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March 26, 2004
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CHACTERIZATION OF SOME PROBABILITY DISTRIBUTIONS THROUGH BINARY ASSOCIATIVE OPERATION

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Abstract: A binary operation * over real numbers is said to be associative if $(x*y)*z = x*(y*z)$ and it is said to be reducible if $x*y = x*z$ or $y*w = z*w$ if and only if $z = y$. The operation * is said have an identity element $\tilde{e}$ if $x*\tilde{e} = x$ for all x. We characterize different probability distributions under binary operations on the random variables.

AMS (2000) Subject Classification: Primary 62E10.

Keywords and phrases: Characterization, Binary associative operation, Exponential distribution, Beta distribution, Gamma distribution, Uniform distribution, Weibull distribution, Pareto distribution.

1 Introduction

Suppose U is uniformly distributed on the interval $[0,1]$. Let Y_1 and Y_2 be independent non-negative random variables independent of a random variable Y and U. Suppose further that

$$Y \stackrel{d}{=} U(Y_1 + y_2)$$

in the sense that the random variables Y and $U(Y_1 + Y_2)$ have the same distribution. Kotz and Stuetel (1988) proved that the above equation characterizes the exponential distribution. We now extend such results to other distributions such as Pareto distribution and Weibull distribution by unifying the results via binary associative operation on the random variables.

A binary operation * over real numbers is said to be associative if $(x*y)*z = x*(y*z)$ and it is said to be reducible if $x*y = x*z$ or $y*w = z*w$ if and only if $z = y$. The operation * is said have an identity element $\tilde{e}$ if $x*\tilde{e} = x$ for all x. it is known that the general reducible continuous solution of the functional equation is $x*y = g^{-1}(g(x)+g(y))$ where $g(.)$ is a continuous and strictly monotone function provided $x, y, x*y$ belong to a fixed possibly infinite interval A(cf. Aczel (1966)). The function g is determined up to a multiplicative constant, that is, $g_1^{-1}(g_1(x)+g_1(y)) = g_2^{-1}(g_2(x)+g_2(y))$ for all x and y in a fixed interval A imples $g_2(x) = \alpha g_1(x)$ for all x in that interval for some $\alpha \neq 0$. We assume here after that the binary operation is reducible and associative with the function continuous and strictly increasing. Further asume that there exists an identity element $\tilde{e} \in \bar{R}$ such that $x*\tilde{e} = x, x \in A$. It is also known that every continuous reducible and associative operation defined on an interval A is commutative (cf. Aczel (1966), p.267).

Examples of such binary operations are given in Muliere and Scarsini (1987). For instance (i) if $A = (-\infty, \infty)$ and $x * y = x + y$, then $g(x) = x$, (ii) if $A = (0, \infty)$ and $x * y = xy, x > 0, y > 0$, then $g(x) = \log x$, (iii) if $A = (0, \infty)$ and $x * y = (x^\alpha + y^\alpha)^{1/\alpha}, x > 0, y > 0$ for some $\alpha > 0$, then $g(x) = x^\alpha$, (iv) if $A = (-1, \infty)$ and $x * y = x + y + xy + 1, x > -1, y > -1$, then $g(x) = \log(1 + x)$,(v) if $A = (0, \infty)$ and $x * y = xy/(x + y), x > 0, y > 0$, then $g(x) = 1/x$, and (vi) if $A = (0, \infty)$ and $x * y = (x + y)/(1 + xy), x > 0, y > 0$, then $g(x) = arth\ \ x$.

A characterization of the multivariate normal distribution through a binary associative operation which is associative was given in Prakasa Rao (1974) and in Prakasa Rao (1977) for gaussian measures on locally compact abelian groups. Muliere and Scarsini (1987) characterize a class of bivariate distributions that generalize the Marshal-Olkin bivariate exponential distribution through a functional equation invoving two binary associative operations. Some general results on characterization of probability distributions through binary associative operayions are studied in Muliere and Prakasa Rao (2003). A characterization of bivariate probability distributions using the bivariate lack of memory property under binary associative operation is discussed in Prakasa Rao (2004).

We now study an extension of results in Kotz and Stuetel (1968) through binary associative operation there by giving characterizations of Weibull, Pareto and exponential distributions. By choosing the binary associative operation appropriately, we obtain characterizations for different classes of distributions. For instance, (i) if $x * y = x + y$, then we obtain the characterization of exponential distribution; (ii) if $xy = x * y$, then we obtain the characterization of the Pareto distribution ; and (iii) if $x * y = (x^\alpha + y^\alpha)^{1/\alpha}$, then we obtain a characterization of the Weibull distribution. Inter alia, we derive some properties of the uniform, Beta and Gamma distributions.

2 Preliminaries

Suppose $f(x)$ is a real-valued function that is defined almost everywhere for $x \geq 0$ and is such that

$$\int_0^1 |f(x)| x^{c_1 - 1} dx < \infty \text{ and } \int_1^\infty |f(x)| x^{c_2 - 1} dx < \infty$$

for some real numbers c_1 and c_2 with $c_1 < c_2$. Then the Mellin tranform of $f(x)$ is defined by

$$\hat{f}(s) = \int_0^\infty x^{s-1} f(x) dx$$

where $s = c + i\tau$ is a complex variable with $c_1 < c < c_2$.

If the Mellin transform exists and is an analytic function of the complex variable s for $c_1 \le Re(s) \le c_2$, where c_1 and c_2 are real, then the inversion integral converges to the function $f(x)$, that is,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)x^{-s}ds$$

where $c_1 < c < c_2$ where the integration path is parellel to the imaginary axis of the complex plane s and the integral is iunderstood in the sense of the Cauchy principal value (cf. Polyanin and Manzirov (1998), pp. 433-434; Springer (1979), pp.30-31.)

3 Main result

Let * be a binary associative operation as described in Section 1 and $g(.)$ be the associated function corresponding to the operation *. Suppose that $g(X_1)$ and $g(X_2)$ are independent and identically distributed (i.i.d) non-negative random variables which are independent of another random variable W. Our problem is to determine the distribution of the random variable W such that

$$g(X) \stackrel{d}{=} W\left[g\left(X_1 * X_2\right)\right]. \tag{1}$$

For any random variables X and Y, we write $X \stackrel{d}{=} Y$ if X and Y have the same distribution. From (1) we obtain

$$g(X) \stackrel{d}{=} W\left[g(X_1) + g(X_2)\right]. \tag{2}$$

Let $F_X(.)$ denote the distribution function of any random variable X and $f_X(.)$ denote the probability density function of X whenever it exists. For any random variable X, let

$$\phi_g(s) = E\left(e^{s\,g(X)}\right) \equiv \int_R e^{s\,g(x)}dF_X(x)$$

denote the integral tranform of the distribution of $g(X)$. Then (2) is equivalent to

$$\phi_g(s) = \int_R \phi_g^2(sw)f_W(w)dw. \tag{3}$$

Then, in order to find $f_W(.)$, we need to solve the integral equation (3).

We give a solution of (3) using the technique of Mellin transforms (see A. D. Polyamin and A. V. Manzhirov (1998), pp. 495-496). For completness, we give also the derivation of the solution. Multiplying on both sides of the equation (3) by s^{t-1} and integrating with respect to s from 0 to ∞, we obtain that

$$\int_0^\infty f_W(w)dw\int_0^\infty \phi_g^2(sw)s^{t-1}ds = \int_0^\infty \phi_g(s)s^{t-1}ds. \tag{4}$$

In the equation (4), we note that

(i) $\hat{\phi}_g(t) = \int_0^\infty \phi_g(s)s^{t-1}ds$ is the Mellin transform of the integral transform $\phi_g(s)$ of the distribution of $g(X)$, and

(ii) $\hat{\phi}_g^2(t) = \int_0^\infty \phi_g^2(s)s^{t-1}ds$ is the Mellin transform of the integral transform $\phi_g^2(s)$ of the distribution of the random variable $g(X_1) + g(X_2)$ with X_1 and X_2 independent.

Now we make the change of variable $z = sw$ in the inner integral of the double integral. This implies the relation

$$\hat{\phi}_g^2(s) \int_0^\infty f_W(w)w^{-s}dw = \hat{\phi}_g(s). \tag{5}$$

Taking into account the formula

$$\int_0^\infty f_W(w)w^{-s}dw = \hat{f_W}(1-s),$$

we can write the equation (5) in the form

$$\hat{\phi}_g^2(s)\hat{f_W}(1-s) = \hat{\phi}_g(s). \tag{6}$$

Replacing $1-s$ by s in the equation (6) and solving the resulting equation for $\hat{f_W}(s)$, we obtain the Mellin transform

$$\hat{f_W}(s) = \frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)} \tag{7}$$

of the required solution $f_W(w)$ of the equation (4).

Applying now the inversion formula for the Mellin transforms, we obtain the solution of the integral equation (3) in the form

$$f_W(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)} w^{-s}ds. \tag{8}$$

If (i) $f_W(w) \geq 0$, and (ii) $\int_0^\infty f_W(w)dw = 1$, then we have the answer to our question stated by the equation (1).

Remark 3.1: Obviously we can generalize the above discussion to find the distribution of a random variable W independent of n i.i.d. random variables $X_i,\ 1 \leq i \leq n$ satisfying

$$g(X) \stackrel{d}{=} W\left[g(X_1) + g(X_2) + \cdots + g(X_n)\right].$$

In analogy with (3), we obtain the integral equation

$$\phi_g(s) = \int_0^\infty \phi_g^n(sw)f_W(w)dw.$$

4 Examples

We present some examples in this section and derive some known results.

Example 4.1: Kotz and Steutel (1988) presented a new approach to characterize the exponential distributions. They have proved that if U, X_1 and X_2 are independent random variables, U is uniformly distributed in $(0, 1)$ and X_1, X_2 distributed as X, a nonnegative random variable, satisfying $X \stackrel{d}{=} U\left[X_1 + X_2\right]$, then X is exponentialy distributed. Following our approach, suppose that

(i) $g(X)$ has the same distribution as $g(X_1)$ and $g(X_2)$, which are independent, and

(ii) $g(X)$ is exponentially distributed, and

(iii) $g(X) \stackrel{d}{=} W\left[g(X_1 * X_2)\right]$ with the random variable W independent of X_1 and X_2.

We will now show that there is a unique solution to the above problem and that W has the uniform distribution on $(0, 1)$.

Without loss of generality, we assume that $E(g(X)) = 1$. Suppose that $g(X) = Y$ has the probability density function $f_y(y)$ given by

$$\begin{aligned} f_y(y) &= e^{-y} \text{ if } y > 0 \\ &= 0 \text{ if } y \leq 0. \end{aligned}$$

The Laplace transform of Y is

$$L_Y(s) = \frac{1}{1+s}$$

and the corresponding Mellin transform of this Laplace transform $L_Y(s)$ is

$$\hat{\phi}_g(s) = \frac{\pi}{\sin(\pi s)}, \quad 0 < Re(s) < 1.$$

The Laplace transform of $g\left(X_1\right) + g\left(X_2\right)$ is

$$L_{g(X_1)+g(X_2)}(s) = \frac{1}{(1+s)^2}$$

and the corresponding Mellin transform of this Laplace transform $L_{g(X_1)+g(X_2)}(s)$is

$$\hat{\phi}_g^2(s) = \frac{(-1)\pi}{\sin(\pi s)} C_{s-1}^{n}$$

where $C_{s-1}^n = \frac{\Gamma(s-1+1)}{\Gamma(n)\Gamma(s-1-n+1)}$ (cf. Polyanin and Manzhirov (1998), p.757). Observe that, for $n = 1$,

$$\begin{aligned} C_{s-1}^n &= \frac{\Gamma(s)}{\Gamma(1)\Gamma(s-1-1+1)} = \frac{\Gamma(s)}{\Gamma(s-1)} = \frac{(s-1)\Gamma(s-1)}{\Gamma(s-1)} \\ &= (s-1). \end{aligned}$$

Hence

$$\hat{\phi}_g^2(s) = -\frac{\pi}{\sin(\pi s)}(s-1)$$

and

$$\begin{aligned} \hat{f_W}(s) &= -\frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)} \\ &= \left(\frac{\pi}{\sin(\pi(1-s))}\right)\left(-\frac{\sin(\pi(1-s))}{(1-s-1)\pi}\right) = \frac{1}{s}. \end{aligned}$$

Using the inversion formula for the Mellin transform, we obtain that

$$f_W(w) = \begin{cases} 1, & 0 < w < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have the following result.

Proposition 4.1: Suppose that $g(X_1)$ and $g(X_2)$ are independent and identically distributed non-negative random variables which are independent of another random variable W and that

$$g(X) \stackrel{d}{=} W\left[g(X_1 * X_2)\right].$$

Then

i) $g(X)$ has the same distribution as that of $g(X_1)$ and $g(X_2)$, and that

ii) $g(X)$ is exponentially distributed,

imply that the random variable W has the uniform distribution on $(0, 1)$.

Example 4.2: Suppose that $g(X_1)$ and $g(X_2)$ are independent identically distributed (absolutely continuous) nonnegative random variables and W is a random variable, independent of X_1 and X_2, with bounded support. Suppose that

$$g(X) \stackrel{d}{=} W\left[g(X_1) + g(X_2)\right]$$

Then, if

i) $g(X)$ has the same distribution as that of $g(X_1)$ and $g(X_2)$, and

ii) $g(X) \sim \Gamma(1, a)$, Gamma distribution with parameters 1 and a;

then $W \sim \text{Beta}(a, a)$, Beta distribution with parameters a and a.

We write that $Z \sim \Gamma(1, a)$, if the probability density function of Z is given by

$$
\begin{aligned}
f_Z(z) &= \frac{z^{a-1}e^{-z}}{\Gamma(a)}, \quad z > 0 \\
&= 0, \qquad z \leq 0.
\end{aligned}
$$

The Laplace transform of Z is $L_Z(s) = \left(\frac{1}{1+s}\right)^a$. The Mellin transform of this Laplace transform $L_Z(s)$ is

$$\hat{\phi}_g(s) = \int_0^\infty \left(\frac{1}{1+x}\right)^a x^{s-1} dx.$$

With the change of variable $\frac{1}{1+x} = y$, we obtain that

$$
\begin{aligned}
\hat{\phi}_g(s) &= \int_0^1 y^{a-s-1}(1-y)^{s-1} dy \\
&= B(a-s, s) = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}
\end{aligned}
$$

where $B(.,.)$ is the Beta-function.

Suppose $g(X_1) \sim \Gamma(1, a)$ and $g(X_2) \sim \Gamma(1, a)$ and X_1 and X_2 are independent. Then the Laplace transform of $g(X_1) + g(X_2)$ is

$$L_{g(X_1)+g(X_2)}(s) = \left(\frac{1}{1+s}\right)^{2a}$$

and the Mellin transform of this Laplace transform is

$$B(2a-s, s) = \frac{\Gamma(s)\Gamma(2a-s)}{\Gamma(2a)}.$$

From the result obtained in Section 2, we observe that the Mellin transform of W is

$$
\begin{aligned}
\hat{f_W}(s) &= \frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)} \\
&= \frac{B(a-(1-s), (1-s))}{B(2a-(1-s), (1-s))} \\
&= \frac{\Gamma(1-s)\Gamma(a+s-1)}{\Gamma(a)} \frac{\Gamma(2a)}{\Gamma(2a+s-1)\Gamma(1-s)} \\
&= \frac{\Gamma(2a)\Gamma(a+s-1)}{\Gamma(s)\Gamma(2a+s-1)}.
\end{aligned}
$$

Using the inversion formula for Mellin transforms, we obtain that

$$f_W(w) = \frac{1}{B(a,a)} w^{a-1}(1-w)^{a-1}, \quad 0 \le w \le 1$$

which in turn proves that the random variable $W \sim \text{Beta}(a,a)$, that is, W has the Beta disribution with parameters a and a.

If, in the above example, we assume that

i) $g(X)$, $g(X_1)$ and $g(X_2)$ are independent but not necessarily identically distributed such that $g(X) \sim \Gamma(1,a)$, $g(X_1) \sim \Gamma(1,a)$ and $g(X_2) \sim \Gamma(1,b)$,

then , it can be shown that $W \sim \text{Beta}(a,b)$.

This can be seen from the facts

$$\begin{aligned} \hat{\phi}_{g(X_1)}(s) &= \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}, \\ \hat{\phi}_{g(X_1)+g(X_2)}(s) &= \frac{\Gamma(s)\Gamma(a+b-s)}{\Gamma(a+b)}, \end{aligned}$$

then

$$\begin{aligned} \hat{f_W}(s) = \frac{\hat{\phi}_{g(X_1)}(1-s)}{\widehat{\phi}_{g(X_1)+g(X_2)}(1-s)} &= \frac{\Gamma(1-s)\Gamma(a+s-1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(1-s)\Gamma(a+b+s-1)} \\ &= \frac{\Gamma(a+b)\Gamma(a+s-1)}{\Gamma(a)\Gamma(a+b+s-1)}. \end{aligned}$$

Using the inversion formula for Mellin transforms, we obtain that

$$f_W(w) = \frac{1}{B(a,b)} w^{a-1}(1-w)^{b-1}, \quad 0 \le w \le 1.$$

Characterization of exponential distribution

We now combine the results obtained above in the following proposition.

Proposition 4.2: Suppose that $g(X_1)$ and $g(X_2)$ are independent and identically distributed nonnegative random variables which are independent of another random variable W and suppose that

$$g(X) \stackrel{d}{=} W\,[g(X_1) + g(X_2)]$$

Then any two of the following three conditions imply the third condition:

i) $g(X)$ has the same distribution as that of $g(X_1)$ and $g(X_2)$;

ii) W has a uniform distribution on $(0,1)$; and

iii) $g(X)$ is exponentially distributed.

Proof: We have observed that the conditions (i) and (iii) imply (ii) from Proposition 4.1. It is clear that the conditions (ii) and (iii) imply (i). This can be seen by the following arguments. From the equation (6), we have

$$\hat{\phi}_g^2(s)\hat{f_W}(1-s) = \hat{\phi}_g(s).(10)$$

Since W is uniform on $[0,1]$, the Mellin tranform of W is $\hat{f_W}(s) = \frac{1}{s}$. Since the random variable $g(X)$ is standard exponential, the Mellin transform of the Laplace transform of $g(X)$ is

$$\hat{\phi}_g(s) = \frac{\pi}{\sin(\pi s)}.$$

Therefore

$$\hat{\phi}_g^2(s) = \frac{\pi(1-s)}{\sin(\pi s)} = -\frac{\pi(s-1)}{\sin(\pi s)}$$

from the equation (10). Using the inversion formula of Mellin and Laplace transforms, we obtain that the random variables $g(X_1)$ and $g(X_2)$ have standard exponential distributions.

We now prove that (i), (ii) imply (iii). From (9) and (i), we have

$$\phi_g(s) = \int_R [\phi_g(sw)]^2 f_W(w)dw.$$

Condition (ii) implies that

$$\begin{aligned}\phi_g(s) &= \int_0^1 (\phi_g(sw))^2 dw \\ &= \frac{1}{s}\int_0^1 \phi_g^2(w)dw.\end{aligned}$$

Following Kotz and Steutel (1998) and Alamatsaz (1985), we obtain that

$$\phi_g(s) = \frac{1}{1+ag(s)}$$

with $a \geq 0$. Hence $g(X)$ has an exponential distribution. It is immediate to verify that

$$\frac{1}{s} = \frac{\hat{\phi}_g(1-s)}{\hat{\phi}_g^2(1-s)}.$$

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