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Ordering convolutions of gamma random variables

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Abstract

Let $a_{(i)}$ and $b_{(i)}$ be the ith smallest components of $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ respectively, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{+n}$. The vector $\mathbf{a}$ is said to be p-larger than the vector $\mathbf{b}$ (denoted by $\mathbf{a} \overset{p}{\succeq} \mathbf{b}$) if $\prod_{i=1}^{k} a_{(i)} \leq \prod_{i=1}^{k} b_{(i)}$, for $k = 1, \ldots, n$. Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, $i = 1, \ldots, n$. It is shown that if $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$, then $\sum_{i=1}^{n} X_{\lambda_i}$ is greater than $\sum_{i=1}^{n} X_{\lambda_i^*}$ according to dispersive as well as hazard rate orderings. This strengthens the results of Kochar and Ma [Statistics & Probability Letters 43 (1999), 321-324] and Korwar [J. Multivariate Analysis 80 (2002), 344-357] from usual majorization to p-larger ordering and leads to better bounds on various quantities of interest.

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Key words : Schur functions, majorization, p-larger ordering, log-concave density, dispersive ordering and hazard rate ordering.

1 Introduction

Convolutions of independent random variables occur quite frequently in statistics, applied probability, operations research and in many other areas. Their distribution theory is quite complicated when the convoluting random variables are *not* identically distributed. In such situations, it is of interest to obtain bounds and approximations on moments and other characteristics of interest for such statistics.

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Kochar and Ma (1999) proved that a convolution of independent exponential random variables with unequal hazard rates is stochastically larger in the sense of *dispersive* ordering when the parameters of the exponential distributions are more dispersed in the sense of *majorization*. Korwar (2002) extended this result to the case of gamma random variables with different scale parameters but with a common shape parameter whose value is at least one. In this paper we further investigate this problem when the scale parameters of the gamma random variables satisfy some constraints which are weaker than majorization. The new results obtained in this paper lead to better bounds on various quantities of interest. Other related work on this problem where some other stochastic orders are also considered is by Bock et al. (1987), Boland, El-Neweihi and Proschan (1994) and Bon and Paltanea (1999).

A random variable X is said to be more dispersed than another random variable Y (denoted by $X \geq_{disp} Y$) if $F^{-1}(v) - F^{-1}(u) \geq G^{-1}(v) - G^{-1}(u)$, for $0 \leq u \leq v \leq 1$, where F^{-1} and G^{-1} are the right continuous inverses of the distribution functions F and G of X and Y, respectively. It is well known that $X \geq_{disp} Y \Rightarrow var(X) \geq var(Y)$. For further properties of dispersive order, see Shaked and Shanthikumar (1994).

Let $\{x_{(1)} \leq \ldots \leq x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \cdots, x_n)$. The vector $\mathbf{x}$ is said to majorize another vector $\mathbf{y}$ of the same dimension (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. Functions that preserve the majorization ordering are called Schur-convex functions. Bon and Paltanea (1999) define a vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ to be be p-larger than another vector $\mathbf{y}$ also in $\mathbb{R}^{+n}$ (written $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, j = 1, \ldots, n$. It can be shown that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{+n}$, $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y}$. The converse, however, may not be true.

Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that for $i = 1, \ldots, n$, X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i so that its density function is given by $f(x; a, \lambda_i) = \{1/\Gamma(a)\} \lambda_i{}^a \, x^{a-1} exp(-\lambda_i x),\ x > 0; 0, otherwise$. We prove in Theorem 2.1 in the next section that when the common shape parameter a is a positive integer, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} X_{\lambda_i^*}$. To prove it, we shall be repeatedly using the following result due to Lewis and Thompson (1981).

THEOREM **1.1.** *Let Z be a random variable independent of random variables X and Y. If $X \geq_{disp} Y$ and Z has a log-concave density, then $X + Z \geq_{disp} Y + Z$.*

2 Main Results

The next theorem states the main result of this paper.

THEOREM **2.1.** *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\lambda_1^*, \ldots, \lambda_n^*)$, where $S(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} X_{\lambda_i}$.*

PROOF : To prove this theorem we follow the same technique as used by Korwar (2002) in his paper. First we consider the case when $n = 2$. Without loss of generality we assume that $\lambda_1 \geq \lambda_2$ and $\lambda_1^* \geq \lambda_2^*$. The proofs for the cases when $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_1^*$, $\lambda_1 = \lambda_2^*$ and $\lambda_1^* > \lambda_2^*$ follow from Theorem 1.1 while for the the case when $\lambda_1 = \lambda_2$ and $\lambda_1^* = \lambda_2^*$ follows from the fact that gamma random variables with a common shape parameter are dispersive ordered with respect to their scale parameters. Now we only need to consider the following two cases separately in order to prove the desired result.

Case (a) $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$.

First we consider the case when $\lambda_1 \neq \lambda_1^*$ and $\lambda_2 \neq \lambda_2^*$ and will discuss the other possibilities later. Let $f(y; a, \lambda_1, \lambda_2)$ and $F(y; a, \lambda_1, \lambda_2)$ denote the density function and the distribution function of $S(\lambda_1, \lambda_2)$, respectively. To prove the required result, in the light of Lemma 2.1 in Khaledi and Kochar (2002, p.16), it is sufficient to show that for $0 < y \leq x < 1$,

(i) $F^{-1}(x; a, e^{a_1}, e^{a_2}) - F^{-1}(y; a, e^{a_1}, e^{a_2})$ is Schur-convex in (a_1, a_2),

(ii) $F^{-1}(x; a, e^{a_1}, e^{a_2}) - F^{-1}(y; a, e^{a_1}, e^{a_2})$ is decreasing in a_1 and a_2,

where $a_i = \log \lambda_i$, $i = 1, 2$.

From the definition of dispersive ordering it follows that (i) is equivalent to

$$(a_1, a_2) \overset{m}{\succeq} (a_1^*, a_2^*) \Longrightarrow S(e^{a_1}, e^{a_2}) \geq_{disp} S(e^{a_1^*}, e^{a_2^*}), \tag{2.1}$$

where $a_i^* = log \lambda_i^*$, $i = 1, 2$.

As seen in Korwar (2002) the density function of $S(\lambda, c - \lambda)$ for $y > 0$, can be written as

$$f(y\ ; a, \lambda) = \sqrt{\pi}(\lambda(c - \lambda))^a / \Gamma(a) \{y/(2\lambda - c)\}^{a-1/2} \exp(-cy/2) I_{a-1/2}((\lambda - c/2)y), \tag{2.2}$$

where

$$I_{a-1/2}(y) = \{2(y/2)^{a-1/2} / \sqrt{\pi}\Gamma(a)\} \int_0^1 (1 - t^2)^{(a-1)} \cosh(ty) dt.$$

Let $a_i = \log \lambda_i$ and $a_i^* = \log \lambda_i^*$, $i = 1, 2$ and $a_1 + a_2 = d$. The constraints $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$ respectively, are equivalent to $a_1 > a_2$ and $a_1^* > a_2^*$. Using these in (2.2), we get

$$f(y; a, a_1) = \sqrt{\pi} e^{ad} / \Gamma(a) \{y/(e^{a_1} - e^{d-a_1})\}^{(a-1/2)} \exp(-\frac{e^{a_1} + e^{d-a_1}}{2} y) I_{a-1/2}(\frac{e^{a_1} - e^{d-a_1}}{2} y).$$

Using the recurrence formula $I'_v(z) = I_{v+1}(z) + (v/z) I_v(z)$ in $f'(y; a, a_1)$, the derivative of f and after simplifications, we get

$$f'(y; a, a_1) a e^{-d} (\frac{e^{2a_1} - e^{2(d-a_1)}}{2}) f(y; a{+}1, = \frac{e^{d-a_1} - e^{a_1}}{2} \{y f(y; , a, a_1) - a e^{-d} (e^{a_1} + e^{d-a_1}) f(y; a{+}1, a_1)\}. \tag{2.3}$$

The Laplace transform of $f(y; a, a_1)$, denoted by $g_{a,a_1}(s)$, is

$$g_{a,a_1}(s) = \frac{e^{ad}}{\{(s + e^{a_1})(s + e^{d-a_1})\}^a}, \tag{2.4}$$

and that of $yf(y;a,a_1)$ is

$$\begin{aligned} -g'_{a,a_1}(s) &= -\frac{\partial g_{a,a_1}(s)}{\partial s} \\ &= \frac{ae^{ad}(2s+e^{a_1}+e^{d-a_1})}{\{(s+e^{a_1})(s+e^{d-a_1})\}^{a+1}}. \end{aligned} \tag{2.5}$$

Taking Laplace transforms of both sides of (2.3) and using the above relations, we get

$$L(f'(y;a,a_1)) = \frac{e^{d-a_1}-e^{a_1}}{2}\{-g'_{a,a_1}(s) - ae^{-d}(e^{a_1}+e^{d-a_1})g_{a+1,a_1}(s)\}. \tag{2.6}$$

Using the equality $L(\int_0^y f'(z;a,a_1)dz) = L(f'(y;a,a_1))/s$; and (2.6), (2.4) and (2.5), we obtain

$$\begin{aligned} L(F'(y;a,a_1)) &= \frac{e^{d-a_1}-e^{a_1}}{2s}\{-g'_{a,a_1}(s) - ae^{-d}(e^{a_1}+e^{d-a_1})g_{a+1,a_1}(s)\} \\ &= ae^{-d}(e^{d-a_1}-e^{a_1})L(f(y;a+1,a_1)). \end{aligned} \tag{2.7}$$

Now, by taking the inverse Laplace transforms of both sides of (2.7) and dividing it by $f(y;a,a_1)$, we get

$$\frac{F'(y;a,a_1)}{f(y;a,a_1)} = ae^{-d}(e^{d-a_1}-e^{a_1})\frac{f(y;a+1,a_1)}{f(y;a,a_1)} \tag{2.8}$$

Korwar (2002) has shown that $f(y;a+1,a_1)/f(y;a,a_1)$ is increasing in y. That is, the R.H.S. of (2.8) is decreasing in y, since $a_1 > d - a_1$. Then (2.1) follows from Saunders and Moran (1978, p. 428).

Note that (ii) is equivalent to saying that $S(\lambda_1, \lambda_2)$ is decreasing in λ_1 and λ_2 according to dispersive ordering. Since $X_{\lambda'_1} \geq_{disp} X_{\lambda_1}$ for $\lambda_1 > \lambda'_1$ and X_{λ_2} has a log-concave density, it follows from Theorem 1.1 that $S(\lambda'_1, \lambda_2) \geq_{disp} S(\lambda_1, \lambda_2)$. Similarly one can prove that $S(\lambda_1, \lambda_2)$ is decreasing in λ_2.

Case (b) $\lambda_1 > \lambda_2$ and $\lambda_1^* = \lambda_2^*$.

Again the required result for the case when $\lambda_1 = \lambda_1^*$ immediately follows from Theorem 1.1. Now let $\lambda_1 \neq \lambda_1^*$. In this case $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies that $\lambda_1^* \geq \tilde{\lambda}$, where $\tilde{\lambda} = (\lambda_1\lambda_2)^{1/2}$, the geometric mean of λ_1, λ_2. First we prove the result for the case when $\lambda_1^* = \tilde{\lambda}$. It is easy to see that, for $m \geq 1$, $(\lambda_1, \lambda_2) \overset{p}{\succeq} (\tilde{\lambda}, \tilde{\lambda} + 1/m)$. Using this observation, it follows from case (a) that, for $m \geq 1$, $X_{\lambda_1} + X_{\lambda_2} \geq_{disp} X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/m}$. Using the fact that $X_{\tilde{\lambda}+1/m} \overset{\mathcal{L}}{\to} X_{\tilde{\lambda}}$ as $m \to \infty$, it follows that $X_{\tilde{\lambda}} + X_{\tilde{\lambda}+1/m} \overset{\mathcal{L}}{\to} Y$, as $m \to \infty$, where Y is a gamma random variable with shape parameter 2a and scale parameter $\tilde{\lambda}$. Combining these observations, the required result in this case follows from Theorem 3 of Lewis and Thompson (1981).

The result for the case when $\lambda_1^* > \tilde{\lambda}$ follows from the above case and the fact that gamma random variables with a common shape parameters are decreasing according to dispersive ordering with respect to their scale parameters. This completes the proof of this case.

Now we give the proof for $n > 2$. As in the case of $n = 2$, it is easy to see that $S(\lambda_1, \ldots, \lambda_n)$ is decreasing in λ_i according to dispersive ordering, for $i = 1, \ldots, n$. It remains to show that $\mathbf{a} \overset{m}{\succeq} \mathbf{a}^* \implies S(e^{a_1}, \ldots, e^{a_n}) \geq_{disp} S(e^{a_1^*}, \ldots, e^{a_n^*})$, where $a_i = \log\lambda_i$ and $a_i^* = \log\lambda_i^*$,

$i = 1,, \ldots, n$. To prove this, it is sufficient to consider the case when $(a_1, a_2) \overset{m}{\succeq} (a_1^*, a_2^*)$, and $a_i = a_i^*$, $i = 3, \ldots, n$. Note that $S(e^{a_3}, \ldots, e^{a_n})$ has a log-concave density because a class of distributions with log-concave densities is closed under convolutions (cf. Dharmadhiakri and Joag-dev, 1988, p. 17). Since $S(e^{a_1}, e^{a_2}) \geq_{disp} S(e^{a_1^*}, e^{a_2^*})$, adding $S(e^{a_3}, \ldots, e^{a_n})$ to both sides of the above inequality, we find that the required result follows from Theorem 1.1. ■

A random variable X with survival function $\overline{F}$ is said to be larger than another random variable Y with survival function $\overline{G}$ according to *hazard rate* ordering (denoted by $X \geq_{hr} Y$) if $\overline{F}(x)/\overline{G}(x)$ is increasing in x. The following result is an immediate consequence of Theorem 2.1, Theorem 2.1 of Bagai and Kochar(1986, p.1381) and the fact a convolution of IFR distributions is IFR.

COROLLARY 2.1. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then, $\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^*$ implies $S(\lambda_1, \ldots, \lambda_n) \geq_{hr} S(\lambda_1^*, \ldots, \lambda_n^*)$.*

A special case of the above corollary when random variables X_{λ_i}'s have exponential distributions, has been proved by Bon and Paltanea (1999). Since for a positive vector $\boldsymbol{\lambda}$, $(\lambda_1, \ldots, \lambda_n) \overset{p}{\succeq} (\tilde{\lambda}, \ldots, \tilde{\lambda})$, where $\tilde{\lambda}$ is the geometric mean of the $\tilde{\lambda}$'s, we get the following lower bounds on various quantities of interest associated with convolutions of gamma random variables.

COROLLARY 2.2. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*
(a) $S(\lambda_1, \ldots, \lambda_n) \geq_{disp} S(\tilde{\lambda}, \ldots, \tilde{\lambda})$
(b) $S(\lambda_1, \ldots, \lambda_n) \geq_{hr} S(\tilde{\lambda}, \ldots, \tilde{\lambda})$ which implies
(c) $S(\lambda_1, \ldots, \lambda_n) \geq_{st} S(\tilde{\lambda}, \ldots, \tilde{\lambda})$,
where $\tilde{\lambda}$ is the geometric mean of the λ_i's.

The bounds given by Korwar (2002) are in terms of arithmetic mean $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i$ instead of the geometric mean on the right hand sides of the above inequalities.

In Figures 2.1 and 2.2, we plot the distribution functions of convolutions of two independent gamma random variables along with the bounds given by Corollary 2.2 (c) and by Korwar (2002). In Figures 2.3 and 2.4, we plot the hazard functions of convolutions of two independent gamma random variables along with the bounds given by Corollary 2.2 (b) and by Korwar (2002). The vector of parameters in Figures 2.1 and 2.3 is $\boldsymbol{\lambda_1} = (1, 2)$ and that in Figures 2.2 and 2.4 is $\boldsymbol{\lambda_2} = (0.25, 2.75)$. Note that $\boldsymbol{\lambda_2} \overset{m}{\succeq} \boldsymbol{\lambda_1}$. It appears from these figures that the improvements on the bounds are relatively more if λ_i's are more dispersed in the sense of majorization. This is due to the fact that the geometric mean is Schur concave whereas the arithmetic mean is Schur constant and the distribution (hazard rate) of convolutions of i.i.d. gamma random variables with common parameter $\tilde{\lambda}$ is decreasing (increasing) in $\tilde{\lambda}$.

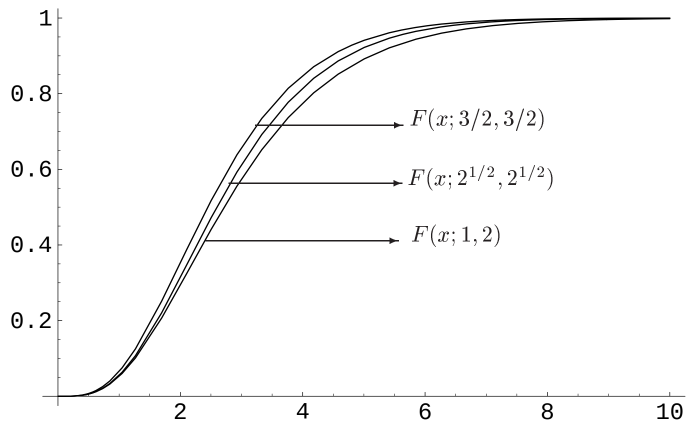

Figure 2.1. Graphs of distribution functions of $S(\lambda_1, \lambda_2)$

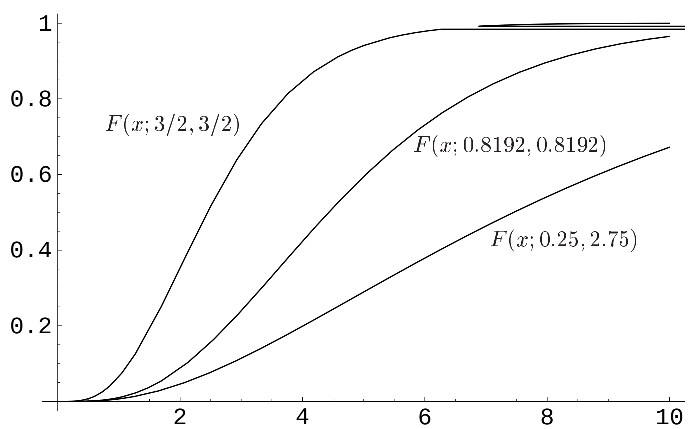

Figure 2.2. Graphs of distribution functions of $S(\lambda_1, \lambda_2)$

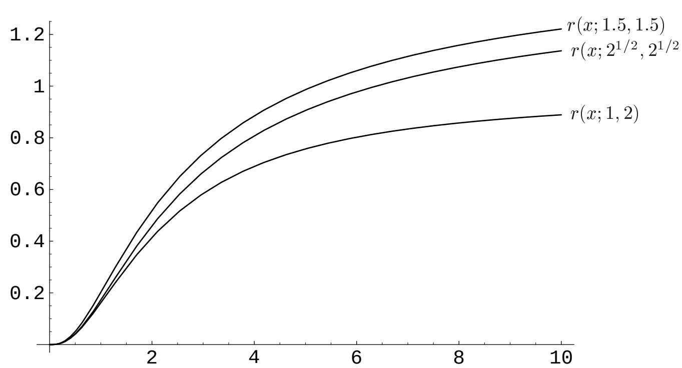

Figure 2.3. Graphs of hazard rates of $S(\lambda_1, \lambda_2)$

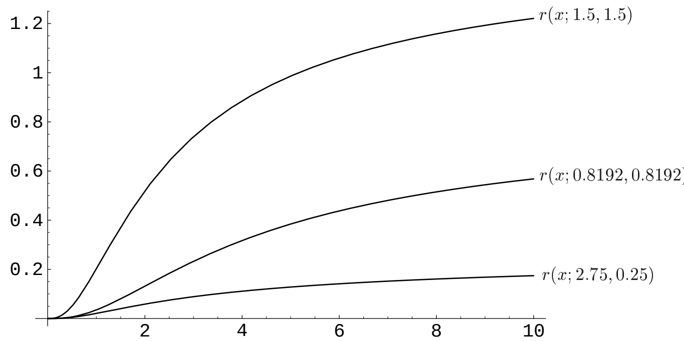

Figure 2.4. Graphs of hazard rates of $S(\lambda_1, \lambda_2)$

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References

1. Bagai, I. and Kochar, S. C. (1986). On tail ordering and comparison of failure rates. *Commun. Statist. Theory and Methods* **15**, 1377-1388.
2. Bock, M. E., Diaconis, P., Huffer, H. W. and Perlman, M. D. (1987). Inequalities for linear combinations of gamma random variables. *Canad. J. Statist.* **15**, 387-395.
3. Boland, P.J., El-Neweihi, E. and Proschan, F. (1994). Schur properties of convolutions of exponential and geometric random variables. *J. Mult. Anal.* **48**, 157-167.
4. Bon, J. L. and Paltanea, E. (1999). Ordering properties of convolutions of exponential random variables. *Lifetime Data Anal.* **5**, 185-192.
5. Dharmadhikari, S. and Joeg-dev, K. (1988). Unimodality, Convexity and Applications. *Academic press, INC.*
6. Khaledi, B. E. and Kochar, S. (2002). Dispersive ordering among linear combinations of uniform random variables. *J. Statist. Plann. Inference.* **100** 13-21.
7. Kochar, S. C. and Ma, C. (1999). Dispersive ordering of convolutions of exponential random variables. *Statist. Probab. Lett.* **43**, 321-324.
8. Korwar, R. M. (2002). On stochastic orders for sums of independent random variables. *J. Mult. Anal.* **80**, 344-357.
9. Lewis, T. and Thompson, J. W. (1981). Dispersive distribution and the connection between dispersivity and strong unimodality. *J. Appl. Probab.* **18**, 76-90.
10. Marshall, A. W. and Olkin, I. (1979). *Inequalities : Theory of Majorization and Its Applications.* Academic Press, New York.
11. Saunders, I. W. and Moran, P. A. P. (1978). On quantiles of the gamma and F distributions. *J. Appl. Probab.* **15**, 426-432.
12. Shaked, M. and Shanthikumar, J. G. (1994). *Stochastic Orders and their Applications.* Academic Press, San Diego, CA.