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Estimation for Translation of a Process Driven by Fractional Brownian Motion

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Abstract

We investigate the general problem of estimating the translation of a stochastic process governed by a stochastic differential equation driven by a fractional Brownian motion. The special case of the Ornstein-Uhlenbeck process is discussed in particular.

Keywords and phrases: Stochastic differential equation ; fractional Ornstein-Uhlenbeck type process; fractional Brownian motion; Estimation for translation ; Maximum likelihood estimation.

AMS Subject classification (2000): Primary 62M09, Secondary 60G15.

1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion (fBm). Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. Sequential estimation problem for the fractional Ornstein-Uhlenbeck type process was discussed in Prakasa Rao (2004). This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by an fBm $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1. 1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. We discussed more general classes of stochastic processes satisfying linear stochastic differential equations driven an fBm and studied the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes in Prakasa Rao

(2003a,b). Recently Baran and Pap (2003) considered the problem of estimation of the mean for the translation of an Ornstein-Uhlenbeck process. We now consider similar problems for processes governed by stochastic differential equations driven by a fBm. Interalia we obtain sufficient conditions for the absolute continuity of the measures generated by a stochastic process $\{Y(t), 0 \leq t \leq T\}$ driven by a fBm with Hurst index $H \in (0,1)$ and its translation $\{\tilde{Y}(t), 0 \leq t \leq T\}$ with $\tilde{Y}(t) = Y(t) + g(t)$ and $g(t)$ nonrandom and obtain the Radon-Nikodym derivative in case the measures are absolutely continuous . As a consequence we study the maximum likelihood estimation of the parameter m when the function $g(t) = mh(t)$ with a known function $h(.)$ satisfying$h(0) = 0$ and unknown parameter m. We consider the special case of the fractional Ornstein-Uhlenbeck type process with the Hurst index $H \in (\frac{1}{2}, 1)$ in more detail. Hu (2001) studied the prediction and translation problems for fractional Brownian motion using fractional calculus methods. However our approach to the problem is via the techniques developed by Kleptsyna et al. (2000). Norros et al. (1999) considered the case of constant drift or equivalently the case when $\tilde{Y}(t) = Y(t) + mt$ and derived the maximum likelihood estimator of the parameter m when Y is a fBm with Hurst index $H \in [\frac{1}{2}, 1)$.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions.The natural fitration of a stochastic process is understood as the P-completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

(2. 1) $$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0.$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

(2. 2) $$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience we write the above integral equation in the form of a stochastic differential equation

(2. 3) $$dY_t = C(t)dt + B(t)dW_t^H, Y(0) = 0, t \geq 0$$

driven by the fractional Brownian motion W^H. The integral

(2. 4) $$\int_0^t B(s)dW_s^H$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBM in a natural sense (cf. Norros et al. (1999)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2. 5}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2. 6}$$

$$\lambda_H = \frac{2H\;\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2. 7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2. 8}$$

and

$$M_t^H = \int_0^t k_H(t,s)dW_s^H, t \geq 0. \tag{2. 9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variation $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural fitration of the fBM W^H. In fact the stochastic integral

$$\int_0^t B(s)dW_s^H \tag{2. 10}$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr, 0 \leq s \leq t \tag{2. 11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Therorem 2.1: Let M^H be the fundamental martingale associated with the fBM W^H defined by (2.9). Then

$$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, t \in [0,T] \tag{2. 12}$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. (1993)) so that

$$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}ds, t \in [0,T] \tag{2. 13}$$

is well-defined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absoulute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2: Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by

(2. 14) $$Z_t = \int_0^t k_H(t,s) B^{-1}(s) dY_s$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold:
(i) The process Z is an $(\mathcal{F}_t)$ -semimartingale with the decomposition

(2. 15) $$Z_t = \int_0^t Q_H(s) dw_s^H + M_t^H$$

where M^H is the fundamental martingale defined by (2.9), (ii) the process Y admits the representation

(2. 16) $$Y_t = \int_0^t K_H^B(t,s) dZ_s$$

where the function K_H^B is as defined in (2.11), and (iii) the natural fitrations of $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000) derived the following Girsanov type formula as a consequence of the Theorem 2.2.

Theorem 2.3: Suppose the assumptions of Theorem 2.2 hold. Define

(2. 17) $$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t) dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t) dw_t^H\}.$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

(2. 18) $$V_t = \int_0^t B(s) dW_s^H, 0 \le t \le T.$$

.

Consider now the process $\tilde{Y}(t) = Y(t) + g(t), t \ge 0$ where $g(.)$ is an absolutely continuous function with $g(0) = 0$. Note that the function $g(.)$ is almost everywhere differentiable. Let $g'(t)$ denote the derivative of $g(t)$ wherever it exists and define it to be zero elsewhere. The process $\tilde{Y}(t)$ satisfies the integral equation

(2. 19) $$\tilde{Y}_t = g(t) + \int_0^t C(s) ds + \int_0^t B(s) dW_s^H, t \ge 0$$

where $C = \{C(t), t \ge 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a nonvanishing nonrandom function. For convenience, we write the above integral equation in the form of a stochastic differential equation

(2. 20) $$d\tilde{Y}_t = (C(t) + g'(t))dt + B(t) dW_t^H, \tilde{Y}(0) = 0, t \ge 0$$

driven by the fractional Brownian motion W^H. Let

$$\tilde{C}(t) = C(t) + g'(t) \tag{2. 21}$$

and define

$$\tilde{Q}_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{\tilde{C}(s)}{B(s)}ds, t \geq 0 \tag{2. 22}$$

Observe that

$$\begin{aligned}\tilde{Q}_H(t) &= Q_H(t) + \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{g'(s)}{B(s)}ds \\ &= Q_H(t) + g^*_{H,B}(t) \ \ \text{(say)}.\end{aligned} \tag{2. 23}$$

Furthermore we have the following analogue of Theorem 2.3.

Theorem 2.4: Suppose the function $g(.)$ is such that the sample paths of the process $\tilde{Q}_H$ defined by (2.23) belong P-a.s to $L^2([0,T], dw^H)$. Define

$$\tilde{\Lambda}_H(T) = \exp\{-\int_0^T \tilde{Q}_H(t)dM_t^H - \frac{1}{2}\int_0^t \tilde{Q}_H^2(t)dw_t^H\}. \tag{2. 24}$$

Suppose that $E(\tilde{\Lambda}_H(T)) = 1$. Then the measure $\tilde{P}^* = \tilde{\Lambda}_H(T)P$ is a probability measure and the probability measure of the process $\tilde{Y}$ under $\tilde{P}^*$ is the same as that of the process V defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \leq t \leq T. \tag{2. 25}$$

. As a consequence of Theorems 2.3 and 2.4, we obtain the following result.

Theorem 2.5: Let $\tilde{P}_T^*$ and P_T^* be the probability measures generated by the processes $\tilde{Y}$ and Y respectively on the interval $[0,T]$. Then the measures are absolutely continuous with respect to each other and the Radon-Nikodym derivative of $\tilde{P}_T^*$ with respect to P_T^* is given by

$$\frac{d\tilde{P}_T^*}{dP_T^*} = \exp\{-\int_0^T \{\tilde{Q}_H(t) - Q_H(t)\}dM_t^H - \frac{1}{2}\int_0^t \{\tilde{Q}_H^2(t) - Q_H^2(t)\}dw_t^H\}. \tag{2. 26}$$

3 Maximum Likelihood Estimation of translation

Let us now suppose that $g(t) = m\ h(t)$ with $h(0) = 0$ in the discussion in the previous section and suppose that the functions $h(.)$ and $B(.)$ are *known* and $h(.)$ is differentiable everywhere but the constant m is *unknown*. The problem is to estimate the parameter m based on the observation $\{\tilde{Y}_t, 0 \leq t \leq T\}$. Observe that

$$\begin{aligned}\tilde{Q}_H(t) &= Q_H(t) + m(\frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{h'(s)}{B(s)}ds) \\ &= Q_H(t) + m\ h^*_{H,B}(t) \\ &= Q_H(t) + m\ h^{**}(t) \ \ \text{(say)}.\end{aligned} \tag{3. 1}$$

Following the notation used above, it follows that

(3. 2)
$$\tilde{Z}_t = \int_0^t \tilde{Q}_H(s) dw_s^H + M_t^H, t \geq 0.$$

Applying Theorem 2.5, we get that

(3. 3)

$$\begin{aligned}
\frac{d\tilde{P}_T^*}{dP_T^*} &= \exp\{-m \int_0^T h^{**}(t) dM_t^H - \frac{1}{2} \int_0^T (2m\tilde{Q}_H(t)h^{**}(t) - m^2(h^{**}(t))^2)\} dw_t^H\} \\
&= \exp\{-m \int_0^T h^{**}(t)(d\tilde{Z}_t - \tilde{Q}_H(t)dw_t^H) - \frac{1}{2} \int_0^T (2m\tilde{Q}_H(t)h^{**}(t) - m^2(h^{**}(t))^2) dw_t^H\} \\
&= \exp\{-m \int_0^T h^{**}(t) d\tilde{Z}_t + \frac{1}{2} m^2 \int_0^T (h^{**}(t))^2 dw_t^H\}.
\end{aligned}$$

Suppose that

$$0 < \int_0^T (h^{**}(t))^2 dw_t^H < \infty.$$

Then we obtain that the maximum likelihood estimator of m based on the process $\{\tilde{Y}(t), 0 \leq t \leq T\}$ is given by

(3. 4)
$$\hat{m}_T = \frac{\int_0^T h^{**}(t) d\tilde{Z}_t}{\int_0^T (h^{**}(t))^2 dw_t^H}.$$

Remarks: Observe that the estimator $\hat{m}_T$ does not directly depend on the process $\{C(t)\}$ but through the observation of the process $\{\tilde{Z}_t, 0 \leq t \leq T\}$.

Suppose m_0 is the true value of m. Then it follows that

(3. 5)
$$\hat{m}_T - m_0 = \frac{\int_0^T h^{**}(t) dZ_t}{\int_0^T (h^{**}(t))^2 dw_t^H}.$$

Suppose that $h(t) \equiv t$ in the above discussion which reduces to the constant drift case. Then

(3. 6)
$$h^{**}(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{1}{B(s)} ds$$

and the corresponding estimator $\hat{m}_T$ for the parameter m can be computed using the equation (3.4) once the function $B(.)$ is known.

4 Fractional Ornstein-Uhlenbeck type process

Suppose the process $\{Y_t, t \geq 0\}$ satisfies the stochastic integral equation

(4. 1)
$$Y_t = \theta \int_0^t Y_s ds + \sigma W_t^H, t \geq 0$$

or equivalently the stochastic differential equation

$$dY_t = \theta Y_t dt + \sigma dW_t^H, Y_0 = 0, t \geq 0 \tag{4. 2}$$

with known Hurst index $H \in [\frac{1}{2}, 1)$. Such a process is called a *fractional Ornstein-Uhlenbeck type process.* Suppose we observe the process $\{\tilde{Y}_t, 0 \leq t \leq T\}$ where $\tilde{Y}_t = Y_t + m\ h(t)$ with $h(0) = 0$. Further suppose that the function $h(.)$ is *known* and everywhere differentiable and satisfies the condition

$$0 < \int_0^T (h^{**}(t))^2 dw_t^H < \infty.$$

but the parameter m is *unknown.* The problem is to estimate the parameter m based on the observation of the process $\{\tilde{Y}_t, 0 \leq t \leq T\}$. Following the results given in the previous section, we obtain that

$$\hat{m}_T = \frac{\int_0^T h^{**}(t) d\tilde{Z}_t}{\int_0^T (h^{**}(t))^2 dw_t^H} \tag{4. 3}$$

where

$$\begin{aligned} d\tilde{Z}_t &= \tilde{Q}_H(t) dw_t^H + dM_t^H \\ &= (Q_H(t) + mh^{**}(t)) dw_t^H + dM_t^H \\ &= dZ_t + mh^{**}(t) dw_t^H, \end{aligned} \tag{4. 4}$$

$$\tilde{Q}_H(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{\theta Y_s + mh'(s)}{\sigma} ds, \;\; Q_H(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{\theta Y_s}{\sigma} ds \tag{4. 5}$$

and

$$h^{**}(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) \frac{h'(s)}{\sigma} ds \tag{4. 6}$$

Suppose m_0 is the true value of m. Then it follows that

$$\hat{m}_T - m_0 = \frac{\int_0^T h^{**}(t) dZ_t}{\int_0^T (h^{**}(t))^2 dw_t^H}. \tag{4. 7}$$

It is easy to see that the solution of the stochastic differential equation (4.1) is given by

$$Y_t = \sigma \int_0^t e^{\theta(t-u)} dW_u^H, t \geq 0. \tag{4. 8}$$

Hence the process $Y_t, t \geq 0$ with $Y_0 = 0$ is a zero mean gaussian process with covariance function given by

$$\begin{aligned} Cov(Y_t, Y_s) &= \sigma^2 E\{\int_0^t e^{\theta(t-u)} dW_u^H \int_0^s e^{\theta(s-v)} dW_v^H\} \\ &= \sigma^2 H(2H-1) e^{\theta(t+s)} \int_0^\infty \int_0^\infty f(u) g(v) |u-v|^{2H-2} du dv \end{aligned} \tag{4. 9}$$

where

$$\begin{aligned} f(u) &= e^{-\theta u} \text{ if } 0 \leq u \leq t \\ &= 0 \text{ otherwise} \end{aligned} \tag{4. 10}$$

and

(4. 11)
$$g(v) = e^{-\theta v} \text{ if } 0 \leq v \leq s$$
$$= 0 \text{ otherwise.}$$

This follows by results in Pipiras and Taqqu (2000). From the representation (2.14), we obtain that $\{Z_t, t \geq 0\}$ is a zero mean gaussian process. Hence it follows from the representation given by (4.7) that $\hat{m}_T - m_0$ has the gaussian distribution with mean zero and variance given by

(4. 12)
$$Var(\hat{m}_T) = \frac{E(\int_0^T h^{**}(t)dZ_t)^2}{(\int_0^T (h^{**}(t))^2 dw_t^H)^2}.$$

Observe that

(4. 13)
$$\int_0^T h^{**}(t)dZ_t = \int_0^T h^{**}(t)Q_H(t)dw_t^H + \int_0^T h^{**}(t)dM_t^H$$

from (2.15) and hence

(4. 14)
$$\begin{aligned}
E(\int_0^T h^{**}(t)dZ_t)^2 &= Var(\int_0^T h^{**}(t)dZ_t) \\
&= Var(\int_0^T h^{**}(t)Q_H(t)dw_t^H) + Var(\int_0^T h^{**}(t)dM_t^H) \\
&\quad +2\, Cov(\int_0^T h^{**}(t)Q_H(t)dw_t^H, \int_0^T h^{**}(t)dM_t^H) \\
&= Var(\int_0^T h^{**}(t)Q_H(t)dw_t^H) + 2\, Cov(\int_0^T h^{**}(t)Q_H(t)dw_t^H, \int_0^T h^{**}(t)dM_t^H) \\
&\quad + \int_0^T (h^{**}(t))^2 dw_t^H.
\end{aligned}$$

Therefore

(4. 15)
$$\begin{aligned}
Var(\hat{m}_T) &= \frac{Var(\int_0^T h^{**}(t)Q_H(t)dw_t^H) + 2\, Cov(\int_0^T h^{**}(t)Q_H(t)dw_t^H, \int_0^T h^{**}(t)dM_t^H)}{(\int_0^T (h^{**}(t))^2 dw_t^H)^2} \\
&\quad + \frac{1}{\int_0^T (h^{**}(t))^2 dw_t^H}.
\end{aligned}$$

Suppose that $h(t) \equiv t$ in the above discussion which reduces to the constant drift case. Then

(4. 16)
$$\begin{aligned}
h^{**}(t) &= \frac{1}{\sigma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)ds \\
&= \frac{1}{\sigma}
\end{aligned}$$

and the corresponding estimator $\hat{m}_T$ for the parameter m can be computed using the equation (4.3) once the constant σ is known. In fact

$$\hat{m}_T = \sigma \frac{\tilde{Z}_T}{w_T^H}, \tag{4. 17}$$

$$\hat{m}_T - m_0 = \sigma \frac{Z_T}{w_T^H}. \tag{4. 18}$$

Further more

(4. 19)

$$\begin{aligned}
Var(\hat{m}_T) &= \sigma^2 \frac{Var(\int_0^T Q_H(t)dw_t^H) + 2\, Cov(\int_0^T Q_H(t)dw_t^H, M_T^H)}{(w_T^H)^2} \\
&\quad + \sigma^2 \frac{1}{w_T^H} \\
&\leq \sigma^2 \frac{E(\int_0^T Q_H(t)dw_t^H)^2 + 2\ ([Var(\int_0^T Q_H(t)dw_t^H)][Var(M_T^H)])^{1/2}}{(w_T^H)^2} \\
&\quad + \sigma^2 \frac{1}{w_T^H} \\
&\leq \sigma^2 \frac{E(\int_0^T Q_H^2(t)dw_t^H)w_T^H + 2\ ([E(\int_0^T Q_H(t)dw_t^H)^2][Var(M_T^H)])^{1/2}}{(w_T^H)^2} \\
&\quad + \sigma^2 \frac{1}{w_T^H} \\
&\leq \sigma^2 \frac{E(\int_0^T Q_H^2(t)dw_t^H)w_T^H + 2\ ([E(\int_0^T Q_H^2(t)dw_t^H)w_T^H][Var(M_T^H)])^{1/2}}{(w_T^H)^2} \\
&\quad + \sigma^2 \frac{1}{w_T^H} \\
&\leq \sigma^2 \frac{E(\int_0^T Q_H^2(t)dw_t^H)w_T^H + 2\ ([E(\int_0^T Q_H^2(t)dw_t^H)w_T^H][[w_T^H])^{1/2}}{(w_T^H)^2} \\
&\quad + \sigma^2 \frac{1}{w_T^H} \\
&\leq \sigma^2 \frac{E(\int_0^T Q_H^2(t)dw_t^H) + 2\ (E(\int_0^T Q_H^2(t)dw_t^H))^{1/2}}{w_T^H} + \sigma^2 \frac{1}{w_T^H}
\end{aligned}$$

from the representation (2.9) and an application of the Cauchy-Schwartz inequality and the Fubini's theorem. If a bound on the term

$$E(\int_0^T Q_H^2(t)dw_t^H)$$

can be obtained as a function of T, then it is posible to obtain an upper bound on the variance term given above. It is possible to get an explicit expression for

$$\Psi_T(\theta; a) = E[\exp\{-a \int_0^T Q_H^2(t)dw_t^H\}], a > 0$$

as given in Proposition 3.2 of Kleptsyna and Le Breton (2002) and hence

$$E(\int_0^T Q_H^2(t)dw_t^H) = -\lim_{a\to 0+} \Psi'_T(\theta : a).$$

It is known from the arguments given in Kleptsyna and Le Breton (2002) that

$$\int_0^T Q_H^2(t)dw_t^H \to \infty \text{ a.s as } T \to \infty.$$

However explicit computation of the expectation defined above seems to be difficult. If

$$E(\int_0^T Q_H^2(t)dw_t^H) = o(w_T^H)$$

as $T \to \infty$, then we obtain that

$$Var(\hat{m}_T) \to 0 \text{ as } T \to \infty$$

and hence $\hat{m}_T \xrightarrow{p} m_0$ as $T \to \infty$ since $E(\hat{m}_T) = m_0$ for all T. Hence $\hat{m}_T$ is a consistent estimator of m_0 under the above condition.

An alternate way of viewing the equation (4.18) is by writing it in the form

$$\frac{w_T^H(\hat{m}_T - m_0)}{\sigma} = Z_T = \frac{1}{\sigma}\int_0^T k_H(T,s)dY_s \tag{4. 20}$$

or equivalently

$$w_T^H(\hat{m}_T - m_0) = \int_0^T k_H(T,s)dY_s \tag{4. 21}$$

which in turn shows that the distrbution of the estimator $\hat{m}_T$ is normal with the mean m_0 and the variance

$$(w_T^H)^{-2}E[\int_0^T k_H(T,s)dY_s]^2.$$

Remarks: If the parameter $\theta = 0$, then the process $\{Y_t, t \geq 0\}$ reduces to the fBm and

$$Var(\hat{m}_T) = \sigma^2\frac{1}{w_T^H} = \sigma^2\lambda_H T^{2H-2} \tag{4. 22}$$

from (2.8) and the definition of the process $\{Q_H(t), t \geq 0\}$. Since the Hurst index $H \in [\frac{1}{2}, 1)$, it follows that

$$Var(\hat{m}_T) \to 0 \text{ as } T \to \infty. \tag{4. 23}$$

Combining this observation with the fact that $E(\hat{m}_T) = m_0$, it follows that $\hat{m}_T \xrightarrow{p} m_0$ as $T \to \infty$. In other words, the estimator $\hat{m}_T$ is a consistent estimator for m_0. A stronger result also follows from the fact that, in case $\theta = 0$,

$$\hat{m}_T - m_0 = \sigma\frac{M_T^H}{w_T^H} \tag{4. 24}$$

and the last term tends to zero almost surely as $T \to \infty$ by the Strong law of large numbers for martingales (cf. Prakasa Rao (1999b), p.61) since the quadratic variation of the martingale M^H is w^H and $w_T^H \to \infty$ as $T \to \infty$. The strong consistency of the estimator $\hat{m}_T$, for the case of the fBm, was earlier proved in Norros et al. (1999).

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