0, y > 0$ then $g(x) = \log x$, (iii) if $A = (0, \infty)$ and $x * y = (x^\alpha + y^\alpha)^{1/\alpha}, x > 0, y > 0$ for some $\alpha > 0$, then $g(x) = x^\alpha$, (iv) if $A = (-1, \infty)$ and $x * y = x + y + xy + 1, x > -1, y > -1$, then $g(x) = \log(1 + x)$ (v)if $A = (0, \infty)$ and $x * y = xy/(x + y), x > 0$, then $g(x) = 1/x$ and (vi)if $A = (0, \infty)$ and $x * y = (x + y)/(1 + xy), x > 0, y > 0$, then $g(x) = arth\ x$.

A characterization of the multivariate normal distribution through a binary operation which is associative is given in Prakasa Rao (1974) and in Prakasa Rao (1977) for Gaussian measures on locally compact abelian groups. Some characterizations of probability distributions through binary associative operations have been studied in Muliere and Prakasa Rao (2002) extending earlier results in Prakasa Rao (1992, 1997).

Let $X = (X_1, X_2)$ be a bivariate nonnegative random vector with the bivariate survival function

$$S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2), x_1 \geq 0, x_2 \geq 0 \tag{2. 4}$$

satisfying the functional equation

$$S(x_1 + t, x_2 + t) = S(x_1, x_2)\ S(t, t), x_1 \geq 0, x_2 \geq 0, t \geq 0. \tag{2. 5}$$

The above functional equation represents a particular type of bivariate lack of memory property ($BLMP$). Marshall and Olkin (1967) characterized the class of bivariate distributions with the exponential marginal distributions satisfying the above functional equation. They have shown that the unique solution of the above functional equation is the bivariate distribution with the survival function given by

$$S(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}. \tag{2. 6}$$

This bivariate distribution is a mixture of an absolutely continuous distribution and a singular part that concentrates its mass on the line $x_1 = x_2$. This family of distributions have been found useful for applications in reliability theory (cf. Basu and Block (1975); Galambos and Kotz (1978)).

Muliere and Scarsini (1987) characterized a class of bivariate Marshall-Olkin type distributions that generalize the Marshall-Olkin bivariate exponential distribution through a functional equation involving binary associative operations. These classes of bivariate distributions do not necessarily have exponential distributions as their marginal distributions and their form depends on the associative operation. They concentrate positive mass on the line $x_1 = x_2$ as in the case of bivariate exponential distribution introduced by Marshall and Olkin (1967).

Let $*$ be a binary associative operation with an identity element $\tilde{e}$. Suppose that the survival function $S(x_1, x_2)$ satisfies the functional equations

$$S(x_1 * t, x_2 * t) = S(x_1, x_2)\ S(t, t), \tag{2. 7}$$

$$S_1(x_1 * t) = S_1(x_1)\ S_1(t), S_1(x_1) = S(x_1, \tilde{e}), \tag{2. 8}$$

and

$$S_2(x_2 * t) = S_2(x_2)\ S_2(t), S_2(x_2) = S(\tilde{e}, x_2) \tag{2. 9}$$

for all $x_1, x_2, t > \tilde{e}$.

Muliere and Scarsini (1987) prove that the only continuous solution of the functional equations (2.7)-(2.9) is

$$S(x_1, x_2) = \exp\{-\lambda_1 g(x_1) - \lambda_2 g(x_2) - \lambda_{12}\ g(\max(x_1, x_2))\} \tag{2. 10}$$

with $\lambda_1, \lambda_2, \lambda_{12} > 0$ wher $g(.)$ is the function corresponding to the binary associative operation $*$.

Different specializations of the binary associative operation * lead to different bivariate survival functions.

Example 1: If $x * y = x + y$, then $g(x) = x$ and

$$S(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2))\}. \tag{2. 11}$$

This is the Marshall-Olkin distribution.

Example 2: If $x * y = xy$, then $g(x) = \log x$ and

$$S(x_1, x_2) = x_1^{-\lambda_1} x_2^{-\lambda_2} (\max(x_1, x_2))^{-\lambda_{12}}. \tag{2. 12}$$

This is the bivariate Pareto distribution over the set $(1, \infty) \times (1, \infty)$.

Example 3: If $x * y = (x^\alpha + y^\alpha)^{1/\alpha}$, then $g(x) = x^\alpha$ and

$$S(x_1, x_2) = \exp\{-\lambda_1 x_1^\alpha - \lambda_2 x_2^\alpha - \lambda_{12} \max(x_1^\alpha, x_2^\alpha)\}. \tag{2. 13}$$

This is the bivariate Weibull distribution (cf. Marshall and Olkin (1967); Moeschberger (1974)).

Recently Roy (2002) introduced a new concept of bivariate lack of memory property.

Definition : Let $S(x_1, x_2)$ be the bivariate survival function of a nonnegative bivariate random vector (X_1, X_2). The survival function $S(x_1, x_2)$ is said to possess bivariate lack of memory property $BLMP_2$ if and only if for all x_1, x_2, y_1 and y_2,

$$S(x_1 + y_1, x_2)\ S(0, x_2) = S(x_1, x_2)\ S(y_1, x_2) \tag{2. 14}$$

and

$$S(x_1, x_2 + y_2)\ S(x_1, 0) = S(x_1, x_2)\ S(x_1, y_2). \tag{2. 15}$$

It is easy to see that if a bivariate distribution has the $BLMP_2$ property, then the marginals posess univariate lack of memory property (LMP). This can be seen by substituting $x_2 = 0$ in (2.14) or $x_1 = 0$ in (2.15).

Roy (2002) proved that a bivariate random vector $\mathbf{X}$=(X_1, X_2) follows $BLMP_2$ if and only if $\mathbf{X}$ follows the bivariate exponential distribution introduced by Gumbel (1960) with the survival function

$$S(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}. \tag{2. 16}$$

It was shown further by Roy (2002) that the failure rates

$$r_i(x_1, x_2) = \frac{\partial}{\partial x_i}[-\log S(x_1, x_2)], i = 1, 2 \tag{2. 17}$$

are locally constant in the sense that $r_1(x_1, x_2) = r_1(0, x_2)$ for all x_1 and x_2 and similarly $r_2(x_1, x_2) = r_2(x_1, 0)$ for all x_1 and x_2. Similarly the man residual lives

$$M_i(x_1, x_2) = E(X_i - x_i | X_1 \geq x_1, X_2 \geq x_2), i = 1, 2 \tag{2. 18}$$

are locally constant.

3 Main Results

Let $\mathbf{X}$= (X_1, X_2) be a bivariate random vector with the survival function $S(x_1, x_2)$. Let * be a binary associative operation with the identity $\tilde{e}$ satisfying the equations

$$S(x_1 * y_1, x_2)\ S(\tilde{e}, x_2) = S(x_1, x_2)\ S(y_1, x_2) \tag{3. 1}$$

and

$$S(x_1, x_2 * y_2)\ S(x_1, \tilde{e}) = S(x_1, x_2)\ S(x_1, y_2). \tag{3. 2}$$

Suppose that $S(x_1, x_2) > 0$ for all $x_1 \geq \tilde{e}$ and $x_2 \geq \tilde{e}$. Let $g(.)$ be a continuous strictly increasing function associated with the binary associative operation as described earlier. It is known that the function $g(.)$ is unique up to a multiplicative constant. Note that $g(\tilde{e}) = 0$. Further more

$$x_1 * x_2 = g^{-1}(g(x_1) + g(x_2)). \tag{3. 3}$$

Suppose that the function $g(.)$ defined above is differentiable with the derivative $g'(x) > 0$ for all $x \geq \tilde{e}$. We now characterize the class of all bivariate distributions satisfying the functional equations (3.1) and (3.2).

Observe that

$$\begin{aligned} M_1(x_1, x_2) &\equiv \int_{\tilde{e}}^{\infty} \frac{S(x_1 * t, x_2)}{S(x_1, x_2)}\ dt \\ &= \int_{\tilde{e}}^{\infty} \frac{S(t, x_2)}{S(\tilde{e}, x_2)}\ dt \end{aligned} \tag{3. 4}$$

from the equation (3.2) and the right side is a function of x_2 alone. Let us denote it by $K_1(x_2)$. Hence

$$\frac{1}{K_1(x_2)} = \frac{1}{M_1(x_1, x_2)} = \frac{S(x_1, x_2)}{\int_{\tilde{e}}^{\infty} S(x_1 * t, x_2)\ dt} \tag{3. 5}$$

for all $x_1 \geq \tilde{e}$ and $x_2 \geq \tilde{e}$. Let

$$\begin{aligned} A_1(x_1, x_2) &\equiv \int_{\tilde{e}}^{\infty} S(x_1 * t, x_2)\ dt \\ &= \int_{x_1}^{\infty} S(u, x_2) \frac{g'(u)}{g'(g^{-1}(g(u) - g(x_1)))}\ du. \end{aligned} \tag{3. 6}$$

Observe that

$$\begin{aligned} \frac{\partial A_1}{\partial x_1} &= -\frac{S(x_1, x_2) g'(x_1)}{g'(g^{-1}(0))} \\ &= -\frac{S(x_1, x_2) g'(x_1)}{g'(\tilde{e})}. \end{aligned} \tag{3. 7}$$

Combining relations (3.4) to (3.7), we get that

$$\frac{1}{K_1(x_2)} = \frac{S(x_1, x_2)}{A_1(x_1, x_2)} \tag{3. 8}$$

$$= -\frac{\frac{\partial A_1}{\partial x_1} \frac{g'(\tilde{e})}{g'(x_1)}}{A_1(x_1, x_2)}. \tag{3. 9}$$

Therefore

$$\begin{aligned} \frac{g'(x_1)}{g'(\tilde{e})} \frac{1}{K_1(x_2)} &= -\frac{\frac{\partial A_1}{\partial x_1}}{A_1(x_1, x_2)} \\ &= -\frac{\partial \log A_1}{\partial x_1}. \end{aligned} \tag{3. 10}$$

Solving this differential equation, we obtain that

$$\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(x_2)} = -\log A_1 + B_1(x_2) \tag{3. 11}$$

for some function $B_1(.)$. Therefore

$$A_1(x_1, x_2) = \exp[-\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(x_2)} + B_1(x_2)] \tag{3. 12}$$

which implies that

$$S(x_1, x_2) = \frac{1}{K_1(x_2)}\exp[-\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(x_2)} + B_1(x_2)]. \tag{3. 13}$$

Therefore

$$S(x_1, x_2) = L_1(x_2)\exp[-\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(x_2)}] \tag{3. 14}$$

for some functions $K_1(x_2)$ and $L_1(x_2)$. This relation was derived from (2.14).Similarly it follows from (2.15)that

$$S(x_1, x_2) = L_2(x_1)\exp[-\frac{g(x_2)}{g'(\tilde{e})}\frac{1}{K_2(x_1)}] \tag{3. 15}$$

for some functions $K_2(x_1)$ and $L_2(x_1)$. Hence

$$L_1(x_2)\exp[-\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(x_2)}] = L_2(x_1)\exp[-\frac{g(x_2)}{g'(\tilde{e})}\frac{1}{K_2(x_1)}] \tag{3. 16}$$

for all x_1, x_2. Let $x_2 = \tilde{e}$ in the above identity. Then it follows that

$$L_1(\tilde{e})\exp[-\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(e)}] = L_2(x_1)\exp[-\frac{g(\tilde{e})}{g'(\tilde{e})}\frac{1}{K_2(x_1)}] \tag{3. 17}$$

or equivalently

$$\log L_1(\tilde{e}) - [\frac{g(x_1)}{g'(\tilde{e})}\frac{1}{K_1(e)}] = \log L_2(x_1) \tag{3. 18}$$

for all x_1 since $g(\tilde{e}) = 0$. Therefore

$$L_2(x_1) = \exp[\alpha_1 + \beta_1 g(x_1)] \tag{3. 19}$$

for all x_1 for some constants α_1 and β_1 and hence

$$S(x_1, x_2) = \exp[\alpha_1 + \beta_1 g(x_1) + \gamma_1 g(x_2)H_2(x_1)] \tag{3. 20}$$

for some constants $\alpha_1, \beta_1, \gamma_1$ and some function H_2 depending on x_1 only. A similar analysis starting with substituting $x_1 = \tilde{e}$ in the identity (3.16) shows that

$$S(x_1, x_2) = \exp[\alpha_2 + \beta_2 g(x_2) + \gamma_2 g(x_1)H_1(x_2)] \tag{3. 21}$$

for some constants $\alpha_2, \beta_2, \gamma_2$ and some function H_1 depending on x_2 only.

Equating the relations (3.20) and (3.21), it follows that

$$\alpha_1 + \beta_1 g(x_1) + \gamma_1 g(x_2) H_2(x_1) = \alpha_2 + \beta_2 g(x_2) + \gamma_2 g(x_1) H_1(x_2) \tag{3. 22}$$

for all x_1 and x_2. Let $x_1 = x_2 = \tilde{e}$. in the equation (3.22). Then it follows that $\alpha_1 = \alpha_2$ since $g(\tilde{e}) = 0$. Hence

$$\beta_1 g(x_1) + \gamma_1 g(x_2) H_2(x_1) = \beta_2 g(x_2) + \gamma_2 g(x_1) H_1(x_2) \tag{3. 23}$$

for all x_1 and x_2. Fix a value of $x_2 = x_{20}$ such that $g(x_{20}) \neq 0$. Then it follows that

$$\begin{aligned} \beta_1 g(x_1) + \gamma_1 g(x_{20}) H_2(x_1) &= \beta_2 g(x_{20}) + \gamma_2 g(x_1) H_1(x_{20}) \\ &= c_1 + c_2 g(x_1) \end{aligned} \tag{3. 24}$$

for some constants c_1 and c_2. Therefore

$$\begin{aligned} H_2(x_1) &= \frac{c_1 + c_2 g(x_1) - \beta_1 g(x_1)}{\gamma_1 g(x_{20})} \\ &= c_3 + c_4 g(x_1) \end{aligned} \tag{3. 25}$$

when $\gamma_1 \neq 0$ for some constants c_3 and c_4. In particular, we have

$$\begin{aligned} S(x_1, x_2) &= \exp[\alpha_1 + \beta_1 g(x_1) + \gamma_1 g(x_2)(c_3 + c_4 g(x_1))] \\ &= \exp[\alpha_1 + \beta_1 g(x_1) + \beta_2 g(x_2) + \beta_3 g(x_1) g(x_2)] \end{aligned} \tag{3. 26}$$

It is clear that the above representation holds even if $\gamma_1 = 0$ from the equation (3.20). Let $x_1 = x_2 = \tilde{e}$ in the equation (3.26). Then it follows that $\alpha_1 = 0$ since $g(\tilde{e}) = 0$ and $S(\tilde{e}, \tilde{e}) = 1$. Suppose that $S_(x_1, \tilde{e}) < 1$ and $S(\tilde{e}, x_2) < 1$ for all $x_1 > \tilde{e}$ or $x_2 > \tilde{e}$. Then it follows that $\beta_1 < 0, \beta_2 < 0$ and $0 \leq -\beta_3 \leq \beta_1 \beta_2$. Hence we have the foloowing theorem.

Theorem 2.1: Suppose a bivariate random vector $\mathbf{X}$ has the $BLMP_2$ property under a binary associative operation * with an identity $\tilde{e}$, that is, its survival function $S(x_1, x_2)$ satisfies the conditions

$$S(x_1 * y_1, x_2)\ S(\tilde{e}, x_2) = S(x_1, x_2)\ S(y_1, x_2) \tag{3. 27}$$

and

$$S(x_1, x_2 * y_2)\ S(x_1, \tilde{e}) = S(x_1, x_2)\ S(x_1, y_2). \tag{3. 28}$$

for all $x_1 \geq \tilde{e}$ and $x_2 \geq \tilde{e}$. Further suppose that $S_(x_1, \tilde{e}) < 1$ and $S(\tilde{e}, x_2) < 1$ for all $x_1 > \tilde{e}$ or $x_2 > \tilde{e}$. Then there exists constants $\lambda_1 > 0, \lambda_2 > 0, 0 \leq \lambda_3 \leq \lambda_1 \lambda_2$ such that

$$S(x_1, x_2) = \exp[-\lambda_1 g(x_1) - \lambda_2 g(x_2) - \lambda_3 g(x_1) g(x_2)] \tag{3. 29}$$

for all $x_1 \geq \tilde{e}$ and $x_2 \geq \tilde{e}$ where $g(.)$ is the function corresponding to the binary associate operation *.

Remark 1: By choosing the binary associate operation * as the addition operation on the set of real numbers with the identitity $\tilde{e} = 0$, we obtain that $g(x) = x$ and hence derive the characterization of the bivariate exponential distribution given in Theorem 3.1 of Roy (2002) as a corollory to Theorem 2.1 given above. As pointed out by Roy (2002), the characterization results obtained in Johnson and Kotz (1975), Zahedi (1985) and Roy and Gupta (1996) also follow as special cases of our results.

Remark 2: A multivariate extension of Theorem 2.1 can be obtained by mathematical induction. The multivariate version of $BLMP_2$ property under the binary associative operation * with an identity $\tilde{e}$ can be defined as follows. A k-dimensional random vector $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_i}, \ldots, \mathbf{X_k})$ is said to have the $MLMP_2$ property under a binary associative operation * with an identity $\tilde{e}$ if its survival function $S(x_1, x_2, \ldots, x_k)$ satisfies the conditions

$$\begin{aligned} &S(x_1, \ldots, x_{i-1}, x_i * y_i, x_{i+1}, \ldots, x_k)S(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k) \\ &\quad = S(x_1, \ldots, x_i, \ldots, x_k)S(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k) \end{aligned}$$

for all $x_i \geq \tilde{e}, i = 1, 2, \ldots, k$. Further suppose that

$$S(\tilde{e}, \ldots, \tilde{e}, x_i, \tilde{e}, \ldots, \tilde{e}) < 1, i = 1, 2, \ldots, k.$$

Then it can be shown that the class of all such multivariate distributions are those with the survival functions of the form

$$S(x_1, \ldots, x_i, \ldots, x_k) = \exp[-\Sigma\lambda_i x_i - \Sigma\Sigma\lambda_{ij} x_i x_j - \ldots - \lambda_{12\ldots k} x_1 x_2 \ldots x_k]. \tag{3. 30}$$

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