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Characterization of spectral triples: A combinatorial approach

Partha Sarathi Chakraborty
Arupkumar Pal

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Abstract

We describe a general technique to study Dirac operators on noncommutative spaces under some additional assumptions. The main idea is to capture the compact resolvent condition in a combinatorial set up. Using this, we then prove that for type A_ℓ compact quantum groups, if $\ell > 1$ then the L_2-space does not have any equivariant Dirac operator with nontrivial sign acting on it. As a second illustration of the technique, we prove that if $\ell > 1$, then for a certain class of representations of the C^*-algebra $C(SU_q(\ell+1))$, there does not exist any Dirac operator that diagonalises with respect to the natural basis of the underlying Hilbert space and has nontrivial sign.

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Contents

1 Introduction

The purpose of this paper is twofold — one, to present a combinatorial technique of characterizing spectral triples for a given representation of a C^*-algebra, and two, to investigate the existence of meaningful spectral triples for type A quantum groups. A spectral triple, one might recall, is the starting point in noncommutative geometry (NCG) where a geometric space is described by a triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A}$ being an involutive algebra represented as bounded operators on a Hilbert space $\mathcal{H}$, and D being a selfadjoint operator with compact resolvent and having bounded commutators with the algebra elements. This D should be nontrivial in the sense that the associated Kasparov module should give a nontrivial element in K-homology. Groups have always played a very crucial role in the study of geometry of a space, mainly as objects that govern the symmetry of the space. One would expect the same in NCG also. Moreover, since one now deals with a larger class of spaces, mainly noncommutative ones, it is natural to expect that one would require a larger class, Hopf algebras or the quantum groups, to play a similar role. In the classical case, groups which govern symmetry are themselves nice geometric objects. Here we want to look at quantum groups from the same angle. In [1], the authors treated the case of the quantum $SU(2)$ group and characterized all spectral triples acting on its L_2-space that are equivariant with respect to its natural left (co)action. This was subsequently used by Connes ([6]) to make elaborate computations and illustrate that most of the machinery of NCG work very well for $SU_q(2)$. One of our objectives in this paper is to investigate other quantum groups from the same point of view. In particular, we will formulate the notion of equivariance under a (quantum) group action, and systematically look for equivariant spectral triples for a large class of compact quantum groups.

The other main objective is to present a technique that will enable us to do this. The (selfadjoint) operator D in a spectral triple comes with two restrictions on it, namely, it has to have compact resolvent, and must have bounded commutators with algebra elements. Various analytic consequences of the compact resolvent condition (growth properties of the commutators of the algebra elements with the sign of D) have been used in the past by various authors. Here we will take a new approach that will help us exploit it from a combinatorial point of view. The idea is very simple. Given a selfadjoint operator with compact resolvent, one can associate with it a certain graph in a natural way. This makes it possible to do a detailed combinatorial analysis of the growth restrictions (on the eigenvalues of D) that come from the boundedness of the commutators, and to characterize the sign of the operator D completely.

In the remaining part of this section, we will outline the above technique. We then use this general scheme in the remaining sections in two specific cases, both involving type A compact quantum groups. In section 2, we take $\mathcal{A}$ to be the C^*-algebra of continuous functions on $G = SU_q(\ell+1)$, the Hilbert space to be $L_2(G)$, the L_2-space of the haar state for G, where elements of $\mathcal{A}$ act by left multiplication, and assume D to be equivariant under the natural left action of G on $\mathcal{A}$ which is implemented on $L_2(G)$. We show that for $\ell > 1$, *there does not exist any equivariant Dirac operator with nontrivial sign* acting on $L_2(G)$, which in particular shows that no equivariant spectral triple exists on $L_2(G)$ with nontrivial K-homology class. This behaviour is very different from the case $\ell = 1$, $q \neq 1$, which was established in [1].

In section 3, we deal with a case analogous to the one for $SU_q(2)$ treated in [2]. We take the 'standard representations', which are obtained by integrating certain families of irreducible representations of the C^*-algebra $C(SU_q(\ell+1))$, and show the nonexistence of Dirac operators that diagonalise nicely and has nontrivial sign.

1.1 The general scheme

Suppose $\mathcal{H}$ is a Hilbert space, and D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent. Then D admits a spectral resolution $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$, where the d_γ's are all distinct and each P_γ is a finite dimensional projection. Let c be a positive real. Let us now define a graph $\mathcal{G}_c$ as follows: take the vertex set V to be Γ. Connect two vertices γ and γ' by an edge if $|d_\gamma - d_{\gamma'}| < c$. Assume now onward that all the d_γ's are nonzero. Let $V^+ = \{\gamma \in V : d_\gamma > 0\}$ and $V^- = \{\gamma \in V : d_\gamma < 0\}$. This will give us a partition of V.

Definition 1.1.1 Let $\mathcal{G} = (V, E)$ be an infinite graph. A pair (V_1, V_2) of disjoint subsets of V is said to admit an **infinite ladder** if there are two sequences of points $\gamma_n \in V_1$, $\delta_n \in V_2$, and a sequence of disjoint paths p_n joining γ_n to δ_n.

Lemma 1.1.2 *The pair (V^+, V^-) does not admit any infinite ladder.*

Proof: Observe that if there is a path from γ to δ and $d_\gamma > 0$, $d_\delta < 0$, then for some α on the path, one must have $d_\alpha \in [-c, c]$. Therefore the existence of an infinite ladder would contradict the compact resolvent condition. □

Definition 1.1.3 Let $\mathcal{G} = (V, E)$ be an infinite graph. We call it **sign determining** if there is a partition of the vertex set that does not admit any infinite ladder.

Thus the previous lemma says that the graph $\mathcal{G}_c$ is sign determining.

The idea here will be to start from the opposite direction. Suppose $\mathcal{A}$ is a C^*-algebra represented on a Hilbert space, and suppose we want to have an idea about all operators D that will make $(\mathcal{A}, \mathcal{H}, D)$ into a spectral triple. Of course, in this generality, the problem would be intractable in most cases. We will impose some extra conditions on this D that will be natural from the context. For example, if a group or a quantum group has an action on $\mathcal{A}$, we might demand equivariance under that action. This would give some idea about the spectral resolution $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$, more specifically some idea about how the set Γ and the projections P_γ look like. In other words, this would provide us with a diagonalising basis for D. Note that since D is known to be self-adjoint with discrete spectrum, there always exists such a basis. Next we construct a family of graphs $\mathcal{G}_c$ depending on a positive real parameter c by deciding to join two points γ and γ' in the vertex set $V := \Gamma$ if $|d_\gamma - d_{\gamma'}| < c$. Since this comes from D, the graph must be sign-determining. Of course, for a given c, the graph $\mathcal{G}_c$ may have no edges, or too few edges (if the singular values of D happen to grow too fast), in which case, the statement that $\mathcal{G}_c$ is sign-determining will not provide us with anything worthwhile. Fortunately, the operators we are interested in are meant to be the Dirac operators of some commutative/noncommutative manifold. Therefore the singular values of D will grow at the rate of $O(n^{1/d})$ for some $d \geq 1$. So one can choose a large enough c and work with the graph $\mathcal{G}_c$. We call this the **growth graph** for the operator D. Now one looks at the actions of the elements from $\mathcal{A}$ on $\mathcal{H}$ and try and see what the boundedness of the commutators $[D, a]$ (for $a \in \mathcal{A}$) tell us. These conditions will give some growth restrictions on the quantities d_γ, which, in turn, will give some information about the set of edges in the graph. Using this information, we then characterize those partitions of the vertex set that do not admit any infinite ladder.

2 The equivariant case

Suppose G is a compact group, quantum or classical, and $\mathcal{A}$ is a unital C^*-algebra. Assume that G has an action on $\mathcal{A}$ given by $\tau : \mathcal{A} \to \mathcal{A} \otimes C(G)$, so that $(\mathrm{id} \otimes \Delta)\tau = (\tau \otimes \mathrm{id})\tau$, Δ being the coproduct. In other words, we have a C^*-dynamical system $(\mathcal{A}, G, \tau)$. Our goal is to study spectral triples for $\mathcal{A}$ equivariant under this action. Let us first say what we mean by 'equivariant' here.

A covariant representation (π, u) of $(\mathcal{A}, G, \tau)$ consists of a unital *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, a unitary representation u of G on $\mathcal{H}$, i.e. a unitary element of the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that they obey the condition $(\pi \otimes \mathrm{id})\tau(a) = u(\pi(a) \otimes I)u^*$ for all $a \in \mathcal{A}$.

Definition 2.0.1 Suppose $(\mathcal{A}, G, \tau)$ is a C^*-dynamical system. An operator D acting on a Hilbert space $\mathcal{H}$ is said to be **equivariant** with respect to a covariant representation (π, u) of the system if $D \otimes I$ commutes with u.

Since the operator D is self-adjoint with compact resolvent, it will admit a spectral resolution $\sum_\lambda d_\lambda P_\lambda$, where the d_λ's are distinct and each P_λ is finite dimensional. Also, D has been assumed to be equivariant — so that the P_λ's commute with u (to be precise, the $(P_\lambda \otimes I)$'s do), i.e. u keeps each $P_\lambda\mathcal{H}$ invariant. As G is compact, each $P_\lambda\mathcal{H}$ will decompose further as $\oplus_\mu P_{\lambda\mu}\mathcal{H}$ such that the restriction of u to each $P_{\lambda\mu}$ is irreducible. In other words, one can now write D in the form $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$ for some index set Γ and a family of finite dimensional projections P_γ such that each P_γ commutes with u and the restriction of u to each P_γ is irreducible.

In this section, we will deal with the case $G = SU_q(\ell + 1)$, $\mathcal{A} = C(G)$, τ is the natural left action coming from the coproduct, $\mathcal{H}$ is $L_2(G)$, π is the representation of $\mathcal{A}$ on $\mathcal{H}$ by left multiplication, and u is the left regular representation. Structure of the regular representation of a compact (quantum) group along with the remarks made above tell us the following. Let Λ be the set of unitary irreducible representation-types for G. Then $\mathcal{H}$ decomposes as $\oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$, where the restriction of u to $\mathcal{H}_\lambda$ is equivalent to $\dim\lambda$ copies of the irreducible λ, and also that D respects this decomposition. Further, restriction of D to $\mathcal{H}_\lambda$ is of the form $\sum_\mu d_{\lambda\mu} P_{\lambda\mu}$, u commutes with each of these $P_{\lambda\mu}$'s, and the restriction of u to $P_{\lambda\mu}\mathcal{H}$ is equivalent to λ. Let N_λ be any set with $|N_\lambda| = \dim\lambda$. One can then choose an orthonormal basis $\{e^\lambda_{ij} : i, j \in N_\lambda\}$ such that the spaces $P_{\lambda\mu}\mathcal{H}$ are precisely $\mathrm{span}\{e^\lambda_{ij} : j \in N_\lambda\}$ for distinct values of $i \in N_\lambda$. Since D is of the form $\sum_\lambda \sum_\mu d_{\lambda\mu} P_{\lambda\mu}$, in this system of bases, D will look like $e^\lambda_{ij} \mapsto d(\lambda, i) e^\lambda_{ij}$. In what follows, we will make a special choice of N_λ, which will make the combinatorial analysis very convenient.

2.1 Preliminaries

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ. let $((a_{ij}))$ be the associated Cartan matrix, q be a real number lying in the interval $(0,1)$ and let $q_i = q^{(\alpha_i,\alpha_i)/2}$, where α_i's are the simple roots of $\mathfrak{g}$. Then the quantised universal envelopping algebra (QUEA) $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i and K_i^{-1}, $i = 1, \ldots, \ell$, satisfying the following relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q_i^{\frac{1}{2}a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-\frac{1}{2}a_{ij}} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0 \quad \forall i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0 \quad \forall i \neq j,$$

where $\binom{n}{r}_q$ denote the q-binomial coefficients. Hopf *-structure comes from the following maps:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,$$

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0 = \epsilon(F_i),$$

$$S((K_i) = K_i^{-1}, \quad S(E_i) = -q_i E_i, \quad S(F_i) = -q_i^{-1} F_i,$$

$$K_i^* = K_i, \quad E_i^* = -q_i^{-1} F_i, \quad F_i^* = -q_i E_i.$$

In the type A case, the associated Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $(\alpha_i, \alpha_i) = 2$ so that $q_i = q$ for all i. The QUEA in this case is denoted by $u_q(su(\ell+1))$.

Take the collection of matrix entries of all finite-dimensional unitarizable $u_q(su(\ell+1))$-modules. The algebra generated by these gets a natural Hopf*-structure as the dual of $u_q(su(\ell+1))$. One can also put a natural C^*-norm on this. Upon completion with respect to this norm, one gets a unital C^*-algebra that plays the role of the algebra of continuous functions on $SU_q(\ell+1)$. For a detailed account of this, refer to chapter 3, [9]. In [11], Woronowicz gave a different description of this C^*-algebra. which was later shown by Rosso ([10]) to be equivalent to the earlier one.

For remainder of this article, we will take G to be $SU_q(\ell+1)$ and $\mathcal{A}$ will be the C^*-algebra of continuous functions on G.

Gelfand-Tsetlin tableaux. Irreducible unitary representations of the group $SU_q(\ell+1)$ are indexed by Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$, where λ_i's are nonnegative integers, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell+1}$ (Theorem 1.5, [11]). Write $\mathcal{H}_\lambda$ for the Hilbert space where the irreducible λ acts. There are various ways of indexing the basis elements of $\mathcal{H}_\lambda$. The one we will use is due to Gelfand and Tsetlin. According to their prescription, basis elements for $\mathcal{H}_\lambda$ are parametrized by arrays of the form

$$\mathbf{r} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,\ell} & r_{1,\ell+1} \\ r_{21} & r_{22} & \cdots & r_{2,\ell} & \\ & \cdots & & & \\ r_{\ell,1} & r_{\ell,2} & & & \\ r_{\ell+1,1} & & & & \end{pmatrix},$$

where r_{ij}'s are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \ldots, \ell+1$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j. Such arrays are known as Gelfand-Tsetlin tableaux, to be abreviated as GT tableaux for the rest of this section. For a GT tableaux $\mathbf{r}$, the symbol $\mathbf{r}_{i\cdot}$ will denote its i^{th} row. It is well-known that two representations indexed respectively by λ and λ' are equivalent if and only if $\lambda_j - \lambda'_j$ is independent of j ([11]). Thus one gets an equivalence relation on the set of Young tableaux $\{\lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell+1}, \lambda_j \in \mathbb{N}\}$. This, in turn, induces an equivalence relation on the set of all GT tableaux $\Gamma = \{\mathbf{r} : r_{ij} \in \mathbb{N}, r_{ij} \geq r_{i+1,j} \geq r_{i,j+1}\}$: one says $\mathbf{r}$ and $\mathbf{s}$ are equivalent if $r_{ij} - s_{ij}$ is independent of i and j. By Γ we will mean the above set modulo this equivalence.

We will denote by u^λ the irreducible unitary indexed by λ, $\{e^\lambda_{\mathbf{r}} : \mathbf{r}_{1\cdot} = \lambda\}$ will denote an orthonormal basis for $\mathcal{H}_\lambda$ and $u^\lambda_{\mathbf{rs}}$ will stand for the matrix entries of u^λ in this basis. The symbol $\mathbb{1}$ will denote the Young tableaux $(1, 0, \ldots, 0)$. We will often omit the symbol $\mathbb{1}$ and just write u in order to denote $u^{\mathbb{1}}$. Notice that any GT tableaux $\mathbf{r}$ with first row $\mathbb{1}$ must be, for some $i \in \{1, 2, \ldots, \ell+1\}$, of the form (r_{ab}), where

$$r_{ab} = \begin{cases} 1 & \text{if } 1 \leq a \leq i \text{ and } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus such a GT tableaux is uniquely determined by the integer i. We will often write just i for this GT tableaux $\mathbf{r}$. Thus for example, a typical matrix entry of $u^{\mathbb{1}}$ will be written simply as u_{ij}.

Let $\mathbf{r} = (r_{ab})$ be a GT tableaux. Let $H_{ab}(\mathbf{r}) := r_{a+1,b} - r_{a,b+1}$ and $V_{ab}(\mathbf{r}) := r_{ab} - r_{a+1,b}$.

An element $\mathbf{r}$ of Γ is completely specified by the following differences

$$\mathbf{D}(\mathbf{r}) = \begin{pmatrix} V_{11}(\mathbf{r}) & H_{11}(\mathbf{r}) & H_{12}(\mathbf{r}) & \cdots & H_{1,\ell-1}(\mathbf{r}) & H_{1,\ell}(\mathbf{r}) \\ V_{21}(\mathbf{r}) & H_{21}(\mathbf{r}) & H_{22}(\mathbf{r}) & \cdots & H_{2,\ell-1}(\mathbf{r}) & \\ & \cdots & & & & \\ V_{\ell,1}(\mathbf{r}) & H_{\ell,1}(\mathbf{r}) & & & & \end{pmatrix}.$$

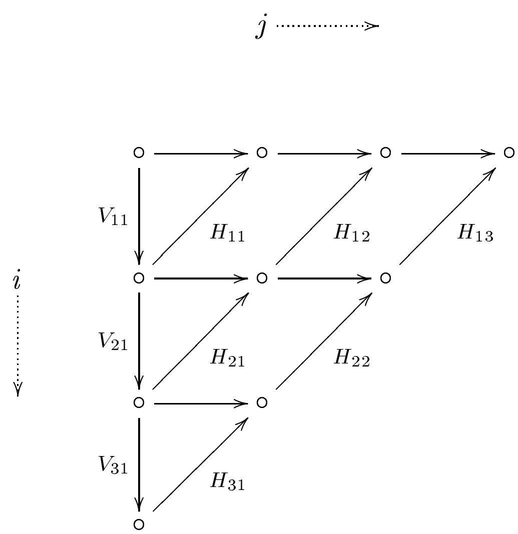
The differences satisfy the following inequalities

$$\sum_{k=0}^{b} H_{a-k,k+1}(\mathbf{r}) \leq V_{a+1,1}(\mathbf{r}) + \sum_{k=0}^{b} H_{a-k+1,k+1}(\mathbf{r}), \quad 1 \leq a \leq \ell, \ \ 0 \leq b \leq a-1. \qquad (2.1.1)$$

Conversely, if one has an array of the form

$$\begin{pmatrix} V_{11} & H_{11} & H_{12} & \cdots & H_{1,\ell-1} & H_{1,\ell} \\ V_{21} & H_{21} & H_{22} & \cdots & H_{2,\ell-1} & \\ & \cdots & & & & \\ V_{\ell,1} & H_{\ell,1} & & & & \end{pmatrix},$$

where V_{ij}'s and H_{ij}'s are in $\mathbb{N}$ and obey the inequalities (2.1.1), then the above array is of the form $\mathbf{D}(\mathbf{r})$ for some GT tableaux $\mathbf{r}$. Thus the quantities V_{a1} and H_{ab} give a coordinate system for elements in Γ. The following diagram explains this new coordinate system. The hollow circles stand for the r_{ij}'s. The entries are decreasing along the direction of the arrows, and the V_{ij}'s and the H_{ij}'s are the difference between the two endpoints of the corresponding arrows.



Clebsch-Gordon coefficients. Look at the representation $u^{\mathbb{1}} \otimes u^{\lambda}$ acting on $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_{\lambda}$. The representation decomposes as a direct sum $\oplus_{\mu} u^{\mu}$, i.e. one has a corresponding decomposition $\oplus_{\mu} \mathcal{H}_{\mu}$ of $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_{\lambda}$. Thus one has two orthonormal bases $\{e^{\mu}_{\mathbf{s}}\}$ and $\{e^{\mathbb{1}}_{i} \otimes e^{\lambda}_{\mathbf{r}}\}$. The Clebsch-Gordon coefficient $C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{s})$ is defined to be the inner product $\langle e^{\mu}_{\mathbf{s}}, e^{\mathbb{1}}_{i} \otimes e^{\lambda}_{\mathbf{r}} \rangle$. Since $\mathbb{1}$, λ

and μ are just the first rows of i, $\mathbf{r}$ and $\mathbf{s}$ respectively, we will often denote the above quantity just by $C_q(i,\mathbf{r},\mathbf{s})$.

Next, we will compute the quantities $C_q(i,\mathbf{r},\mathbf{s})$. We will use the calculations given in ([8], pp. 220), keeping in mind that for our case (i.e. for $SU_q(\ell+1)$), the top right entry of the GT tableaux is zero.

Let $M=(m_1,m_2,\ldots,m_i)\in\mathbb{N}^i$ be such that $1\le m_j\le \ell+2-j$. Denote by $M(\mathbf{r})$ the tableaux $\mathbf{s}$ defined by

$$s_{jk}=\begin{cases} r_{jk}+1 & \text{if } k=m_j,\ 1\le j\le i,\\ r_{jk} & \text{otherwise.}\end{cases} \tag{2.1.2}$$

With this notation, observe now that $C_q(i,\mathbf{r},\mathbf{s})$ will be zero unless $\mathbf{s}$ is $M(\mathbf{r})$ for some $M\in\mathbb{N}^i$. (One has to keep in mind though that not all tableaux of the form $M(\mathbf{r})$ is a valid GT tableaux)

From ([8], pp. 220), we have

$$C_q(i,\mathbf{r},M(\mathbf{r}))=\prod_{a=1}^{i-1}\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_{m_a}\\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_{m_{a+1}}\end{array}\right\rangle\times\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{i\cdot} & \mathbf{r}_{i\cdot}+e_{m_i}\\ (0,\mathbf{0}) & \mathbf{r}_{i+1\cdot} & \mathbf{r}_{i+1\cdot}\end{array}\right\rangle, \tag{2.1.3}$$

where e_k stands for a vector (in the appropriate space) whose k^{th} coordinate is 1 and the rest are all zero, and

$$\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j\\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_k\end{array}\right\rangle^2 = q^{-r_{aj}+r_{a+1,k}-k+j}\times\prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a}\frac{[r_{a,i}-r_{a+1,k}-i+k]_q}{[r_{a,i}-r_{a,j}-i+j]_q}$$
$$\times\prod_{\substack{i=1\\ i\neq k}}^{\ell+1-a}\frac{[r_{a+1,i}-r_{a,j}-i+j-1]_q}{[r_{a+1,i}-r_{a+1,k}-i+k-1]_q}, \tag{2.1.4}$$

$$\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j\\ (0,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}\end{array}\right\rangle^2 = q^{\left(j+1+\sum_{i=1}^{\ell+1-a}r_{a+1,i}-\sum_{\substack{i=1\\ i\neq j}}^{\ell+2-a}r_{a,i}\right)}$$
$$\times\left(\frac{\prod_{i=1}^{\ell+1-a}[r_{a+1,i}-r_{aj}-i+j-1]_q}{\prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a}[r_{a,i}-r_{aj}-i+j]_q}\right), \tag{2.1.5}$$

where for an integer n, $[n]_q$ denotes the q-number $(q^n-q^{-n})/(q-q^{-1})$. After some lengthy but straightforward computations, we get the following two equations:

$$\left|\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j\\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_k\end{array}\right\rangle\right| = A'q^A, \tag{2.1.6}$$

$$\left|\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j\\ (0,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}\end{array}\right\rangle\right| = B'q^B, \tag{2.1.7}$$

where

$$A=\begin{cases}\sum_{j\wedge k<b<j\vee k}(r_{a+1,b}-r_{a,b})+(r_{a+1,j\wedge k}-r_{a,j\vee k}) & \text{if } j\neq k,\\ 0 & \text{if } j=k.\end{cases}$$

$$= \sum_{j\wedge k\le b<j\vee k} (r_{a+1,b} - r_{a,b+1}) + 2 \sum_{k<b<j} (r_{a,b} - r_{a+1,b})$$
$$= \sum_{j\wedge k\le b<j\vee k} H_{ab}(\mathbf{r}) + 2 \sum_{k<b<j} V_{ab}(\mathbf{r}). \tag{2.1.8}$$
$$B = \sum_{j\le b<\ell+2-a} H_{ab}(\mathbf{r}), \tag{2.1.9}$$

and A' and B' both lie between two positive constants independent of i, $\mathbf{r}$ and M.

Combining these, one gets

$$C_q(i, \mathbf{r}, M(\mathbf{r})) = \text{const} \cdot q^{C(\mathbf{r},M)}, \tag{2.1.10}$$

where

$$C(\mathbf{r}, M) = \sum_{a=1}^{i-1} \left(\sum_{m_a\wedge m_{a+1}\le b<m_a\vee m_{a+1}} H_{ab}(\mathbf{r}) + 2 \sum_{m_{a+1}<b<m_a} V_{ab}(\mathbf{r}) \right) + \sum_{m_i\le b<\ell+2-i} H_{ib}(\mathbf{r}). \tag{2.1.11}$$

2.2 Left multiplication operators

The matrix entries $u^\lambda_{\mathbf{rs}}$ form a complete orthogonal set of vectors in $L_2(G)$. Write $e^\lambda_{\mathbf{rs}}$ for $\|u^\lambda_{\mathbf{rs}}\|^{-1} u^\lambda_{\mathbf{rs}}$. Then the $e^\lambda_{\mathbf{rs}}$'s form a complete orthonormal basis for $L_2(G)$. Let π denote the representation of $\mathcal{A}$ on $L_2(G)$ by left multiplications. We will now derive an expression for $\pi(u_{ij}) e^\lambda_{\mathbf{rs}}$.

From the definition of matrix entries and that of the CG coefficients, one gets

$$u^\rho e(\rho, \mathbf{t}) = \sum_{\mathbf{s}} u^\rho_{\mathbf{st}} e(\rho, \mathbf{s}), \tag{2.2.1}$$

$$e(\mu, \mathbf{n}) = \sum_{j,\mathbf{s}} C_q(j, \mathbf{s}, \mathbf{n}) e(1\!\!1, j) \otimes e(\lambda, \mathbf{s}). \tag{2.2.2}$$

Apply $u \otimes u^\lambda$ on both sides and note that $u \otimes u^\lambda$ acts on $e(\mu, \mathbf{n})$ as u^μ:

$$\sum_{\mathbf{m}} u^\mu_{\mathbf{mn}} e(\mu, \mathbf{m}) = \sum_{j,\mathbf{s}} \sum_{i,\mathbf{r}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}} e(1\!\!1, i) \otimes e(\lambda, \mathbf{r}). \tag{2.2.3}$$

Next, use (2.2.2) to expand $e(\mu, \mathbf{m})$ on the left hand side to get

$$\sum_{i,\mathbf{r},\mathbf{m}} u^\mu_{\mathbf{mn}} C_q(i, \mathbf{r}, \mathbf{m}) e(1\!\!1, i) \otimes e(\lambda, \mathbf{r}) = \sum_{j,\mathbf{s}} \sum_{i,\mathbf{r}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}} e(1\!\!1, i) \otimes e(\lambda, \mathbf{r}). \tag{2.2.4}$$

Equating coefficients, one gets

$$\sum_{\mathbf{m}} C_q(i, \mathbf{r}, \mathbf{m}) u^\mu_{\mathbf{mn}} = \sum_{j,\mathbf{s}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}}. \tag{2.2.5}$$

Now using orthogonality of the matrix $((C_q(1\!\!1, \lambda, \mu; j, \mathbf{s}, \mathbf{n})))_{(\mu,\mathbf{n}),(j,\mathbf{s})}$, we obtain

$$u_{ij} u^\lambda_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(i, \mathbf{r}, \mathbf{m}) C_q(j, \mathbf{s}, \mathbf{n}) u^\mu_{\mathbf{mn}}. \tag{2.2.6}$$

From ([8], pp. 441), one has $\|u^{\lambda}_{\mathbf{rs}}\| = d_{\lambda}^{-\frac{1}{2}} q^{-(\rho,\lambda(\mathbf{r}))}$, where ρ is the half-sum of positive roots, $\lambda(\mathbf{r})$ is the weight such that $e(\lambda,\mathbf{r})$ belongs to the corresponding weight space (of V_λ), and $d_\lambda = \sum_{\mathbf{r}:\lambda(\mathbf{r})\in P(\lambda)} q^{2(\rho,\lambda(\mathbf{r}))}$, $P(\lambda)$ being the set of weights corresponding to the weight space decomposition of V_λ.

Therefore

$$\pi(u_{ij})e^{\lambda}_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n}) d_\lambda^{\frac{1}{2}} d_\mu^{-\frac{1}{2}} q^{(\rho,\lambda(\mathbf{r}))-(\rho,\mu(\mathbf{m}))} e^{\mu}_{\mathbf{mn}}. \tag{2.2.7}$$

Write

$$k(\mathbf{r},\mathbf{m}) = d_\lambda^{\frac{1}{2}} d_\mu^{-\frac{1}{2}} q^{(\rho,\lambda(\mathbf{r}))-(\rho,\mu(\mathbf{m}))}. \tag{2.2.8}$$

Lemma 2.2.1 *There exist constants $K_2 > K_1 > 0$ such that $K_1 < k(\mathbf{r}, M(\mathbf{r})) < K_2$ for all $\mathbf{r}$.*

Proof: Observe that ([3], pp-365)

$$d_\lambda = \prod_{1\le i\le j\le \ell+1} \frac{(\lambda_i-\lambda_j+j-i)_q}{(j-i)_q}.$$

Therefore one gets

$$\frac{d_\lambda}{d_{\lambda+e_k}} = \prod_{j:k<j} \frac{(\lambda_k-\lambda_j+j-k)_q}{(\lambda_k-\lambda_j+j-k+1)_q} \times \prod_{i:i<k} \frac{(\lambda_i-\lambda_k+k-i)_q}{(\lambda_i-\lambda_k+k-i-1)_q}.$$

There are ℓ terms in the above product, and each term lies between two positive quantities that depend just on q. Also one can compute the quantity $(\rho,\lambda(\mathbf{r}))-(\rho,\mu(\mathbf{m}))$ directly, and it turns out to be bounded. Therefore the result follows. □

2.3 Boundedness of commutators

Let D be an equivariant Dirac operator acting on $L_2(G)$. It follows from the discussion in the beginning of this section that D must be of the form

$$e^{\lambda}_{\mathbf{rs}} \mapsto d(\mathbf{r}) e^{\lambda}_{\mathbf{rs}}, \tag{2.3.1}$$

(Here, for a Young tableaux λ, N_λ is the set of all GT tableaux, modulo the appropriate equivalence relation, with top row λ). Then we have

$$[D,\pi(u_{ij})]e^{\lambda}_{\mathbf{rs}} = \sum (d(\mathbf{m})-d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})k(\mathbf{r},\mathbf{m})e^{\mu}_{\mathbf{mn}}. \tag{2.3.2}$$

Therefore the condition for boundedness of commutators reads as follows:

$$|(d(\mathbf{m})-d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})k(\mathbf{r},\mathbf{m})| < c, \tag{2.3.3}$$

where c is independent of i, j, λ, μ, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{m}$ and $\mathbf{n}$.

Using lemma 2.2.1, we get

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(1\!\!1, \lambda, \mu; i, \mathbf{r}, \mathbf{m})C_q(1\!\!1, \lambda, \mu; j, \mathbf{s}, \mathbf{n})| < c. \tag{2.3.4}$$

Choosing j, $\mathbf{s}$ and $\mathbf{n}$ suitably, one can ensure that (2.3.4) implies the following:

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(1\!\!1, \lambda, \mu; i, \mathbf{r}, \mathbf{m})| < c. \tag{2.3.5}$$

It is not too difficult to show that this condition is also sufficient for the boundedness of the commutators $[D, u_{ij}]$.

From (2.1.10), one gets

$$|d(\mathbf{r}) - d(M(\mathbf{r}))| \leq cq^{-C(\mathbf{r},M)}. \tag{2.3.6}$$

Let us next form a graph $\mathcal{G}_c$ as described in section 1 by connecting two elements $\mathbf{r}$ and $\mathbf{r}'$ if $|d(\mathbf{r}) - d(\mathbf{r}')| < c$. We will assume this graph to be sign determining. In other words, we will assume the existence of a partition (Γ^+, Γ^-) that does not admit any infinite ladder. For any subset F of Γ, we will denote by $F^\pm$ the sets $F \cap \Gamma^\pm$. Our next job is to study this graph in more detail using the boundedness conditions above. Let us start with a few definitions and notations. By an **elementary move**, we will mean a map M from some subset of Γ to Γ such that γ and $M(\gamma)$ are connected by an edge. A **move** will mean a composition of a finite number of elementary moves. If M_1 and M_2 are two moves, M_1M_2 and M_2M_1 will in general be different. For a family of moves $M_1, M_2, \ldots, M_r$, we will denote by

$$\sum_{\substack{j=1\\ \rightarrow}}^{r} M_j \quad \text{and} \quad \sum_{\substack{j=1\\ \leftarrow}}^{r} M_j$$

the moves $M_1M_2\ldots M_r$ and $M_r\ldots M_2M_1$ respectively. For a nonnegative integer n and a move M, we will denote by nM the move obtained by applying M successively n times. Of special interest to us will be moves of the form $M : \mathbf{r} \mapsto \mathbf{s}$, where $\mathbf{s}$ is given by (2.1.2) We will use the vector $(m_1, \ldots, m_k)$ to denote M. The following families of moves will be particularly useful to us:

$$M_{ik} = (i, i-1, \ldots, i-k+1) \in \mathbb{N}^k, \quad N_{ik} = (\underbrace{i+1, \ldots, i+1}_{k}, i, i, \ldots, i) \in \mathbb{N}^{\ell+2-i}.$$

For describing a path in our graph, we will often use phrases like 'apply the move $\sum_{\substack{j=1\\ \rightarrow}}^{k} M_j$ to go from $\mathbf{r}$ to $\mathbf{s}$'. This will refer to the path given by

$$\Big(\mathbf{r},\ M_k(\mathbf{r}), M_{k-1}M_k(\mathbf{r}),\ \ldots,\ M_1M_2\ldots M_k(\mathbf{r}) = \mathbf{s}\Big).$$

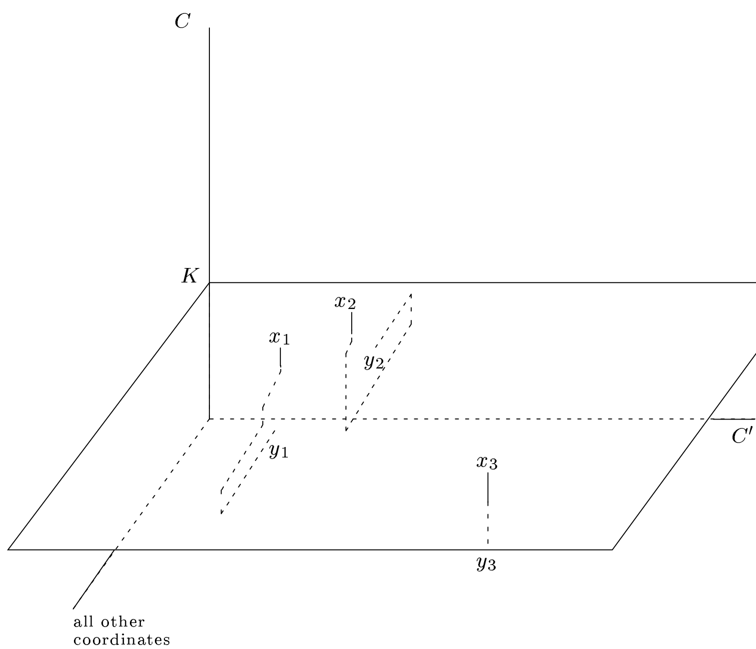
We will need the following consequence of (2.3.6) subsequently.

Lemma 2.3.1 *Let N_{jk} and M_{ik} be the moves defined above. Then*

1. $|d(\mathbf{r}) - d(N_{j0}(\mathbf{r}))| \leq c$,
2. $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq cq^{-\sum_{a=1}^{k-1} H_{a,i+1-a} - \sum_{b=i}^{\ell} H_{k,b+k-1}}$. *In particular, if* $H_{a,i+1-a}(\mathbf{r}) = 0$ *for* $1 \leq a \leq k-1$ *and* $H_{k,b+k-1}(\mathbf{r}) = 0$ *for* $i \leq b \leq \ell$, *then* $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq c$.

2.4 Characterization of sign D

In this section, we will use lemma 2.3.1 to prove a characterization theorem for the sign of the operator D. The main ingredients in the proof are the finiteness of exactly one of the sets F^+ and F^- for appropriately chosen subsets F of Γ. General form of the argument for proving this will be as follows: for a carefully chosen coordinate C (in the present case, C would be one of the V_{a1}'s or H_{ab}'s), a sweepout argument will show that any γ can be connected by a path, throughout which $C(\cdot)$ remains constant, to another point γ' for which $C(\gamma') = C(\gamma)$ and all other coordinates of γ' are zero. This would help connect any two points γ and δ by a path such that $C(\cdot)$ would lie between $C(\gamma)$ and $C(\delta)$ on the path. This would finally result in the finiteness of at least one (and hence exactly one) of $C(F^+)$ and $C(F^-)$. Next, assuming one of these, say $C(F^-)$ is finite, one shows that for any other coordinate C', $C'(F^-)$ is also finite. This is done as follows. If $C'(F^-)$ is infinite, one chooses elements $y_n \in F^-$ with $C'(y_n) < C'(y_{n+1})$ for all n. Now starting at each y_n, produce paths keeping the C'-coordinate constant and taking the C-coordinate above the plane $C(\cdot) = K$, where $C(F^-) \subseteq [-K, K]$. This will produce an infinite ladder. The argument is explained in the following diagram.



Our next job is to define an important class of subsets of Γ. Observe that lemma 2.3.1 tells us that for any $\mathbf{r}$ and any j, the points $\mathbf{r}$ and $N_{j0}(\mathbf{r})$ are connected by an edge, whenever $N_{j0}(\mathbf{r})$ is a GT tableaux. Let $\mathbf{r}$ be an element of Γ. Define the **free plane passing through r** to be the minimal subset of Γ that contains $\mathbf{r}$ and is closed under application of the moves N_{j0}. We will denote this set by $\mathscr{F}_{\mathbf{r}}$. The following is an easy consequence of this definition.

Lemma 2.4.1 *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be two GT tableaux. Then* $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$ *if and only if* $V_{a,1}(\mathbf{r}) = V_{a,1}(\mathbf{s})$ *for all* a *and for each* b*, the difference* $H_{a,b}(\mathbf{r}) - H_{a,b}(\mathbf{s})$ *is independent of* a.

Corollary 2.4.2 *Let $\mathbf{r}, \mathbf{s} \in \Gamma$. Then either $\mathscr{F}_{\mathbf{r}} = \mathscr{F}_{\mathbf{s}}$ or $\mathscr{F}_{\mathbf{r}} \cap \mathscr{F}_{\mathbf{s}} = \phi$.*

Let $\mathbf{r} \in \Gamma$. For $1 \leq j \leq \ell + 1$, define a_j to be an integer such that $H_{a_j,j}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Note three things here:
1. definition of a_j depends on $\mathbf{r}$,
2. for a given j and given $\mathbf{r}$, a_j need not be unique, and
3. if $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$, then for each j, the set of k's for which $H_{kj}(\mathbf{s}) = \min_i H_{ij}(\mathbf{s})$ is same as the set of all k's for which $H_{kj}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Therefore, the a_j's can be chosen in a manner such that they remain the same for all elements lying on a given free plane.

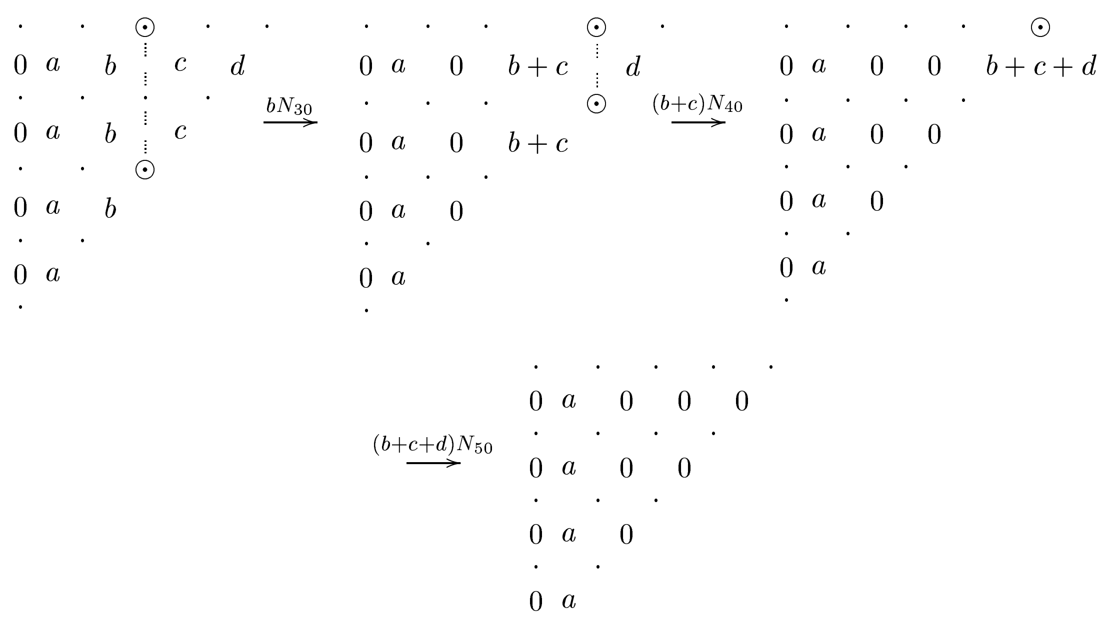
Lemma 2.4.3 *Let $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$. Let $\mathbf{s}'$ be another GT tableaux given by*

$$V_{a1}(\mathbf{s}') = V_{a1}(\mathbf{s}) \textit{ and } H_{a1}(\mathbf{s}') = H_{a1}(\mathbf{s}) \textit{ for all } a, \quad H_{a_b,b}(\mathbf{s}') = 0 \textit{ for all } b > 1,$$

where the a_j's are as defined above. Then there is a path in $\mathscr{F}_{\mathbf{r}}$ from $\mathbf{s}$ to $\mathbf{s}'$ such that $H_{11}(\cdot)$ remains constant throughout this path.

Proof: Apply the move $\sum_{\underset{\leftarrow}{b=2}}^{\ell} \left(\sum_{j=2}^{b} H_{a_j,j}(\mathbf{s}) \right) N_{b+1,0}$. $\square$

The following diagram will help explain the steps involved in the above proof in the case where $\mathbf{r}$ is the constant tableaux.



A dotted line joining two circled dots signifies a move that increases the r_{ij}'s lying on the dotted line by one. Where there is one circled dot and no dotted line, it means one applies the move that raises the r_{ij} corresponding to the circled dot by one.

Proposition 2.4.4 *Let $\mathbf{r}$ be a GT tableaux. Then either $\mathscr{F}_{\mathbf{r}}^{+}$ is finite or $\mathscr{F}_{\mathbf{r}}^{-}$ is finite.*

Proof: Suppose, if possible, both $H_{11}(\mathscr{F}_{\mathbf{r}}^{+})$ and $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ are infinite. Then there exist two sequences of elements $\mathbf{r}_n$ and $\mathbf{s}_n$ with $\mathbf{r}_n \in \mathscr{F}_{\mathbf{r}}^{+}$ and $\mathbf{s}_n \in \mathscr{F}_{\mathbf{r}}^{-}$, such that

$$H_{11}(\mathbf{r}_1) < H_{11}(\mathbf{s}_1) < H_{11}(\mathbf{r}_2) < H_{11}(\mathbf{s}_2) < \cdots.$$

Now starting from $\mathbf{r}_n$, employ the forgoing lemma to reach a point $\mathbf{r}_n' \in \mathscr{F}_{\mathbf{r}}$ for which

$$V_{a1}(\mathbf{r}_n') = V_{a1}(\mathbf{r}_n) \text{ and } H_{a1}(\mathbf{r}_n') = H_{a1}(\mathbf{r}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{r}_n') = 0 \text{ for all } b > 1.$$

Similarly, start at $\mathbf{s}_n$ and go to a point $\mathbf{s}_n' \in \mathscr{F}_{\mathbf{r}}$ for which

$$V_{a1}(\mathbf{s}_n') = V_{a1}(\mathbf{s}_n) \text{ and } H_{a1}(\mathbf{s}_n') = H_{a1}(\mathbf{s}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{s}_n') = 0 \text{ for all } b > 1.$$

Now use the move N_{10} to get to $\mathbf{s}_n'$ from $\mathbf{r}_n'$. The paths thus constructed are all disjoint, because for the path from $\mathbf{r}_n$ to $\mathbf{s}_n$, the H_{11} coordinate lies between $H_{11}(\mathbf{r}_n)$ and $H_{11}(\mathbf{s}_n)$. This means $(\mathscr{F}_{\mathbf{r}}^{+}, \mathscr{F}_{\mathbf{r}}^{-})$ admits an infinite ladder. So one of the sets $H_{11}(\mathscr{F}_{\mathbf{r}}^{+})$ and $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ must be finite. Let us assume that $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ is finite.

Let us next show that for any $b > 1$, $H_{ab}(\mathscr{F}_{\mathbf{r}}^{-})$ is finite. Let K be an integer such that $H_{11}(\mathbf{s}) < K$ for all $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}^{-}$. If $H_{ab}(\mathscr{F}_{\mathbf{r}}^{-})$ was infinite, there would exist elements $\mathbf{r}_n \in \mathscr{F}_{\mathbf{r}}^{-}$ such that

$$H_{ab}(\mathbf{r}_1) < H_{ab}(\mathbf{r}_2) < \cdots.$$

Now start at $\mathbf{r}_n$ and employ the move N_{10} successively K times to reach a point in $\mathscr{F}_{\mathbf{r}}^{+} = \mathscr{F}_{\mathbf{r}} \backslash \mathscr{F}_{\mathbf{r}}^{-}$. These paths will all be disjoint, as throughout the path, H_{ab} remains fixed.

Since the coordinates $(H_{11}, H_{12}, \ldots, H_{1,\ell})$ completely specify a point in $\mathscr{F}_{\mathbf{r}}$, it follows that $\mathscr{F}_{\mathbf{r}}^{-}$ is finite. $\square$

Next we need a set that can be used for a proper indexing of the free planes. Such a set will be called a complementary axis.

Definition 2.4.5 A subset $\mathscr{C}$ of Γ is called a **complementary axis** if

1. $\cup_{\mathbf{r} \in \mathscr{C}} \mathscr{F}_{\mathbf{r}} = \Gamma$,

2. if $\mathbf{r}, \mathbf{s} \in \mathscr{C}$, and $\mathbf{r} \neq \mathbf{s}$, then $\mathscr{F}_{\mathbf{r}}$ and $\mathscr{F}_{\mathbf{s}}$ are disjoint.

Let us next give a choice of a complementary axis.

Lemma 2.4.6 *Define*

$$\mathscr{C} = \{\mathbf{r} \in \Gamma : \Pi_{a=1}^{\ell+1-b} H_{ab}(\mathbf{r}) = 0 \textit{ for } 1 \leq b \leq \ell\}.$$

The set $\mathscr{C}$ defined above is a complementary axis.

Proof: Let $\mathbf{s} \in \Gamma$. A sweepout argument almost identical to that used in lemma 2.4.3 (application of the move $\sum_{\underset{\leftarrow}{b=1}}^{\ell} \left(\sum_{j=1}^{b} H_{a_j,j}(\mathbf{s})\right) N_{b+1,0}$) will connect $\mathbf{s}$ to another element $\mathbf{s}'$ for which

$H_{a_b,b}(\mathbf{s}') = 0$ for $1 \le b \le \ell$ by a path that lies entirely on $\mathscr{F}_{\mathbf{s}}$. Clearly, $\mathbf{s}' \in \mathscr{C}$. Since $\mathbf{s}' \in \mathscr{F}_{\mathbf{s}}$, by corollary 2.4.2, $\mathbf{s} \in \mathscr{F}_{\mathbf{s}'}$.

It remains to show that if $\mathbf{r}$ and $\mathbf{s}$ are two distinct elements of $\mathscr{C}$, then $\mathbf{s} \notin \mathscr{F}_{\mathbf{r}}$. Since $\mathbf{r} \neq \mathbf{s}$, there exist two integers a and b, $1 \le b \le \ell$ and $1 \le a \le \ell + 2 - b$, such that $H_{ab}(\mathbf{r}) \neq H_{ab}(\mathbf{s})$. Observe that $H_{1\ell}(\cdot)$ must be zero for both, as they are members of $\mathscr{C}$. So b can not be ℓ here. Next we will produce two integers i and j such that the differences $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and $H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct. If there is an integer k for which $H_{kb}(\mathbf{r}) = H_{kb}(\mathbf{s}) = 0$, then take $i = a$, $j = k$. If not, there would exist two integers i and j such that $H_{ib}(\mathbf{r}) = 0$, $H_{ib}(\mathbf{s}) > 0$ and $H_{jb}(\mathbf{r}) > 0$, $H_{jb}(\mathbf{s}) = 0$. Take these i and j. Since $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and $H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct, by lemma 2.4.1, $\mathbf{r}$ and $\mathbf{s}$ can not lie on the same free plane. $\square$

Lemma 2.4.7 *Let $\mathbf{r}$ be a GT tableaux. Let $\mathbf{s}$ be the GT tableaux defined by the prescription*

$$V_{a1}(\mathbf{s}) = V_{a1}(\mathbf{r}) \text{ for all } a, \quad H_{ab}(\mathbf{s}) = H_{ab}(\mathbf{r}) \text{ for all } a \ge 2, \text{ for all } b, \quad H_{1,b}(\mathbf{s}) = 0 \text{ for all } b.$$

Then there is a path from $\mathbf{r}$ to $\mathbf{s}$ such that $V_{a1}(\cdot)$ remains constant throughout the path.

Proof: Apply the move $\sum_{\substack{b=1 \\ \rightarrow}}^{\ell} H_{1,b}(\mathbf{r}) M_{b+1,1}$. $\square$

The above lemma is actually the first step in the following slightly more general sweepout algorithm.

Lemma 2.4.8 *Let $\mathbf{r}$ be a GT tableaux. Let $\mathbf{s}$ be the GT tableaux defined by the prescription*

$$V_{11}(\mathbf{s}) = V_{11}(\mathbf{r}), \quad V_{a1}(\mathbf{s}) = 0 \text{ for all } a > 1, \quad H_{ab}(\mathbf{s}) = 0 \text{ for all } a, b.$$

Then there is a path from $\mathbf{r}$ to $\mathbf{s}$ such that $V_{11}(\cdot)$ remains constant throughout the path.

Proof: Apply successively the moves

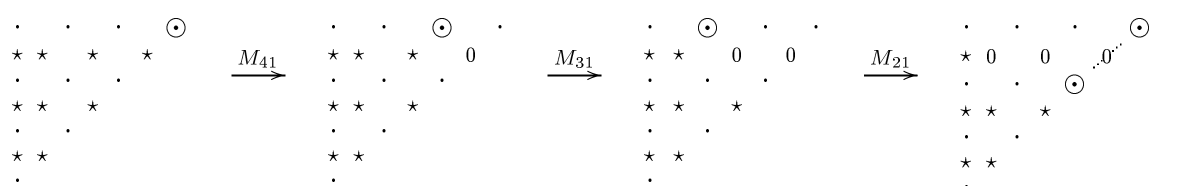
$$\sum_{\substack{b=1 \\ \rightarrow}}^{\ell} H_{1,b}(\mathbf{r}) M_{b+1,1}, \quad \sum_{\substack{b=1 \\ \rightarrow}}^{\ell-1} H_{2,b}(\mathbf{r}) M_{b+2,2}, \quad \ldots, \quad H_{\ell,1}(\mathbf{r}) M_{\ell+1,\ell},$$

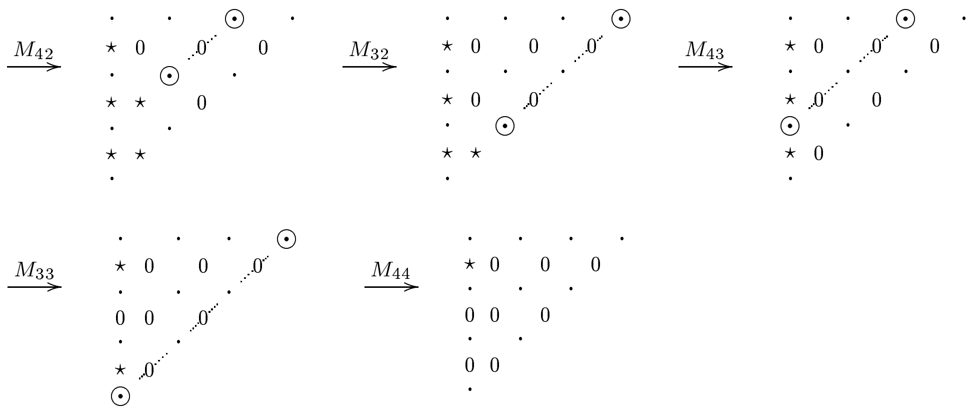
followed by

$$V_{21}(\mathbf{r}) M_{33}, \quad (V_{21}(\mathbf{r}) + V_{31}(\mathbf{r})) M_{44}, \quad \ldots, \quad \left(\sum_{a=2}^{\ell} V_{a1}(\mathbf{r}) \right) M_{\ell+1,\ell+1}. \tag{2.4.1}$$

$\square$

The following diagram will help explain the procedure described above in a simple case.





Corollary 2.4.9 $|d(\mathbf{r})| = O(r_{11})$.

Proof: If one employs the sequence of moves

$$V_{11}(\mathbf{r})M_{22}, \quad (V_{11}(\mathbf{r}) + V_{21}(\mathbf{r}))M_{33}, \quad \ldots, \quad \left(\sum_{a=1}^{\ell} V_{a1}(\mathbf{r})\right) M_{\ell+1,\ell+1}$$

instead of the sequence given in (2.4.1), one would reach the constant (or zero) tableaux. Total length of this path from $\mathbf{r}$ to the zero tableaux is

$$\sum_{a=1}^{\ell} \sum_{b=1}^{\ell+1-a} H_{ab}(\mathbf{r}) + \sum_{b=1}^{\ell} \sum_{a=1}^{b} V_{a1}(\mathbf{r}),$$

which can easily be shown to be bounded by ℓr_{11}. □

Remark 2.4.10 Let D be the following operator:

$$D : e^{\lambda}_{\mathbf{r},\mathbf{s}} \mapsto r_{11} e^{\lambda}_{\mathbf{r},\mathbf{s}} \tag{2.4.2}$$

Then $(\mathcal{A}, \mathcal{H}, D)$ is an equivariant $\ell(\ell+2)$-summable odd spectral triple. Moreover, if D is any equivariant odd Dirac Operator acting on the L_2 space of $SU_q(\ell+1)$, then D cannot be p-summable for $p < \ell(\ell+2)$. However, this D has trivial sign, and consequently trivial K-homology class.

Both the statements above follow from corollary 2.4.9 and the following two elementary facts:
1. The number of Young tableaux with ℓ or less rows and with n cells in the top row is $O(n^{\ell-1})$.
2. the dimension of an irreducible representation corresponding to a Young tableaux with ℓ or less rows and n cells in the top row is $O(n^{\frac{1}{2}\ell(\ell+1)})$.

Lemma 2.4.11 *The sets $V_{11}(\Gamma^+)$ and $V_{11}(\Gamma^-)$ can not both be infinite.*

Proof: If both the sets are infinite, then one can choose two sequences of points $\mathbf{r}_n$ and $\mathbf{s}_n$ such that $\mathbf{r}_n \in \Gamma^+$, $\mathbf{s}_n \in \Gamma^-$ and

$$V_{11}(\mathbf{r}_1) < V_{11}(\mathbf{s}_1) < V_{11}(\mathbf{r}_2) < V_{11}(\mathbf{s}_2) < \ldots.$$

Start at $\mathbf{r}_n$ and use lemma 2.4.8 above to reach a point $\mathbf{r}'_n$ for which $V_{11}(\mathbf{r}'_n) = V_{11}(\mathbf{r}_n)$ and all other coordinates are zero through a path where the V_{11} coordinate remains constant. Similarly, from $\mathbf{s}_n$, go to a point $\mathbf{s}'_n$ for which $V_{11}(\mathbf{s}'_n) = V_{11}(\mathbf{s}_n)$ and all other coordinates are zero. Now apply the move $(V_{11}(\mathbf{s}_n) - V_{11}(\mathbf{r}_n))M_{11}$ to go from $\mathbf{r}'_n$ to $\mathbf{s}'_n$. This will give us a path p_n from $\mathbf{r}_n$ to $\mathbf{s}_n$ on which $V_{11}(\cdot)$ remains between $V_{11}(\mathbf{r}_n)$ and $V_{11}(\mathbf{s}_n)$. Therefore all the paths p_n are disjoint. Thus (Γ^+, Γ^-) admits an infinite ladder. So at least one of $V_{11}(\Gamma^+)$ and $V_{11}(\Gamma^-)$ must be finite. $\square$

Lemma 2.4.12 *Let C be any of the coordinates V_{a1} or H_{ab} where $a > 1$. If $V_{11}(\Gamma^-)$ is finite, then $C(\Gamma^-)$ is also finite.*

Proof: Assume K is a positive integer such that $V_{11}(\Gamma^-) \subseteq [0, K]$. Now suppose, if possible, that $C(\Gamma^-)$ is infinite. Let $\mathbf{r}_n$ be a sequence of points in Γ^- such that

$$C(\mathbf{r}_1) < C(\mathbf{r}_2) < \ldots.$$

Start at $\mathbf{r}_n$, and use lemma 2.4.7 to reach a point $\mathbf{r}'_n$ and then apply M_{11} for $K+1$ times to get to a point $\mathbf{s}_n$ for which $V_{11}(\mathbf{s}_n) > K$. Throughout this path, $C(\cdot)$ is constant, so that the paths are all disjoint. Since $V_{11}(\mathbf{s}_n) > K$, we have $\mathbf{s}_n \in \Gamma^+$. Thus this gives us an infinite ladder for (Γ^+, Γ^-), which is impossible. $\square$

Lemma 2.4.13 *Suppose $H_{1\ell}(F)$ is bounded. If $V_{11}(\Gamma^-)$ is finite, then F^- is finite.*

Proof: The previous lemma, along with the assumption here tells us that the sets $V_{a1}(F^-)$ and $H_{a,\ell+1-a}(F^-)$ are all bounded for $1 \leq a \leq \ell$. Since for an $\mathbf{r} \in V$, one has $r_{11} = \sum_{a=1}^{\ell} V_{a1}(\mathbf{r}) + \sum_{a=1}^{\ell} H_{a,\ell+1-a}(\mathbf{r})$, the set $\{r_{11} : \mathbf{r} \in F^-\}$ is bounded. It follows that F^- is finite. $\square$

Corollary 2.4.14 *If $V_{11}(\Gamma^-)$ is finite, then $\mathscr{C}^-$ is finite.*

Proof: Follows from the observation that $H_{1\ell}(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathscr{C}$. $\square$

Theorem 2.4.15 *Let D be an equivariant Dirac operator on $L_2(SU_q(\ell+1))$. Then* sign *D must be of the form $2P - I$ or $I - 2P$ where P is, up to a compact perturbation, the projection onto the closed span of $\{e^{\lambda}_{\mathbf{r},\mathbf{s}} : \mathbf{r} \in \mathscr{F}_{\mathbf{r}_i}$ for some $i\}$, with $\mathbf{r}_1, \ldots, \mathbf{r}_k$ being a finite collection of GT-tableaux.*

Proof: Let $\mathscr{C}' = \{\mathbf{r} \in \mathscr{C} : \mathscr{F}^+_{\mathbf{r}} \neq \phi \neq \mathscr{F}^-_{\mathbf{r}}\}$. Let us first show that $\mathscr{C}'$ is finite, i.e. except for finitely many $\mathbf{r}$'s in $\mathscr{C}$, one has either $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^+$ or $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^-$. It follows from the argument used in the proof of lemma 2.4.6 that any two points on a free plane can be connected by a path lying entirely on the plane. If $\mathscr{C}'$ is infinite, one can easily produce an infinite ladder using this fact.

Next, observe that

1. either $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^+$ or $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^-$ for all $\mathbf{r} \in \mathscr{C}\backslash\mathscr{C}'$,
2. if $V_{11}(\Gamma^-)$ is finite, then $\mathscr{C}^-$ is finite, and
3. $\mathscr{C} \cap \mathscr{F}_{\mathbf{r}} = \{\mathbf{r}\}$.

It is now clear from the above observations that if $V_{11}(\Gamma^-)$ is finite, then except possibly for finitely many $\mathbf{r}$'s in $\mathscr{C}\backslash\mathscr{C}'$, one has $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^+$.

Finally, employing an appropriate compact perturbation, one can ensure that for each $\mathbf{r} \in \mathscr{C}'$, either $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^+$ or $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^-$. Hence the result. $\square$

As a consequence of this sign characterization, we now get the following theorem.

Theorem 2.4.16 *Let $\ell > 1$. Then there does not exist any equivariant Dirac operator on $L_2(G)$ with nontrivial sign.*

Proof: We will show that if P is as in the earlier theorem, then the commutators $[P, \pi(u_{ij})]$ can not all be compact.

Let us first prove it in the case when P is the projection onto the span of $\{e_{\mathbf{rs}} : \mathbf{r} \in \mathscr{F}_0\}$, where $\mathscr{F}_0$ is the free plane passing through the constant tableaux. We have

$$[P, \pi(u_{ij})]e_{\mathbf{rs}} = \begin{cases} P\pi(u_{ij})e_{\mathbf{rs}} & \text{if } \mathbf{r} \notin \mathscr{F}_0, \\ (P-I)\pi(u_{ij})e_{\mathbf{rs}} & \text{if } \mathbf{r} \in \mathscr{F}_0 \end{cases}.$$

Recall (subsection 2.2) the expression for $\pi(u_{ij})e_{\mathbf{rs}}$:

$$\pi(u_{ij})e_{\mathbf{rs}} = \sum_{\substack{R\in\mathbb{N}^i, S\in\mathbb{N}^j \\ R(1)=S(1)}} C_q(i, \mathbf{r}, R(\mathbf{r}))C_q(j, \mathbf{s}, S(\mathbf{s}))k(\mathbf{r}, R(\mathbf{r}))e_{R(\mathbf{r})S(\mathbf{s})}.$$

Hence for $\mathbf{r} \in \mathscr{F}_0$,

$$\begin{aligned} [P, \pi(u_{ij})]e_{\mathbf{rs}} &= (P-I)\pi(u_{ij})e_{\mathbf{rs}} \\ &= -\sum_{\substack{R\in\mathbb{N}^i, S\in\mathbb{N}^j \\ R(1)=S(1), R\neq N_{i0}}} C_q(i, \mathbf{r}, R(\mathbf{r}))C_q(j, \mathbf{s}, S(\mathbf{s}))k(\mathbf{r}, R(\mathbf{r}))e_{R(\mathbf{r})S(\mathbf{s})}. \end{aligned}$$

In particular, for $i = j = 1$, one gets

$$[P, \pi(u_{11})]e_{\mathbf{rs}} = -\sum_{k=1}^{\ell} C_q(1, \mathbf{r}, M_{k1}(\mathbf{r}))C_q(1, \mathbf{s}, M_{k1}(\mathbf{s}))k(\mathbf{r}, M_{k1}(\mathbf{r}))e_{M_{k1}(\mathbf{r})M_{k1}(\mathbf{s})}.$$

Now suppose $\mathbf{r} \in \mathscr{F}_0$ satisfies

$$r_{1,\ell} = 0 = r_{2,\ell} = r_{1,\ell+1}. \tag{2.4.3}$$

Then

$$\langle e_{M_{\ell 1}(\mathbf{r})M_{\ell 1}(\mathbf{r})}, [P, \pi(u_{11})]e_{\mathbf{rr}}\rangle = -C_q(1, \mathbf{r}, M_{\ell 1}(\mathbf{r}))^2 k(\mathbf{r}, M_{\ell 1}(\mathbf{r})).$$

It follows from (2.1.10) and (2.1.11) that $C_q(1, \mathbf{r}, M_{\ell 1}(\mathbf{r}))$ is bounded away from zero, so long as $\mathbf{r}$ obeys (2.4.3). We have also seen (lemma 2.2.1) that $k(\mathbf{r}, M_{\ell 1}(\mathbf{r}))$ is bounded away from zero. Now it is easy to see that if $\ell > 1$, then there are infinitely many choices of $\mathbf{r}$ satisfying (2.4.3) such that they all lie in $\mathscr{F}_0$. Therefore $[P, \pi(u_{11})]$ is not compact.

For more general P (as in the previous theorem), the idea would be similar, but this time one has to get hold of a positive integer n such that for any $\mathbf{r} \in \cup_{i=1}^k \mathscr{F}_{\mathbf{r}_i}$, $nM_{\ell 1}(\mathbf{r}) \notin \cup_{i=1}^k \mathscr{F}_{\mathbf{r}_i}$, and then compute $\langle e_{nM_{\ell 1}(\mathbf{r}) nM_{\ell 1}(\mathbf{r})}, (P - I)\pi(u_{11})^n e_{\mathbf{rr}} \rangle$. □

3 Second illustration

In this section, we will ignore the group nature of $SU_q(\ell+1)$ and focus only on the C^*-algebra $\mathcal{A} = C(SU_q(\ell+1))$. All irreducible representations of this C^*-algebra are well-known ([9]). We will take a large class of representations, which includes the irreducibles in particular, and use the scheme described in section 1 to prove that for a large majority of these representations, no Dirac operator with nontrivial sign exists that diagonalises nicely with respect to the canonical orthonormal basis.

3.1 Irreducible representations

The Weyl group for $SU_q(\ell+1)$ is isomorphic to the permutations group $\mathfrak{S}_{\ell+1}$ on $\ell+1$ symbols. Denote by s_i the transposition $(i, i+1)$. Then $\{s_1, \ldots, s_\ell\}$ form a set of generators for $\mathfrak{S}_{\ell+1}$. Any $\omega \in \mathfrak{S}_{\ell+1}$ can be written as a product

$$\omega = (s_{k_\ell} s_{k_\ell+1} \ldots s_\ell)(s_{k_{\ell-1}} s_{k_{\ell-1}+1} \ldots s_{\ell-1}) \ldots (s_{k_2} s_2)(s_{k_1}),$$

where k_i's are integers satisfying $0 \leq k_i \leq i$, with the understanding that $k_i = 0$ means that the string $(s_{k_i} s_{k_i+1} \ldots s_i)$ is missing. It follows from the strong deletion condition in the characterization of Coxeter system by Tits (see [7]) that the expression for ω given above is a reduced word in the generators s_i. We will denote the length of an element ω by $\ell(\omega)$.

Let S and N be the following operators on $L_2(\mathbb{Z})$:

$$Se_n = e_{n-1}, \quad Ne_n = ne_n.$$

We will denote by the same symbols their restrictions to $L_2(\mathbb{N})$ whenever there is no chance of ambiguity. Denote by ψ_{s_i} the following representation of $\mathcal{A}$ on $L_2(\mathbb{N})$:

$$\psi_{s_i}(u_{ab}) = \begin{cases} \sqrt{I - q^{2N+2}}S & \text{if } a = b = i, \\ S^*\sqrt{I - q^{2N+2}} & \text{if } a = b = i+1, \\ -q^{N+1} & \text{if } a = i,\ b = i+1, \\ q^N & \text{if } a = i+1,\ b = i, \\ \delta_{ab} I & \text{otherwise.} \end{cases}$$

Now suppose $\omega \in \mathfrak{S}_{\ell+1}$ is given by $s_{i_1} s_{i_2} \ldots s_{i_k}$. Define ψ_ω to be $\psi_{s_{i_1}} * \psi_{s_{i_2}} * \ldots * \psi_{s_{i_k}}$ (for two representations ϕ and ψ, $\phi * \psi$ denote the representation $(\phi \otimes \psi)\Delta$).

Next, let $\mathbf{z} = (z_1, \ldots, z_\ell) \in (S^1)^\ell$. Define

$$\chi_{\mathbf{z}}(u_{ab}) = \begin{cases} z_a \delta_{ab} & \text{if } a = 1, \\ \bar{z}_\ell \delta_{ab} & \text{if } a = \ell + 1, \\ \bar{z}_{a-1} z_a \delta_{ab} & \text{otherwise.} \end{cases}$$

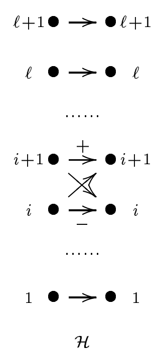
Define χ to be the integral $\int_{\mathbf{z} \in (S^1)^\ell} \chi_{\mathbf{z}} d\mathbf{z}$. Finally, define $\pi_{\omega,\mathbf{z}} = \psi_\omega * \chi_{\mathbf{z}}$ and $\pi_\omega = \psi_\omega * \chi$. It is known ([9]) that $\pi_{\omega,\mathbf{z}}$'s constitute all the ireducible representations of the C^*-algebra $\mathcal{A}$.

Let us introduce a few notations that will be handy later. For a subset $\Lambda = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, \ell\}$, where $i_1 < i_2 < \ldots < i_k$, denote by s_Λ the element $s_{i_1} s_{i_2} \ldots s_{i_k}$ of $\mathfrak{S}_{\ell+1}$. Call a subset J of $\{1, 2, \ldots, \ell\}$ an interval if it is of the form $\{j, j+1, \ldots, j+s\}$. Then for any element ω of the Weyl group, there are intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$ with $\max \Lambda_r > \max \Lambda_s$ for $r > s$ such that

$$\omega = s_{\Lambda_t} s_{\Lambda_{t-1}} \ldots s_{\Lambda_1}. \tag{3.1.1}$$

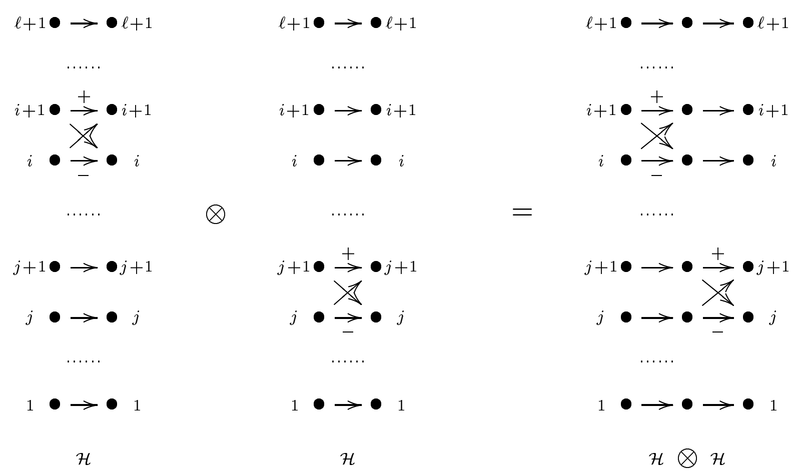
Moreover, as long as we demand that Λ_j's are intervals and obey $\max \Lambda_r > \max \Lambda_s$ for $r > s$, an element ω determines the subsets $\Lambda_t, \ldots, \Lambda_1$ uniquely. Let Λ be the disjoint union of the Λ_j's, that is, $\Lambda = \cup_{j=1}^t \{(j,i) : i \in \Lambda_j\}$. Write $\Lambda_0 = \{(0,i) : i = 1, 2, \ldots, \ell\}$. Often we will identify Λ_0 with the set $\{1, 2, \ldots, \ell\}$. Let $\Gamma = \mathbb{N}^\Lambda \times \mathbb{Z}^{\Lambda_0} = \mathbb{N}^{\Lambda_t} \times \mathbb{N}^{\Lambda_{t-1}} \times \ldots \times \mathbb{N}^{\Lambda_1} \times \mathbb{Z}^{\Lambda_0}$. The Hilbert space on which π_ω acts is $L_2(\Gamma)$. We will denote by $\{e_\gamma : \gamma \in \Gamma\}$ the canonical orthonormal basis for this Hilbert space.

Diagram representation of π_ω. Let us describe how to use a diagram to represent the irreducible ψ_{s_i}.

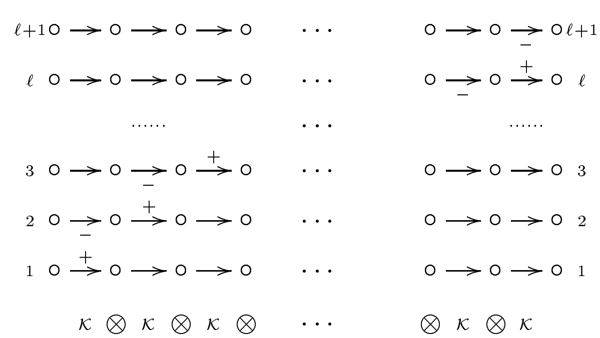


In this diagram, each path from a node k on the left to a node l on the right stands for an operator on $\mathcal{H} = L_2(\mathbb{N})$. A horizontal unlabelled line stands for the identity operator, a horizontal line labelled with a $+$ sign stands for $S^* \sqrt{I - q^{2N+2}}$ and one labelled with a $-$ sign stands for $\sqrt{I - q^{2N+2}} S$. A diagonal line going upward represents $-q^{N+1}$ and a diagonal line going downward represents q^N. Now $\psi_{s_i}(u_{kl})$ is the operator represented by the path from k to l, and is zero if there is no such path. Thus, for example, $\psi_{s_i}(u_{11})$ is I, $\psi_{s_i}(u_{12})$ is zero, whereas $\psi_{s_i}(u_{ii+1}) = -q^{N+1}$, if $i > 1$.

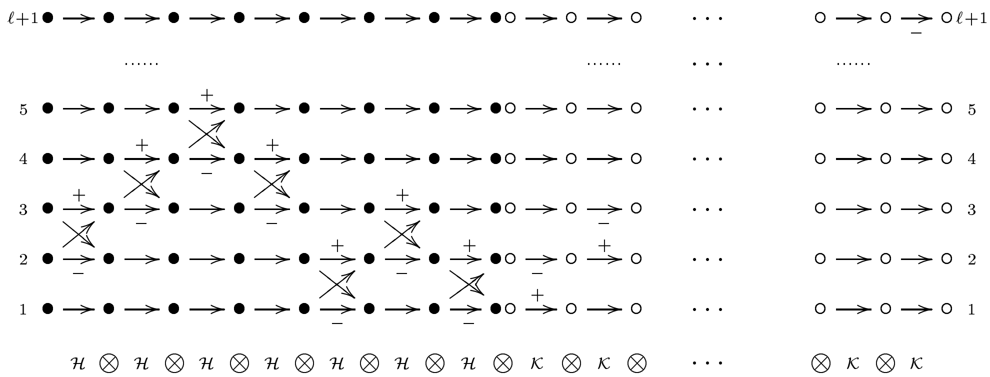
Next, let us explain how to represent $\psi_{s_i} * \psi_{s_j}$. Simply put the two diagrams representing ψ_{s_i} and ψ_{s_j} adjacent to each other, and identify, for each row, the node on the right side of the diagram for ψ_{s_i} with the node on the left in the diagram for ψ_{s_j}. Now, $\psi_{s_i} * \psi_{s_j}(u_{kl})$ would be an operator on $L_2(\mathbb{N}) \otimes L_2(\mathbb{N})$ determined by all the paths from the node k on the left to the node l on the right. It would be zero if there is no such path and if there are more than one paths, then it would be the sum of the operators given by each such path. Thus, we have the following operation on the elementary diagrams described above:



Next, we come to χ. The underlying Hilbert space now is $L_2(\mathbb{Z}^{\Lambda_0}) \cong L_2(\mathbb{Z})^{\otimes \ell}$ (to avoid any ambiguity, we have used hollow circles to denote the nodes as opposed to the bullets used in the earlier case); an unlabelled horizontal arrow stands for I in the corresponding component of $L_2(\mathbb{Z})^{\otimes \ell}$, an arrow labelled with a '+' above it indicates S^* and one labelled '−' below it stands for S. As earlier, $\chi(u_{kl})$ stands for the operator on $L_2(\mathbb{Z})^{\otimes \ell}$ represented by the path from k on the left to l on the right. In the diagram below, $\mathcal{K}$ will stand for $L_2(\mathbb{Z})$.



Finally, we come to the description of π_ω. As we have already remarked, reduced expression for ω is of the form $\omega = (s_{k_n} s_{k_n+1} \ldots s_n)(s_{k_{n-1}} \ldots s_{n-1}) \ldots (s_{k_2} s_2)(s_{k_1})$. To get the diagram for π_ω, we simply put the diagram for $\psi_{s_{k_n}} * \ldots * \psi_{s_{k_1}}$ and that for χ side by side and identify the nodes on the right of the first diagram with the corresponding ones on the left of the second diagram. Thus for example, if $\omega = (s_2 s_3 s_4)(s_3)(s_1 s_2)(s_1)$, then the following diagram represents π_ω:



The diagram for π_ω introduced above will play an important role in what follows.

3.2 Boundedness of commutators

Our goal is to study operators D on the space $\mathcal{H}_\omega = L_2(\Gamma)$ that diagonalize with respect to the natural canonical basis, and makes $(\pi_\omega(\mathcal{A}), \mathcal{H}_\omega, D)$ a spectral triple. Since D is a self-adjoint operator with discrete spectrum, it is of the form $\sum_{\gamma\in\Gamma} d(\gamma)e_\gamma$.

Definition 3.2.1 A **move** will mean a path from a node on the left to a node on the right in the diagram representing π_ω. More formally, a *move* is a $(t+1)$-tuple of pairs $((i_t, j_t), \ldots, (i_0, j_0))$ such that

1. $j_k = i_{k-1}$ for $k \geq 1$, $i_0 = j_0$,
2. for $k \geq 1$, $j_k < i_k$ implies $j_k = i_k - 1$ and $j_k \in \Lambda_k$,
3. for $k \geq 1$, $j_k > i_k$ implies $i_k, i_k + 1, \ldots, j_k - 1 \in \Lambda_k$.

(the pair (i_k, j_k) will be referred as the k^{th} segment of the *move* lying in the k^{th} string from the right).
We will use the special notation H_r for the *move* for which each i_k and j_k equals r.

Given a *move* p, we will next define an element $m_p \in \mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0}$ whose coordinates are all 0 or ± 1. Let $p = ((i_t, j_t), \ldots, (i_0, j_0))$. Define m_p by the following prescription:

$$m_p(r,s) = \begin{cases} -1 & \text{if } r = 0, s = i_0 - 1 \text{ or } r \geq 1, s = j_r \geq i_r, \\ +1 & \text{if } r = 0, s = i_0 \text{ or } r \geq 1, j_r \geq i_r = s+1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $m_p(r, \cdot)$ will look like

$$\begin{cases} (0,0,\ldots,0) & \text{if } i_r < \min\Lambda_r \text{ or } i_r > \max\Lambda_r + 1 \text{ or } j_r = i_r - 1, \\ (-1,0,\ldots,0) & \text{if } i_r = j_r = \min\Lambda_r, \\ (0,0,\ldots,0,1) & \text{if } i_r = j_r = \max\Lambda_r + 1, \\ (\underbrace{0,\ldots,0}_{i_r-2}, 1, \underbrace{0,\ldots,0}_{j_r-i_r}, -1, 0, \ldots, 0) & \text{if } \min\Lambda_r < i_r \leq j_r, \end{cases}$$

We will often refer to this associated element m_p when we talk about a *move* p.

Let us denote by P_{ij} the set of moves from node i on the left to node j on the right. For a move p, denote by T_p the corresponding operator on $\mathcal{H}^{\otimes \ell(\omega)} \otimes \mathcal{K}^{\otimes \ell}$. Then

$$\pi_\omega(u_{ij}) = \sum_{p \in P_{ij}} T_p. \tag{3.2.1}$$

Denote by W_p the operator obtained from T_p by replacing $\sqrt{I - q^{2N+2}}S$ by S and $S^*\sqrt{I - q^{2N+2}}$ by S^*. One can show easily that m_p is the unique element in $\mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0}$ whose entries are all 0 or ± 1 such that $\langle W_p e_\gamma, e_{\gamma + m_p}\rangle \neq 0$ for some $\gamma \in \Gamma$.

Lemma 3.2.2 *Let $p, p' \in P_{ij}$. If p and p' are different, then for some (r, n), where $1 \leq r \leq t$ and $n \in \Lambda_r$, one has either $m_p(r,n) = 0, m_{p'}(r,n) = \pm 1$ or $m_p(r,n) = \pm 1, m_{p'}(r,n) = 0$.*

Proof: Since p and p' both belong to P_{ij} and are different, $m_p(r,n) \neq m_{p'}(r,n)$ for some pair (r,n). Now look at the coordinate where they are unequal for the first time (from the left), that is, let (r,n) be the pair such that

$$r = \max\{1 \leq j \leq t : m_p(j,i) \neq m_{p'}(j,i) \text{ for some } i\}, \quad n = \min\{i \in \Lambda_r : m_p(r,i) \neq m_{p'}(r,i)\}.$$

It is easy to see now that for this pair (r,n), the required conclusion holds. □

Lemma 3.2.3 *Let F be a finite set of moves. For $p \in F$, let D_p be a (not necessarily bounded) number operator, i.e. an operator of the form $e_\gamma \mapsto t_\gamma e_\gamma$. If $\sum_{p \in F} D_p W_p$ is bounded, then $D_p W_p$ is bounded for each $p \in F$.*

Proof: Take $p' \in F$. Assume that $|F| > 1$. We will show that boundedness of $\sum_{p\in F} D_p W_p$ implies that of $\sum_{p \in F'} D_p W_p$ for some subset F' of F such that $p' \in F'$ and $|F'| < |F|$.

Let $p'' \in F$ be an element of F other than p'. By the previous lemma, there is a pair (r,n) such that either $m_{p'}(r,n) = 0$ and $m_{p''}(r,n) = \pm 1$ or $m_{p'}(r,n) = \pm 1$ and $m_{p''}(r,n) = 0$. For $z \in S^1$, let U_z be the unitary operator on $L_2(\Gamma)$ given by $U_z e_\gamma = z^{\gamma(r,n)} e_\gamma$. Now the proof will follow from the boundedness of the operator $\int_{z \in S^1} U_z (\sum_{p \in F} D_p W_p) U_z^* \, dz$. □

Proposition 3.2.4 *$[D, \pi_\omega(u_{ij})]$ is bounded for all i and j if and only if $[D, W_p]$ is bounded for all moves p.*

Proof: It is enough to show that if $[D, \pi_\omega(u_{ij})]$ is bounded, and if $p \in P_{ij}$, then $[D, W_p]$ is bounded. Since $\pi_\omega(u_{ij}) = \sum_{p \in P_{ij}} T_p$ and each $[D, T_p]$ is of the form $D_p W_p$, it follows from the forgoing lemma that each $[D, T_p]$ is bounded. Since $\sqrt{1 - q^{2n+2}}$ is a bounded quantity whose inverse is also bounded, it follows that $[D, T_p]$ is bounded if and only if $[D, W_p]$ is bounded. □

Thus there is a positive constant c such that D will have bounded commutators with all the $\pi_\omega(u_{ij})$'s if and only if $\|[D, W_p]\| \leq c$.

Let $p = ((i_t, j_t), \ldots, (i_0, j_0))$ be a move. A coordinate (r, n) is said to be a **diagonal component** of p if either $i_r < j_r$ and $s \in \{i_r, i_r + 1, \ldots, j_r - 1\}$, or $j_r = i_r - 1 = s$. One can check that this would correspond exactly to the diagonal parts of the move in the diagram representing ω. Denote by $c(\gamma, p)$ the quantity $\sum_{(j,i)} \gamma(j, i)$, the sum being taken over all diagonal components of p.

Lemma 3.2.5 *$[D, W_p]$ is bounded if and only if $|d(\gamma + m_p) - d(\gamma)| \leq cq^{-c(\gamma,p)}$.*

Proof: Follows easily once one writes down the expression of the commutator. □

An immediate corollary is the following.

Corollary 3.2.6 *Let H_i be as in definition 3.2.1. Then $|d(\gamma + H_i) - d(\gamma)| \leq c$ for all $\gamma \in \Gamma$ and $1 \leq i \leq \ell + 1$.*

3.3 The growth graph and sign characterization

Let us now form the graph $\mathcal{G}_c$ by connecting two vertices γ and γ' if $|d(\gamma) - d(\gamma')| \leq c$. Characterization of $\operatorname{sign} D$ will then proceed as outlined in the beginning of subsection 2.4.

Definition 3.3.1 For $i \in \Lambda_0$, let J_i be the set $\{j \geq 1 : i \in \Lambda_j\}$. The set $\mathscr{F} = \{\gamma \in \mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0} : -\gamma(0, i) = \gamma(0, i-1) = \gamma(j, i)$ for all $j \in J_i\}$ will be called the **free plane**. For a point $\gamma \in \Gamma$, we call the set $\mathscr{F}_\gamma = \{\gamma + \gamma' \in \Gamma : \gamma' \in \mathscr{F}\}$ the **free plane passing through** γ.

Note that for $\gamma \in \mathscr{F}$, the coordinates $\gamma(j, i)$ are all equal for $j \in J_i$.

For $1 \leq i \leq \ell$, define j_i to be 0 if J_i is empty, and to be that element $j \in J_i$ for which $\gamma(j_i, i) = \min\{\gamma(j, i) : j \in J_i\}$.

Remark 3.3.2 1. If J_i is nonempty, j_i need not be unique.
2. If $\gamma' \in \mathscr{F}_\gamma$, then $\min_j \gamma(j, i)$ and $\min_j \gamma'(j, i)$ are attained for the same set of values of j.

Then, given a $\gamma \in \Gamma$, elements in $\mathscr{F}_\gamma$ are determined by the coordinates (j_i, i), $i = 1, \ldots, \ell$.

Lemma 3.3.3 *Let $\gamma \in \Gamma$, and $\gamma' \in \mathscr{F}_\gamma$. Let γ'' be the element in $\mathscr{F}_\gamma$ for which*

$$\gamma''(j_\ell, \ell) = \gamma'(j_\ell, \ell), \quad \gamma''(j_i, i) = 0 \text{ for all } i < \ell.$$

Then there is a path in $\mathscr{F}_\gamma$ joining γ' to γ'' such that throughout this path, the (j_ℓ, ℓ)-coordinate remains constant.

Proof: Apply successively the moves

$$\gamma(j_{\ell-1}, \ell - 1)H_{\ell-1}, \quad (\gamma(j_{\ell-2}, \ell - 2) + \gamma(j_{\ell-1}, \ell - 1))H_{\ell-2}, \quad \ldots, \quad \left(\sum_{i=1}^{\ell-1} \gamma(j_i, i)\right) H_1.$$

As none of these moves touch the (j_ℓ, ℓ)-coordinate, it remains constant throughout the path. □

Lemma 3.3.4 *Let $\gamma \in \Gamma$. Then either $\mathscr{F}_\gamma^+$ is finite or $\mathscr{F}_\gamma^-$ is finite.*

Proof: Write $C(\gamma) = \gamma(j_\ell, \ell)$. We will first show that $C(\mathscr{F}_\gamma^+)$ and $C(\mathscr{F}_\gamma^-)$ can not both be infinite. This is done exactly as in the proof of proposition 2.4.4, using the above sweepout lemma instead of lemma 2.4.3.

Next, suppose $C(\mathscr{F}_\gamma^-) \subseteq [-K, K]$. If $\{\gamma'(j_i, i) : \gamma' \in \mathscr{F}_\gamma\}$ is not bounded for some i with $1 \le i \le \ell - 1$, get a sequence of points $\gamma_n \in \mathscr{F}_\gamma$ such that $\gamma_n(j_i, i) < \gamma_{n+1}(j_i, i)$ for all n. Starting at each γ_n, apply the move $H_{\ell+1}$ enough (e.g. $2K+1$) times to produce an infinite ladder. $\square$

Let us next define a set that will play the role of a complementary axis. Let

$$\mathscr{C} = \{\gamma \in \Gamma : \prod_{j \in J_i} \gamma(j, i) = 0 \text{ for all } i\}.$$

It follows from the sweepout argument used in the proof of lemma 3.3.3 that for any $\gamma' \in \Gamma$, there is a $\gamma \in \mathscr{C}$ such that $\gamma' \in \mathscr{F}_\gamma$. But it is not necessary that for two distinct elements γ and γ' in $\mathscr{C}$, $\mathscr{F}_\gamma$ and $\mathscr{F}_{\gamma'}$ are disjoint. However, this will not be of serious concern to us.

Let

$$i_{min} = \min\{i \in \Lambda_0 : |J_i| > 1\}, \quad j_{min} = \min J_{i_{min}}, \quad j_{max} = \max J_{i_{min}}.$$

Thus i_{min} is the minimum i for which s_i appears more than once in ω, j_{min} and j_{max} are the first and the last string where it appears. Suppose now that we have removed the horizontal arrows labelled $+$ or $-$ corresponding to all the s_i's for which $|J_i| = 1$. Note that this would in particular remove all labelled horizontal lines corresponding to s_i's for $i < i_{min}$. Suppose the j_{min}th segment of a move is (i_{min}, i_{min}). This will uniquely specify the 0^{th} segment which will be of the form (i_0, i_0) for some $i_0 \le i_{min}$. Now define $C_0(\gamma) := \gamma(j_{min}, i_{min}) + \gamma(0, i_0)$ and $C_1(\gamma) = \gamma(j_{max}, i_{min})$ for $\gamma \in \Gamma$.

Lemma 3.3.5 *Let $\gamma \in \mathscr{C}$. Define an element $\gamma' \in \Gamma$ by the following prescription:*

$$\gamma'(j, i) = 0 \textit{ for all } j \ge 1, \quad \gamma'(0, i) = \begin{cases} 0 & \textit{if } i \ne i_0, \\ C_0(\gamma) & \textit{if } i = i_0. \end{cases}$$

Then there is a path connecting γ to γ' such that $C_0(\cdot)$ remains constant throughout this path.

Proof: We will describe a recursive algorithm to go from γ to γ'. Observe that since $\gamma \in \mathscr{C}$, we have $\gamma(t, \max \Lambda_t) = 0$. To begin with, remove all the horizontal arrows labelled $+$ or $-$ corresponding to the s_i's for which $|J_i| = 1$, and work with the resulting diagram.

Now suppose we are at $\delta \in \Gamma$ which satisfies

$$\delta(j, i) = 0 \text{ for all } j > r, \quad \delta(r, i) = 0 \text{ for all } i > n \in \Lambda_r.$$

Step I.

Case I. $r = j_{min}$ and $n = i_{min}$: then apply the move whose $j_{min}{}^{\text{th}}$ segment is (i_{min}, i_{min}). Apply this $\delta(j_{min}, i_{min})$ times. This will make the (j_{min}, i_{min})-coordinate zero and the $(0, i_0)$-coordinate $C_0(\gamma)$. Now proceed to step II.

Case II. $r \neq j_{min}$ or $n \neq i_{min}$: Proceed with the following algorithm.

Algorithm A(r, n)**.** $(\min \Lambda_r \leq n \leq \max \Lambda_r)$ Remove all horizontal arrows labelled $+$ or $-$ from the s_i's in the strings s_{Λ_t}, $s_{\Lambda_{t-1}}$, …, $s_{\Lambda_{r+1}}$ as well as from the s_i's corresponding to $i \in \Lambda_r$, $i > n$. What this will achieve is the following: any permissible move in the resulting diagram will not change the coordinates (j, i) where either $r + 1 \leq j \leq t$ or $j = r$ and $i > n$. Apply the negative of the move whose r^{th} segment is $(n + 1, \max \Lambda_r + 1)$ for $\delta(r, n)$ number of times. This would kill the (r, n)-coordinate, i.e. will make it zero. Now remove the two horizontal lines labelled '$+$' and '$-$' corresponding to s_n appearing in the string s_{Λ_r}.

Step II.

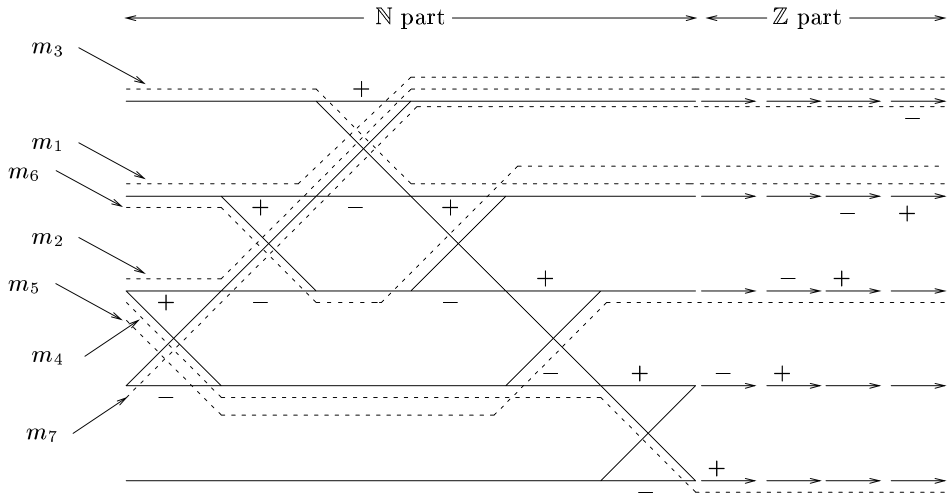
Case I. $n > \min \Lambda_r$: keep r intact, reduce the value of n by 1 and go back to step I.

Case II. $r > 1$ and $n = \min \Lambda_r$: change n to $\max \Lambda_{r-1}$, then reduce the value of r by 1, and go back to step I.

Case III. $r = 1$ and $n = \min \Lambda_1$: proceed to step III.

Step III. All the (j, i)-coordinates for $j \geq 1$ are now zero. Next, apply moves ending at i for $i > i_0 + 1$ appropriate number of times starting from the top to kill the coordinates $(0, i)$ for $i > i_0$. Thus we have now reached an element δ for which $\delta(j, i) = 0$ whenever $j \geq 1, i \in \Lambda_j$ or $j = 0, i > i_0$. Therefore we now need to kill the coordinates $(0, i)$ for $i < i_0$. This is achieved as follows. Remove the horizontal arrows labelled $+$ or $-$ from all s_i's. Now apply the moves ending at i for $i < i_0$ appropriate number of times starting from the bottom. $\square$

The next diagram and the table that follows it will explain the proof in a simple case.



$\omega = (s_2 s_3 s_4)(s_3)(s_2)(s_1)$

$t = 4,\ i_{min} = 2,\ j_{min} = 2,\ j_{max} = 4,\ i_0 = 1$

The table below illustrates the sweepout procedure described in the proof of lemma 3.3.5. Starting from a point $\gamma \in \mathscr{C}$, it shows the successive moves applied and how the resulting element looks like at each stage. Observe that for any $\gamma \in \mathscr{C}$, one must have $\gamma(4,4) = 0 = \gamma(1,1)$.

coordinate	(4,2)	(4,3)	(4,4)	(3,3)	(2,2)	(1,1)	(0,1)	(0,2)	(0,3)	(0,4)
γ	$*$	$*$	0	$*$	a	0	b	$*$	$*$	$*$
move m_1	0	$+1$	0	0	0	0	0	0	0	-1
$\gamma_1 = -\gamma(4,3)m_1(\gamma)$	$*$	0	0	$*$	a	0	b	$*$	$*$	$*$
move m_2	$+1$	0	0	0	0	0	0	0	0	-1
$\gamma_2 = -\gamma(4,2)m_2(\gamma_1)$	0	0	0	$*$	a	0	b	$*$	$*$	$*$
move m_3	0	0	0	$+1$	0	0	0	0	-1	$+1$
$\gamma_3 = -\gamma(3,3)m_3(\gamma_2)$	0	0	0	0	a	0	b	$*$	$*$	$*$
move m_4	0	0	0	0	-1	0	$+1$	0	0	0
$\gamma_4 = \gamma(2,2)m_4(\gamma_3)$	0	0	0	0	0	0	$a+b$	$*$	$*$	$*$
move m_5	0	0	0	0	0	0	0	-1	$+1$	0
$\gamma_5 = \gamma_4(0,2)m_5(\gamma_4)$	0	0	0	0	0	0	$a+b$	0	$*$	$*$
move m_6	0	0	0	0	0	0	0	0	-1	$+1$
$\gamma_6 = \gamma_5(0,3)m_6(\gamma_5)$	0	0	0	0	0	0	$a+b$	0	0	$*$
move m_7	0	0	0	0	0	0	0	0	0	-1
$\gamma' = \gamma_6(0,4)m_7(\gamma_6)$	0	0	0	0	0	0	$a+b$	0	0	0

Lemma 3.3.6 *Both $C_0(\mathscr{C}^+)$ and $C_0(\mathscr{C}^-)$ can not be infinite.*

Proof: If both are infinite, there would exist elements $\gamma_n \in \mathscr{C}^+$ and $\delta_n \in \mathscr{C}^-$ such that

$$C_0(\gamma_1) < C_0(\delta_1) < C_0(\gamma_2) < C_0(\delta_2) < \dots .$$

Let γ_n' and δ_n' be given by

$$\gamma_n'(j,i) = 0 \text{ for all } j \geq 1, \quad \gamma_n'(0,i) = \begin{cases} 0 & \text{if } i \neq i_0, \\ C_0(\gamma_n) & \text{if } i = i_0, \end{cases}$$

$$\delta_n'(j,i) = 0 \text{ for all } j \geq 1, \quad \delta_n'(0,i) = \begin{cases} 0 & \text{if } i \neq i_0, \\ C_0(\delta_n) & \text{if } i = i_0. \end{cases}$$

Use the earlier lemma to get paths between γ_n and γ_n' and between δ_n and δ_n'. Remove all the labelled arrows from all the s_i's. Let m_i be the move in the resulting diagram whose 0^{th} segment is (i,i), and let $m = \sum_{\substack{i=1 \\ \leftarrow}}^{i_0} m_i$. Apply this move $C_0(\delta_n) - C_0(\gamma_n)$ times to connect γ_n' and δ_n'. Thus there is a path p_n connecting γ_n and δ_n, and throughout this path, $C_0(\cdot)$ lies between $C_0(\gamma_n)$ and $C_0(\delta_n)$. Therefore the paths p_n are disjoint. □

We will assume from now onward that $C_0(\mathscr{C}^-)$ is finite. We will also assume that $K \in \mathbb{N}$ is such that $C_0(\mathscr{C}^-) \subseteq [-K, K]$.

Lemma 3.3.7 *Let C_1 be as defined prior to lemma 3.3.5, i.e. $C_1(\gamma) = \gamma(j_{max}, i_{min})$. Then the set $C_1(\mathscr{C}^-)$ is finite.*

Proof: If not, get $\gamma_n \in \mathscr{C}^-$ such that

$$C_1(\gamma_1) < C_1(\gamma_2) < C_1(\gamma_3) < \ldots.$$

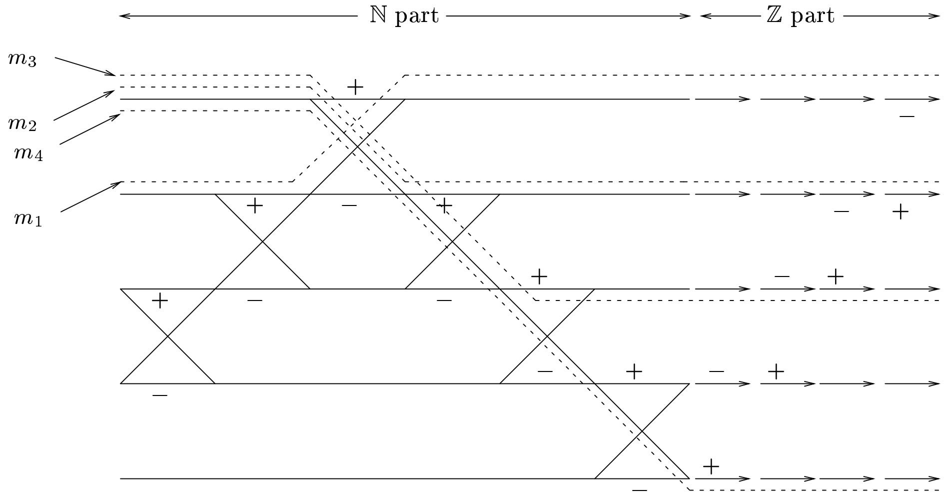
Now the idea is to get a path p_n joining γ_n to some δ_n such that $C_1(\cdot)$ remains constant throughout p_n, and $C_0(\delta_n) > K$, so that each $\delta_n \in \mathscr{C}^+$.

Start at γ_n. Apply algorithm $\mathbf{A}(r,n)$ for

$$\begin{array}{rclll} r &=& t, t-1, \ldots, j_{max}+1, & \min \Lambda_r \leq n \leq \max \Lambda_r, \\ r &=& j_{max}, & i_{min}+1 \leq n, \\ r &<& j_{max}, & \min \Lambda_r \leq n \leq \max \Lambda_r. \end{array}$$

Now apply the move m_{i_0}, where m_i's are the moves described in the proof of the previous lemma, $3K$ times. □

Again we give a diagram and a table to illustrate the above proof for the case $\omega = (s_2 s_3 s_4)(s_3)(s_2)(s_1)$.



$\omega = (s_2s_3s_4)(s_3)(s_2)(s_1)$

$t = 4,\ i_{min} = 2,\ j_{min} = 2,\ j_{max} = 4,\ i_0 = 1$

The next table illustrates the argument in the above proof. Starting from a point $\gamma \in \mathscr{C}^-$, it shows the successive moves applied and how the resulting element looks like at each stage.

coordinate	(4,2)	(4,3)	(4,4)	(3,3)	(2,2)	(1,1)	(0,1)	(0,2)	(0,3)	(0,4)
γ	a	$*$	0	$*$	$*$	0	b	$*$	$*$	$*$
move m_1	0	+1	0	0	0	0	0	0	0	−1
$\gamma_1 = -\gamma(4,3)m_1(\gamma)$	a	0	0	$*$	$*$	0	b	$*$	$*$	$*$
move m_2	0	0	0	+1	0	0	0	0	−1	+1
$\gamma_2 = -\gamma(3,3)m_2(\gamma_1)$	a	0	0	0	$*$	0	b	$*$	$*$	$*$
move m_3	0	0	0	0	+1	0	0	−1	+1	0
$\gamma_3 = -\gamma(2,2)m_3(\gamma_2)$	a	0	0	0	0	0	b	$*$	$*$	$*$
move m_4	0	0	0	0	0	0	+1	0	0	0
$\gamma_4 = 3Km_4(\gamma_3)$	a	0	0	0	0	0	$b + 3K$	$*$	$*$	$*$

Assume without loss in generality that K is a bound for $C_1(\mathscr{C}^-)$ also.

Lemma 3.3.8 *Let $C \equiv (j,i)$ be any coordinate other than $C_1 \equiv (j_{max}, i_{min})$. Then $C(\mathscr{C}^-)$ is finite.*

Proof: The strategy would be the same as in the proof of the earlier lemma with a slight modification. If $C(\mathscr{C}^-)$ is infinite, we can choose $\gamma_n \in \mathscr{C}^-$ such that

$$C(\gamma_n) + K + 1 < C(\gamma_{n+1})$$

for every $n \in \mathbb{N}$. Now connect every γ_n to an element $\delta_n \in \mathscr{C}^+$ by a path p_n such that on p_n, the C_1 coordinate does not vary by more than K. This will ensure that the paths p_n are all disjoint.

For getting p_n as described above, start at γ_n and apply successively the moves

$$H_{\ell+1}, H_{\ell}, \ldots, H_{i_{min}+1},$$

each one $K+1$ times. This will increase the C_1-coordinate by $K+1$. Therefore the endpoint of the path will lie in $\mathscr{C}^+$. $\square$

Thus it now follows that $\mathscr{C}^-$ is finite. This, together with proposition 2.4.4 will give us the following theorem.

Theorem 3.3.9 *Let D be a Dirac operator on $L_2(\Gamma)$ that diagonalises with respect to the canonical orthonormal basis. Then* sign$D$ *has to be of the form $2P - I$ or $I - 2P$ where P is a projection onto the closed linear span of $\{e_\gamma : \gamma \in \cup_{i=1}^k \mathscr{F}_{\gamma_i}\}$ for some finite collection $\gamma_1, \gamma_2, \ldots, \gamma_k$ in Γ.*

Proof: The argument is exactly as in theorem 2.4.15. $\square$

We next show that under this restriction, compactness of the commutator $[\text{sign } D, u_{ij}]$, or, equivalently, that of $[P, u_{ij}]$'s will imply that sign D is trivial.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be elements in Γ and let P be the projection onto span$\{e_\gamma : \gamma \in \cup_i \mathscr{F}_{\gamma_i}\}$. Then for any operator T, we have

$$[P, T]e_\gamma = \begin{cases} PTe_\gamma & \text{if } \gamma \notin \cup_i \mathscr{F}_{\gamma_i}, \\ (P - I)Te_\gamma & \text{if } \gamma \in \cup_i \mathscr{F}_{\gamma_i}. \end{cases}$$

Now let $r = \max \Lambda_t$ and take $T = \pi_\omega(u_{r+1,r})$. Then

$$T(t, r) = q^N, \quad T(0, r-1) = S, \quad T(0, r) = S^*, \tag{3.3.1}$$

and $T(j, i) = I$ for all other pairs (j, i), except possibly $T(t-1, r-1)$, which is S^* if $t - 1 \in J_{r-1}$, and I otherwise. It is easy to check that for $\gamma \in \mathscr{F}_{\gamma_i}$, $\gamma(t, r) + \gamma(0, r) = \gamma_i(t, r) + \gamma_i(0, r)$. Therefore the set $\{\gamma(t, r) + \gamma(0, r) : \gamma \in \cup_i \mathscr{F}_{\gamma_i}\}$ is bounded. Let $n \in \mathbb{N}$ be such that this set is contained in $[-n, n]$. Suppose $\gamma \in \cup_i \mathscr{F}_{\gamma_i}$ obey $\gamma(t, r) = 0$. Then it follows from (3.3.1) that $T^{2n+1}e_\gamma = e_{\gamma'}$, where

$$\gamma'(0, r) = \gamma(0, r) + 2n + 1, \quad \gamma'(0, r-1) = \gamma(0, r-1) - 2n - 1, \quad \gamma'(t, r) = \gamma(t, r).$$

It is clear from this that $\gamma' \notin \cup_i \mathscr{F}_{\gamma_i}$, so that $PT^{2n+1}e_\gamma = 0$. This means $[P, T^{2n+1}]e_\gamma = -e_{\gamma'}$ for all $\gamma \in \cup_i \mathscr{F}_{\gamma_i}$ with $\gamma(t, r) = 0$. Since there are infinitely many choices of such γ, it follows that $[P, T^{2n+1}]$ can not be compact.

We thus have the following theorem.

Theorem 3.3.10 *Let $\ell > 1$. Then there does not exist any Dirac operator on $L_2(\Gamma)$ that diagonalises with respect to the canonical orthonormal basis and has nontrivial sign.*

Remark 3.3.11 Let F be a subset of $\{1, 2, \ldots, \ell\}$. Define $\pi_{\omega,F}$ to be the representation obtained by integrating $\psi_\omega * \chi_{\mathbf{z}}$ with respect to those components z_i of $\mathbf{z}$ for which $i \in F$. If one looks at the representations $\pi_{\omega,F}$ instead of π_ω, a similar analysis will show that nontrivial spectral triples would exist only in the case where ω is of the form s_k (so that $\ell(\omega) = 1$), and $F = \{k\}$. The nontrivial triples in this case will essentially be those of $SU_q(2)$ obtained in [2] and will correspond to the 'k^{th} copy' of $SU_q(2)$ sitting inside $SU_q(\ell + 1)$ via the map

$$u_{ij} \mapsto \begin{cases} \alpha & \text{if } j = i = k, \\ \alpha^* & \text{if } j = i = k+1, \\ -q\beta^* & \text{if } j = k+1, i = k, \\ \beta & \text{if } j = k, i = k+1, \\ I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

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Partha Sarathi Chakraborty (parthac@isid.ac.in)

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA

Arupkumar Pal (arup@isid.ac.in)

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA