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# On Some Inequalities for N-demimartingales

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**On Some Inequalities for N-demimartingales**

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**Abstract:** Concepts of N-demimartingales, N-demisupermartingales and strong N-demimartingales are introduced. An example of a N-demimartingale is a sequence of the partial sums of a sequence negatively associated zero mean random variables . A Chow type maximal inequality for N-demimartingale is obtained and a comparison theorem on the moment inequalities between a strong N-demimartingale and a sum of independent random variables is derived.

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**Key words and phrases:** N-demimartingale; N-demisupermartingale; Strong N-demimartingale; Negatively associated sequence; Maximal inequality; Comparison theorem; Moment inequality.

# 1 Introduction

A finite family of random variables $\{X_1, \ldots, X_n\}$ is said to *negatively associated* $(NA)$ if for every pair of disjoint subsets $A$ and $B$ of $\{1, 2, \ldots, n\}$ and coordinate wise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$, the inequality

$$\text{cov } (f(X_i, i \in A), g(X_j, j \in B)) \leq 0$$

holds whenever this covariance exists. An infinite family of random variables is said to be *negatively associated* if every finite sub-family is $NA$. Properties of negatively associated random variables and their applications are discussed in Joagdev and Proschan (1983) and Newman (1984). It is known that the class of negatively associated random variables has the property that it is closed under the formation of increasing functions of disjoint sets of random variables (cf. Joagdev and Proschan (1983)). Further more a number of multivariate distributions posess the negatively associated property, such as (a) multinomial distribution (b) multivariate hypergeometric distribution (c) Dirichlet distribution (d) Dirichlet compound multinomial distribution (e) multivariate normal distribution with negatively correlated components (f) distributions arising from random sampling without replacement in survey sampling and (g) the joint distribution of the ranks, in the sense that if $\mathbf{X} = (X_1, \ldots, X_n)$ has the multivariate distribution specified in one of (a) to (g) specified above, then the corresponding components $X_i, 1 \leq i \leq n$ are negatively associated random variables. In view of their applications in multivariate statistical analysis and reliability theory, there is a lot of interest in the study of the probabilistic properties of families of negatively associated random variables. Probability and moment bounds for sums of negatively associated random variables are investigated in Matula (1997). A law of iterated logarithm for negatively associated random variables is proved in Shao and Su (1999). A comparison theorem on moment inequalities between negatively associated random variables and independent random vasriables was proved by Shao (2000). An interesting property of negatively associated random variables is that if $X_i, 1 \leq i \leq n$ are negatively associated, then

$$E(X_1 \ldots X_n) \leq E(X_1) \ldots E(X_n).$$

For proof, see Joagdev and Proschan (1983). We now introduce the concept of an N-demimartingle and N-demisupermartingle and develop a Chow type maximal inequality for an N-demimartingale. We have introduced the concept of a negative demimartingale in Prakasa Rao (2002) which is now termed as N-demimartingale as suggested in Christofides (2003). It can be shown that the partial sums of a sequence of zero mean negatively associated random variables form an N-demimartingle. Other versions of some useful maximal inequalities for N-demimartingales were investigated in Christofides (2003). As opposed to the concepts of negatively associated random variables and N-demimartingales, there is also the notion of associated random variables (cf. Prakasa Rao and Dewan (2001)) and the corresponding concept of demimartingales (cf. Newman and Wright (1982); Prakasa Rao (2002)). A Chow type maximal inequality for demimartingales was derived in Wang (2003). We will not go into the discussion of these concepts and results in this paper. We introduce the concept of a strong N-demimartingale and obtain a comparison theorem on the moment inequalities between a strong N-demimartingale and a sum of independent random varaibles.

## 2 N-demimartingles

A sequence $\{S_n, n \geq 1\}$ is said to be a *N-demimartingale* if, for every coordinate wise nondecreasing function $f$,

$$E[(S_{j+1} - S_j) f(S_1, \ldots, S_j)] \leq 0, \quad j = 1, 2, \ldots \tag{2. 1}$$

whenever the expectation is defined. It is said to be a *N- demisupermartingale* if the relation (2.1) holds for every nonnegative coordinatewise nondecreasing function $f$ whenever the expectation is defined.

**Remarks:** If $\{S_n, n \geq 1\}$ is a N-demimartingale, then $\{-S_n, n \geq 1\}$ is also a N-demimartingale. In fact the sequence $\{aS_n + b, n \geq 1\}$ is a N-demimartingale for every $a$ and $b$ in $R$.(cf. Christofides (2003)).

We now discuss some examples of $N - demimartingales.$

**Lemma 2.1:** Suppose $\{X_i, i \geq 1\}$ are integrable mean zero negatively associated random variables. Let $S_j = X_1 + \ldots + X_j, \quad j \geq 1$ and $S_0 = 0$. Then

$$E((S_{j+1} - S_j) f(S_1, \ldots, S_j)) \leq 0, \quad j \geq 1$$

for every coordinate wise nondecreasing function $f$ and hence $\{S_n, n \geq 0\}$ forms a N-demimartingale.

**Proof:** Note that

$$\begin{aligned} & E[(S_{j+1} - S_j)f(S_1, \ldots, S_j)] \\ &= E[X_{j+1}f(S_1, \ldots, S_j)] \\ &= \operatorname{cov}(X_{j+1}, f(S_1, \ldots, S_j)) \leq 0 \end{aligned}$$

since $f(S_1, \ldots, S_j) = g(X_1, \ldots, X_j)$ where $g$ is a coordinatewise nondecreasing function and $\{X_1, \ldots, X_n\}$ are negatively associated.

**Lemma 2.2:** Let $\{X_i, i \geq 1\}$ be negatively associated random variables. Let $U_n$ be a U-statistic based on $X_1, \ldots, X_n$ and a kernel function $h(x_1, \ldots, x_m) = \Pi_{i=1}^m \tilde{h}(x_i)$ for some nondecreasing function with $E(\tilde{h}(X_i)) = 0, i \geq 1$. Then $S_n = \frac{n!}{m!(n-m)!} U_n, n \geq m$ is an N-demimartingale

**Proof:** See Proposition 1.2 in Christofides (2003).

**Remarks:** A martingale is a demimartingale as well as an N-demimartingale. If $\{S_n, \mathcal{F}_n, n \geq 1\}$ is a super martingale, where $\mathcal{F}_n = \sigma\{S_1, \ldots, S_n\}$, then $\{S_n, n \geq 1\}$ is an N-demisupermartingale. If $\{S_n, n \geq 1\}$ is an N-demisupermartingale, then $\{Y_n, n \geq 1\}$ is an N-superdemimartingale if $Y_n = aS_n + b$ for some $a \geq 0$ and $b$ real. For proofs, see Christofides (2003).

## 3 Chow type maximal inequality for N-demimartingales

We now discuss a maximal inequality for N-demimartingales akin to Chow's maximal inequality for martingales (cf. Chow (1960)). This result complements some recent results om maximal inequalities in Christofides (2003) for N-demimartingales.

**Theorem 3.1:** Let $\{S_n, n \geq 1\}$ be a N-demimartingale. Let $m(.)$ be a nonnegative nondecreasing function on $R$ with $m(0) = 0$. Let $g(.)$ be a function such that $g(0) = 0$ and suppose that

$$g(x) - g(y) \geq (y - x)h(y) \tag{3. 1}$$

for all $x, y$ where $h(.)$ is a nonnegative nondecreasing function. Further suppose that $\{c_k, 1 \leq k \leq n\}$ is a sequence of positive numbers such that $(c_k - c_{k+1})g(S_k) \geq 0, 1 \leq k \leq n-1$. Define

$$Y_k = \max\{c_1 g(S_1), \ldots, c_k g(S_k)\}, k \geq 1, Y_0 = 0.$$

Then

$$E(\int_0^{Y_n} u dm(u)) \leq \sum_{k=1}^{n} c_k E([g(S_k) - g(S_{k-1})] m(Y_n)). \tag{3. 2}$$

**Proof:** Let $Y_0 = 0$. Observe that

$$\begin{aligned} E(\int_0^{Y_n} udm(u)) &= \sum_{k=1}^{n} E(\int_{Y_{k-1}}^{Y_k} udm(u)) \\ &\leq \sum_{k=1}^{n} E[Y_k(m(Y_k) - m(Y_{k+1}))]. \end{aligned} \tag{3. 3}$$

From the definition of $Y_k$, it follows that $m(Y_1) = 0$ for $Y_1 < 0$. Further more $Y_k \geq Y_{k-1}$ and either $Y_k = c_k g(S_k)$ or $m(Y_k) = m(Y_{k-1})$. Hence

$$E(\int_0^{Y_n} udm(u)) \leq \sum_{k=1}^{n} c_k E[g(S_k)(m(Y_k) - m(Y_{k-1}))] \tag{3. 4}$$

since $m(.)$ is a nondecreasing function and $Y_{k-1} \leq Y_k$. Note that

$$\begin{aligned} \sum_{k=1}^{n} c_k E[g(S_k)(m(Y_k) - m(Y_{k-1}))] &= \sum_{k=1}^{n} c_k E[(g(S_k) - g(S_{k-1}))m(Y_n)] \\ &\quad -\{\sum_{k=1}^{n-1} E[(c_{k+1}g(S_{k+1}) - c_k g(S_k))m(Y_k)] \\ &\quad +\sum_{k=1}^{n-1} E[(c_k - c_{k+1})g(S_k)m(Y_n)]\} \end{aligned} \tag{3. 5}$$

Let

$$\begin{aligned} A &= \sum_{k=1}^{n-1} E[(c_{k+1}g(S_{k+1}) - c_k g(S_k))m(Y_k)] \\ &\quad +\sum_{k=1}^{n-1} E[(c_k - c_{k+1})g(S_k)m(Y_n)]. \end{aligned} \tag{3. 6}$$

Since $(c_k - c_{k+1})g(S_k) \geq 0, 1 \leq k \leq n-1$, it follows that

$$\begin{aligned} A &\geq \sum_{k=1}^{n-1} E[(c_{k+1}g(S_{k+1}) - c_k g(S_k))m(Y_k)] \\ &\quad +\sum_{k=1}^{n-1} E[(c_k - c_{k+1})g(S_k)m(Y_k)] \\ &= \sum_{k=1}^{n-1} E[(c_{k+1}g(S_{k+1}) - c_{k+1}g(S_k))m(Y_k)] \\ &= \sum_{k=1}^{n-1} c_{k+1}E[(g(S_{k+1}) - g(S_k))m(Y_k)] \\ &\geq \sum_{k=1}^{n-1} c_{k+1}E[(S_k - S_{k+1})h(S_k)m(Y_k)] \end{aligned} \tag{3. 7}$$

from the property of the function $g(.)$ given by (3.1). Note that $h(S_i)m(Y_i)$ is a nondecreasing function of $S_1, \ldots, S_i$. Since $\{S_i, i \geq 1\}$ forms a N-demimartingale, it follows that

$$E[(S_{k+1} - S_k)h(S_k)m(Y_k)] \leq 0, 1 \leq k \leq n-1$$

and hence

$$\sum_{k=1}^{n-1} c_{k+1} E[(S_k - S_{k+1})h(S_k)m(Y_k)] \geq 0 \tag{3. 8}$$

by the nonnegativity of the sequence $c_i, i \geq 1$. Hence $A \geq 0$. Therefore

$$E(\int_0^{Y_n} u dm(u)) \leq \sum_{k=1}^{n} c_k E([g(S_k) - g(S_{k-1})]m(Y_n)). \tag{3. 9}$$

**Remarks:** Let $\epsilon > 0$ and define $m(t) = 1$ if $t \geq \epsilon$ and $m(t) = 0$ if $t < \epsilon$. Applying the previous theorem, we get that

$$\epsilon P(Y_n \geq \epsilon) \leq \sum_{k=1}^{n} E([g(S_k) - g(S_{k-1})]I(Y_n \geq \epsilon)) \tag{3. 10}$$

where $I(B)$ denotes the indictor function of the set $B$. Eexamples of functions $g$ satisfying (3.1) are $g(x) = -\alpha x$ where $\alpha \geq 0$ and $g(x) = -\alpha x^+$ where $\alpha \geq 0$. Here $x^+ = x$ if $x \geq 0$ and $x^+ = 0$ if $x < 0$.

## 4 Comparison theorem on moment inequalities

We now introduce a stronger condition on an N-demimartingale to derive a comparison theorem. A sequence $\{S_n, n \geq 1\}$ is said to be a *strong N-demimartingale* if, for any two coordinate wise nondecreasing functions $f$ and $g$,

$$Cov[g(S_{j+1} - S_j), f(S_1, \ldots, S_j)] \leq 0, \quad j = 1, 2, \ldots \tag{4. 1}$$

whenever the expectation is defined. It is easy to see that if a sequence $\{X_j, j \geq 1\}$ is a negatively associated sequence of random variables, then the corresponding partial sums form a strong N-demimartingale.

**Theorem 4.1:** Let $\{S_j, 1 \leq j \leq n\}$ be a strong N-demimartingale. Define $X_j = S_j - S_{j-1}, j \geq 1$ with $S_0 = 0$. Let $X_j^*, 1 \leq j \leq n$ be independent random variables such that $X_j$ and $X_j^*$ have the same distribution. Let $S_n^* = X_1^* + \ldots + X_n^*$. Then for any convex function $h$,

$$E(h(S_n)) \leq E(h(S_n^*)) \tag{4. 2}$$

whenever the expectations exist. Further more, if $h(.)$ is a nondecreasing convex function, then

$$E(\max_{1 \leq j \leq n} h(S_j)) \leq E(\max_{1 \leq j \leq n} h(S_j^*)) \tag{4. 3}$$

whenever the expectations exist.

**Proof:** Let a random vector $(Y_1, Y_2)$ be an independent and identically distributed as the random vector $(X_1, X_2)$. It is well known that for any convex function $h$ on the real line, there exists a nondecreasing function $g(.)$ such that for all $a < b$,

$$h(b) - h(a) = \int_a^b g(t)dt.$$

(cf. Roberts and Verberg (1973)).Hence

$$\begin{aligned}
&h(X_1 + X_2) + h(Y_1 + Y_2) - h(X_1 + Y_2) - h(Y_1 + X_2) \\
= &\int_{X_2}^{X_1} [g(Y_1 + t) - g(X_1 + t)] \;\; dt \\
= &\int_{-\infty}^{\infty} [g(Y_1 + t) - g(X_1 + t)](I_{[Y_2 > t]} - I_{[X_2 > t]}) \;\; dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
2E[h(S_2)] - 2E[h(S_2^*)] &= 2(E[h(X_1 + X_2) - E[h(X_1^* + X_2^*)]]) \\
&= E[h(X_1 + X_2) + h(Y_1 + Y_2) - h(X_1 + Y_2) - h(Y_1 + X_2)] \\
&= \int_{-\infty}^{\infty} Cov(g(X_1 + t), I_{[X_2 > t]}) \;\; dt.
\end{aligned}$$

Observe that the functions $g(x + t)$ and $I_{[x>t]}$ are nondecreasing functions in $x$ fr each $t$. Since $S_i, 1 \leq i \leq n$ forms a strong n-demimartingale, it follows that

$$Cov(g(X_1 + t), I_{[X_2 > t]}) \leq 0$$

for each $t$ which in turn implies that

$$E[h(S_2)] - E[h(S_2^*)] \leq 0. \tag{4. 4}$$

This proves the theorem for the case $n = 2$. We now prove the result by induction on $n$. Suppose the result holds for the case of $n - 1$ random variables. Let

$$k(x) = E(h(x + S_{n-1})).$$

Since the function $h(x + .)$ is a convex function for any fixed $x$, it follows that

$$k(x) \leq E(h(x + S_{n-1}^*)) \tag{4. 5}$$

Since $\{S_j, 1 \leq j \leq n\}$ forms a strong N-demimartingale, it follows that$\{S_{n-1}, S_n\}$ also forms a strong N-demimartingale which reduces to negative association of the two random variables $S_{n-1}$ and $X_n$. Therefore

$$\begin{aligned}
E[h(S_n)] &\leq E[h(X_n^* + S_{n-1}^*)] \;\; \text{(by induction hypothesis)} \\
&= E(k(X_n^*)) \\
&\leq E(h(X_n^* + S_{n-1}^*)) \;\; \text{(by (4.5))} \\
&= E[h(S_n^*)].
\end{aligned}$$

This proves the result for the case of $n$ random variables completing the proof by induction. The second part of the theorem can be proved by arguments analagous to those given in Shao (2002). We omit the details.

**Remarks:** The proofs in the above theorem hinge on the fact that two random variables $Z_1, Z_2$ are negatively associated if and only if $Z_1, Z_1 + Z_2$ forms a strong n-demimartingale. This is obvious from the definition of a strong N-demimartingale. As a consequence of the comparison theorem, the Rosenthal maximal inequality and the Kolomogorov exponential inequality hold for strong N-demimartingales.

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