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Large Deviation Probabilities for the MLE and BE of a Parameter for Some Stochastic Partial Differential Equations

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Abstract: This paper is concerned with the study of the probabilty of large deviations of the the maximum likelihood estimator and Bayes estimator of a parameter appearing linearly in the drift coefficients of two types of stochastic partial differential equations.

Key words and phrases : Stochastic partial differential equations, Large Deviation, Maximum likelihood estimator, Inference for stochastic processes.

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1 Introduction

Maximum likelihood estimation and Bayes Estimation of a parameter θ appearing linearly in some stochastic partial differential equations (SPDE) has been considered by Hübner et al. (1993). Detail discussion of these SPDE's and some interesting phenomena arising out of the parameter estimation have been considered by them in two examples. A Berry-Esseen type bound for the distribution of the maximum likelihood estimator (MLE) of the parameter θ for these SPDE's have been considered in Mishra and Prakasa Rao (2004). We now study the bounds on the large deviation probabilities for the maximum likelihood estimator (MLE) $\hat{\theta}_{N,\epsilon}$ and the Bayes estimator (BE) $\tilde{\theta}_{N,\epsilon}$ of a parameter θ occuring linearly in such SPDE's. We follow the method of Ibragimov and Khasminskii (1981) in obtaining these results.

In Section 2, we describe a SPDE with parameter θ such that the corresponding stochastic process u_ϵ generates measures $\{P_\theta^\epsilon, \theta \in \Theta\}$ which are mutually absolutely continuous and the main results pertaining to this section have been described in the Section 3. In Section 4, we describe a SPDE with parameter θ such that the corresponding stochastic process u_ϵ generates measures which form a family of probability measures $\{P_\theta^\epsilon, \theta \in \Theta\}$ which are singular with respect to each other and this section also contains the main results connected to this problem. Comprehensive surveys on statistical inference for such classes of SPDE's are given in Prakasa Rao (2001,2002).

Throughout the paper, we shall denote by C, C_1, C_2 etc. positive constants different at different places of occurence possibly dependent on the initial conditions of the SPDE's.

2 Stochastic PDE with linear drift (Absolutely continuous case) : Estimation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\epsilon(t,x),\ 0 \leq x \leq 1,\ 0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\epsilon(t,x) = (\Delta u_\epsilon(t,x) + \theta u_\epsilon(t,x))\, dt + \epsilon dW_Q(t,x) \tag{2. 1}$$

with the initial and boundary conditions given by

$$\begin{aligned} u_\epsilon(0,x) &= f(x),\ f \in L_2[0,1], \\ u_\epsilon(t,0) &= u_\epsilon(t,1) = 0,\ 0 \leq t \leq T \end{aligned} \tag{2. 2}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$. Let $\epsilon \to 0$ and $\theta \in \Theta \subset I\!R$. Here Q is a nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$, so that $W_Q(t,x) = Q^{\frac{1}{2}} W(t,x)$ and $W(t,x)$ is a cylindrical Brownian motion in $L_2[0,1]$. Then it is known that (cf. Rozovskii (1990))

$$W_Q(t,x) = \sum_{i=1}^{\infty} q_i^{\frac{1}{2}} e_i(x) W_i(t) \quad \text{a.s.} \tag{2. 3}$$

where $\{W_i(t),\ 0 \leq t \leq T\}, i \geq 1$ are independent one dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system (CONS) in $L_2[0,1]$ consisting of the eigen vectors of Q and $\{q_i\}$ the corresponding eigen values of Q. Let us consider a special covariance operator Q with $e_k = \sin k\pi x,\ k \geq 1$ and $\lambda_k = (\pi k)^2, k \geq 1$. Then $\{e_n\}$ is a CONS with the eigen values $q_i = (1 + \lambda_i)^{-1},\ i \geq 1$ for the operator Q where $Q = (I - \Delta)^{-1}$. Furthermore, $dW_Q = Q^{\frac{1}{2}} dW$. We define a solution $u_\epsilon(t,x)$ of (2.1) as a formal sum

$$u_\epsilon(t,x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t) e_i(x) \tag{2. 4}$$

(cf. Rozovskii (1990)). It is known that the Fourier coefficients $u_{i\epsilon}(t)$ satisfy the stochastic differential equation

$$du_{i\epsilon}(t) = (\theta - \lambda_i) u_{i\epsilon}(t) dt + \frac{\epsilon}{\sqrt{\lambda_i + 1}} dW_i(t),\ 0 < t \leq T \tag{2. 5}$$

with the initial conditions

$$u_{i\epsilon}(0) = v_i, \quad v_i = \int_0^1 f(x) e_i(x) dx. \tag{2. 6}$$

It is further known that the function $u_\epsilon(t,x)$ as defined above belongs to $L_2([0,T] \times \Omega; L_2[0,1])$ together with its derivative in t. Furthermore $u_\epsilon(t,x)$ is the only solution of (2.1) under the boundary condition (2.2).

Let P_θ^ϵ be the measure generated by u_ϵ on $C[0,T]$ when θ is the true parameter. It has been shown by Hübner et al. [3] that the family of measures $\left\{P_\theta^{(\epsilon)}, \theta \in \Theta\right\}$ are mutually absolutely continuous and

$$\begin{aligned}
&\log \frac{dP_\theta^\epsilon}{dP_{\theta_0^\epsilon}}(u_\epsilon) \\
&= \sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\epsilon^2} \left[(\theta - \theta_0) \int_0^T u_{i\epsilon}(t) du_{i\epsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T u_{i\epsilon}^2(t) dt \right].
\end{aligned}$$

The projection of the solution $u_\epsilon(t,x)$ onto the subspace π^N spanned by $\{e_1, e_2, \ldots, e_N\}$ is given by $u_\epsilon^{(N)}(t,x) = \sum_{i=1}^N u_{i\epsilon}(t) e_i(x)$. Let $P_\theta^{\epsilon,N}$ be the probability measure generated by the process $u_\epsilon^N(t,x)$ on $C[0,T]$ when θ is the true parameter. Then the measure $P_\theta^{\epsilon,N}$ is absolutely continuous with respect to the measure $P_{\theta_0}^{\epsilon,N}$ and

$$\begin{aligned}
&\log \frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{\epsilon,N}} \\
&= \sum_{i=1}^{N} \frac{\lambda_i + 1}{\epsilon^2} \left[(\theta - \theta_0) \int_0^T u_{i\epsilon}(t) du_{i\epsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T u_{i\epsilon}(t) dt \right].
\end{aligned} \tag{2. 7}$$

Maximum likelihood estimator:

It is easy to see that the *maximum liklihood estimator* (MLE) of the parameter θ based on the observation $\{u_\epsilon^N(t,x), 0 \le t \le T\}$ is given by

$$\hat{\theta}_{N,T,\epsilon} = \frac{\sum_{i=1}^N (\lambda_i + 1) \int_0^T u_{i\epsilon}(t)(du_{i\epsilon}(t) + \lambda_i u_{i\epsilon}(t) dt)}{\sum_{i=1}^N (\lambda_i + 1) \int_0^T u_{i\epsilon}^2(t) dt} \tag{2. 8}$$

(cf. Hübner et. al. (1993), p.154).

Bayes estimator:

Suppose that Λ is a prior probability measure on $(\Theta, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra of Borel subsets of an open set $\Theta \subset \mathbb{R}$. Further suppose that Λ has a density $\lambda(.)$ with respect to the Lebesgue measure and the density $\lambda(.)$ is continuous and positive in an open neighbourhood of θ_0, the true parameter. The posterior density of θ given $u_\varepsilon^{(N)}(t,x)$, $0 < x < 1$, $0 \le t \le T$ is given by

$$p_T^{(N)}\left(\theta|u_\varepsilon^{(N)}\right) = \frac{\left(dP_\theta^{\varepsilon,N}/dP_{\theta_0}^{\varepsilon,N}\right)(u_\varepsilon^{(N)})\lambda(\theta)}{\int_\Theta \left(dP_\theta^{\varepsilon,N}/dP_{\theta_0}^{\varepsilon,N}\right)(u_\varepsilon^{(N)})\lambda(\theta)d\theta.}$$

We define the *Bayes Estimator* (BE) $\tilde{\theta}_{N,T,\varepsilon}$ of θ, based on the path $u_\varepsilon^{(N)}$ and prior density $\lambda(\theta)$, to be the minimizer of the function

$$B_{N,T,\varepsilon}(\phi) = \int_\Theta L(\theta,\phi)p_T^{(N)}\left(\theta|u_\varepsilon^{(N)}\right)d\theta, \quad \phi \in \Theta$$

where $L(\theta,\phi)$ is a given loss function defined on $\Theta \times \Theta$. In particular, for the quadratic loss function $|\theta-\phi|^2$, the Bayes Estimator $\tilde{\theta}_{N,T,\varepsilon}$ becomes the posterior mean given by

$$\tilde{\theta}_{N,T,\varepsilon} = \int_\Theta u p_T^{(N)}\left(u|u_\varepsilon^{(N)}\right)du \Big/ \int_\Theta p_T^{(N)}\left(v|u_\varepsilon^{(N)}\right)dv.$$

Suppose the loss function $L(\theta,\phi):\Theta\times\Theta\to\mathbb{R})$ satisfies the following conditions:

D(i) $L(\theta,\phi) = L(|\theta-\phi|)$;

D(ii) $L(\theta)$ is non-negative and continuous on $\mathbb{R}$;

D(iii) $L(.)$ is symmetric;

D(iv) the sets $\{\theta : L(\theta) < c\}$ are convex sets and are bounded for all $c > 0$; and

D(v) there exists numbers $\gamma > 0$, $H_0 \geq 0$ such that for $H \geq H_0$,

$$\sup\{L(\theta) : |\theta| \leq H^\gamma\} \leq \inf\{L(\theta) : |\theta| \geq H\}.$$

Obviously, loss functions of the form $L(\theta,\phi) = |\theta-\phi|^2$ satisfy these conditions D(i) - D(v).

3 Main Result (Absolute Continuous Case)

We now prove the following theorem giving the large deviation probability for the MLE and BE discussed in Section 2. Suppose $\theta < \pi^2$.

Theorem 3.1: Under the conditions stated above, there exists positive constants C_1 and C_2, depending on $\theta, \varepsilon, N, T$ and $\|f\|$, such that for every $H > 0$,

$$P_\theta^{\varepsilon,N}\left\{\left|Q_{N,T,\varepsilon}^{1/2}\varepsilon^{-1}\left(\hat{\theta}_{N,T,\varepsilon}-\theta\right)\right| > H\right\} \leq C_1 e^{-C_2H^2}$$

where $\hat{\theta}_{N,T,\varepsilon}$ is the MLE of the parameter θ and

$$Q_{N,T,\varepsilon} = \sum_{i=1}^{N} \left[\frac{\lambda_i + 1}{2(\theta - \lambda_i)} v_i^2 (e^{2(\theta-\lambda_i)T} - 1) - \frac{T\varepsilon^2}{\lambda_i + 1} \right].$$

Theorem 3.2: Under the conditions stated above, there exists positive constants C_1 and C_2, depending on $\theta, \varepsilon, N, T$ and $\|f\|$, such that for every $H > 0$,

$$P_\theta^{\varepsilon,N} \left\{ |Q_{N,T,\varepsilon}^{\frac{1}{2}} \varepsilon^{-1} (\tilde{\theta}_{N,T,\varepsilon} - \theta)| > H \right\} \leq C_1 e^{-C_2 H^2}$$

where $\tilde{\theta}_{N,T,\varepsilon}$ is the BE of the parameter θ with respect to the prior $\lambda(.)$ and the loss function $L(.,.)$ satisfies the conditions D(i)-D(V). Let $E_\theta^{\varepsilon,N}$ denote the expectation with respect to the probability measure $P_\theta^{\varepsilon,N}$.

Fix $\theta \in \Theta$. For the proofs of these theorems, we need the following lemmas. Define

$$Z_{N,T,\varepsilon}(u) = \frac{dP^{\varepsilon,N}_{\theta+\varepsilon u/\sqrt{Q_{N,T,\varepsilon}}}}{dP_\theta^{\varepsilon,N}}.$$

Lemma 3.1 : Under the conditions stated above, there exists positive constants C_1 and C_2 such that

$$E_\theta^{\varepsilon,N}[Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)] \leq C_1 e^{-C_2 u^2}$$

for $-\infty < u < \infty$.

Lemma 3.2 : Under the conditions stated above, there exists positive constant C_1 such that

$$E_\theta^{\varepsilon,N} \left\{ Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_1) - Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_2) \right\}^2 \leq C_1 (u_1 - u_2)^2$$

for $-\infty < u_1, u_2 < \infty$.

Lemma 3.3 : Let $\xi(t)$ be a real valued random function defined on a closed subset F of the Euclidean space R^k. Assume that random process $\xi(t)$ is measurable and separable. Assume that the following conditions are fulfilled : there exists numbers $m \geq \gamma > k$ and a positive continuous function on $G(x) : R^k \to R$ bounded on the compact sets such that for all $x, h \in F$, $x + h \in F$,

$$E|\xi(x)|^m \leq G(x), \; E|\xi(x+h) - \xi(x)|^m \leq G(x)\|h\|^\gamma.$$

Then, with probability 1, the realizations of $\xi(t)$ are continuous functions on F. Moreover let

$$\omega(\delta, \xi, L) = \sup |\xi(x) - \xi(y)|$$

where the upper bound is taken over $x, y \in F$ with $\|x-y\| \leq h,\ \|x\| \leq L,\ \|y\| \leq L$; then

$$E(\omega(h, \xi, L)) \leq B_0 \left(\sup_{\|x\| \leq L} G(x) \right)^{\frac{1}{m}} L^{k/m} h^{\frac{\gamma-k}{m}} \log(h^{-1})$$

where the constant B_0 depends on m, γ and k.

We shall use this lemma with $\xi(u) = Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)$, $m = 2$, $\gamma = 2$, $k = 1$, $G(x) = e^{-cx^2}$ and $L = H + \gamma + 1$.

For the proof of this lemma, see Ibragimov and Khasminskii (1981) (Correction, cf. Kallianpur and Selukar (1993)).

Proof of Lemma 3.1: We know

$$\begin{aligned}
Z_{N,T,\varepsilon}(u) &= \frac{dP^{\varepsilon,N}_{\theta+\varepsilon u/\sqrt{Q_{N,T,\varepsilon}}}}{dP^{\varepsilon,N}_{\theta}} \\
&= \exp\left[\sum_{i=1}^{N} \left\{ \frac{\sqrt{\lambda_i+1}}{\varepsilon} \left(\varepsilon u Q_{N,T,\varepsilon}^{-1/2} \int_0^T u_{i\varepsilon}(t) dW_i(t) \right) \right.\right. \\
&\quad \left.\left. -\frac{1}{2} \frac{\lambda_i+1}{\varepsilon^2} \left(u^2 \varepsilon^2 Q_{N,T,\varepsilon}^{-1} \int_0^T u_{i\varepsilon}^2(t) dt \right) \right\}\right] \\
&= \exp\left[\sum_{i=1}^{N} \left\{ \sqrt{\lambda_i+1}\, u Q_{N,T,\varepsilon}^{-1/2} \int_0^T u_{i\varepsilon}(t) dW_i(t) \right.\right. \\
&\quad \left.\left. -(1/2)(\lambda_i+1) u^2 Q_{N,T,\varepsilon}^{-1} \int_0^T u_{i\varepsilon}^2(t) dt \right\}\right]
\end{aligned}$$

where

$$Q_{N,T,\varepsilon} = \sum_{i=1}^{N} \left[\frac{\lambda_i+1}{2(\theta-\lambda_i)} v_i^2 \left(e^{2(\theta-\lambda_i)T} - 1 \right) - T \frac{\varepsilon^2}{\lambda_i+1} \right].$$

Observe that

$$
\begin{aligned}
& E_\theta^{\varepsilon,N}\left(Z_{N,T,\varepsilon}^{1/2}(u)\right) \\
= \;& E_\theta^{\varepsilon,N} \exp\left[\sum_{i=1}^{N}\left\{\frac{1}{2}\sqrt{\lambda_i+1}\left(uQ_{N,T,\varepsilon}^{-1/2}\right)\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
& \left.\left. -\frac{1}{4}\;(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right\}\right] \\
= \;& E_\theta^{\varepsilon,N}\left\{\exp\left(\frac{1}{2}\sum_{i=1}^{N}\sqrt{\lambda_i+1}uQ_{N,T,\varepsilon}^{-1/2}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
& \left. -\frac{1}{6}\sum_{i=1}^{N}(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right) \\
& \left.\exp\left(-(1/12)\sum_{i=1}^{N}(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right)\right\} \\
\leq \;& \left[E_\theta^{\varepsilon,N}\left\{\exp\left(\frac{1}{2}\sum_{i=1}^{N}\sqrt{\lambda_i+1}uQ_{N,T,\varepsilon}^{-1/2}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right.\right. \\
& \left.\left.\left. -(1/6)\sum_{i=1}^{N}(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right)\right\}^{4/3}\right]^{3/4} \\
& \times\left[E_\theta^{\varepsilon}\left\{\exp\left(-(1/12)(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right)\right\}^4\right]^{1/4} \quad \text{(by Holder's inequality)} \\
= \;& \left\{E_\theta^{\varepsilon,N}\exp\left(\frac{2}{3}\sum_{i=1}^{N}\sqrt{\lambda_i+1}uQ_{N,T,\varepsilon}^{-1/2}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
& \left. -(2/9)\sum_{i=1}^{N}(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}\int_0^T u_{i\varepsilon}^2(t)dt\right)\}^{3/4} \\
& \times\left\{E_\theta^{\varepsilon,N}\exp\left(-(1/3)\sum_{i=1}^{N}\int_0^T(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}u_{i\varepsilon}^2(t)dt\right)\right\}^{1/4} \\
\leq \;& \left[E_\theta^{\varepsilon}\left\{\exp\left(-\frac{1}{3}\sum_{i=1}^{N}\int_0^T(\lambda_i+1)u^2Q_{N,T,\varepsilon}^{-1}u_{i\varepsilon}^2(t)dt\right)\right\}\right]^{1/4} \qquad (3.\;1)
\end{aligned}
$$

(since the first term is less than or equal to one (cf. Gikhman and Skorohod(1972))).

From (2.5), we obtain that

$$
\begin{aligned}
du_{i\varepsilon}(s) &= (\theta-\lambda_i)u_{i\varepsilon}(s)ds+\frac{\varepsilon}{\sqrt{\lambda_i+1}}dW_i(s), \quad 0\leq s\leq T, \\
u_{i\varepsilon}(0) &= v_i.
\end{aligned}
$$

By the Ito's Lemma, we have

$$d(u_{i\varepsilon}(s)e^{-(\theta-\lambda_i)s}) = \frac{\varepsilon}{\sqrt{\lambda_i+1}}e^{-(\theta-\lambda_i)s}dW_i(s)$$

or

$$u_{i\varepsilon}(t)e^{-(\theta-\lambda_i)t} - v_i = \int_0^t \frac{\varepsilon}{\sqrt{\lambda_i+1}}e^{-(\theta-\lambda_i)s}dW_i(s). \tag{3. 2}$$

An application of Lemma 1.13 of Kutoyants (1998), p.26 shows that

$$\sup_{0\le t\le T}|u_{i\varepsilon}(t)-u_{i0}(t)| \le e^{|\theta-\lambda_i|T}\frac{\varepsilon}{\sqrt{\lambda_i+1}}\sup_{0\le t\le T}|W_i(t)|.$$

Note that

$$u_{i0}(t) = v_i e^{(\theta-\lambda_i)t}$$

and hence

$$\int_0^T |u_{i0}(t)|\,|u_{i\varepsilon}(t)-u_{i0}(t)| \; dt \le e^{|\theta-\lambda_i|T}\int_0^T |v_i|e^{-(\lambda_i-\theta)t}\frac{\varepsilon}{\sqrt{\lambda_i+1}}\sup_{0\le t\le T}|W_i(t)|dt.$$

Using the above inequality and (3.1), we obtain that

$$
\begin{aligned}
\left[E_\theta^{\varepsilon,N}\left(Z_{N,T,\varepsilon}^{1/2}(u)\right)\right]^4 &\leq E_\theta^{\varepsilon,N}\left\{\exp\left(-\sum_{i=1}^N \frac{u^2(\lambda_i+1)}{3Q_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}^2(t)dt\right)\right\} \\
&\leq E_\theta^{\varepsilon,N}\exp\left[\sum_{i=1}^N \frac{u^2}{3Q_{N,T,\varepsilon}}(\lambda_i+1)\left\{\left(-\int_0^T u_{i0}^2(t)dt\right)\right.\right. \\
&\quad \left.\left.+\left(\int_0^T 2|u_{i0}(t)|\,|u_{i\varepsilon}(t)-u_{i0}(t)|dt\right)\right\}\right] \\
&= \exp\left\{-\sum_{i=1}^N \frac{u^2}{3Q_{N,T,\varepsilon}}(\lambda_i+1)\int_0^T u_{i0}^2(t)dt\right\} \\
&\quad \times E_\theta^{\varepsilon,N}\exp\left[\frac{u^2}{3Q_{N,T,\varepsilon}}\sum_{i=1}^N(\lambda_i+1)\int_0^T 2|u_{i0}(t)|\,|u_{i\varepsilon}(t)-u_{i0}(t)|dt\right] \\
&\leq \exp\left[\sum_{i=1}^N \frac{-u^2}{3Q_{N,T,\varepsilon}}(\lambda_i+1)\int_0^T v_i^2 e^{-2(\lambda_i-\theta)t}dt\right] \\
&\quad \times E_\theta^{\varepsilon,N}\left\{\exp\left[\sum_{i=1}^N \frac{2u^2}{3Q_{N,T,\varepsilon}}\sqrt{(\lambda_i+1)}|v_i|\varepsilon\eta_i e^{|\theta-\lambda_i|T}\int_0^T e^{-(\lambda_i-\theta)t}dt\right]\right\} \\
&\quad (\text{where } \eta_i = \sup_{0\leq t\leq T}|W_i(t)|) \\
&= \exp\left[-\sum_{i=1}^N\left\{\frac{u^2v_i^2}{3Q_{N,T,\varepsilon}}\frac{\lambda_i+1}{2(\lambda_i-\theta)}\left(1-e^{-2(\lambda_i-\theta)T}\right)\right\}\right] \\
&\quad \times E_\theta^{\varepsilon,N}\left\{\exp\left[2\sum_{i=1}^N \frac{u^2}{3Q_{N,T,\varepsilon}}\frac{\sqrt{\lambda_i+1}|v_i|\varepsilon\eta_i}{\lambda_i-\theta}e^{|\theta-\lambda_i|T}(1-e^{-(\lambda_i-\theta)T)})\right]\right\} \\
&\leq \exp\left[-\sum_{i=1}^N \frac{u^2v_i^2}{3Q_{N,T,\varepsilon}}\frac{\lambda_i+1}{2(\lambda_i-\theta)}\right] \\
&\quad \times E_\theta^{\varepsilon,N}\exp\left[2\sum_{i=1}^N \frac{u^2}{3Q_{N,t,\varepsilon}}\frac{\sqrt{\lambda_i+1}}{\lambda_i-\theta}|v_i|\varepsilon\eta_i e^{|\theta-\lambda_i|T}(1-e^{-(\lambda_i-\theta)T})\right] \\
&\leq \exp\left[-C_1\sum_{i=1}^N \frac{u^2}{Q_{N,T,\varepsilon}}v_i^2\right] \\
&\quad \times E_\theta^{\varepsilon,N}\left(\exp[C_2\sum_{i=1}^N \frac{u^2\varepsilon}{Q_{N,T,\varepsilon}}\frac{1}{\sqrt{\lambda_i}}|v_i|\eta_i e^{|\theta-\lambda_i|T}(1-e^{-(\lambda_i-\theta)T})]\right) \\
&\leq \exp\left[-C_3\frac{u^2}{Q_{N,T,\varepsilon}}\|f\|^2\right] \\
&\quad \times \prod_{i=1}^N E_\theta^{\varepsilon,N}\exp\left[C_2\frac{u^2\varepsilon}{Q_{N,T,\varepsilon}}\frac{1}{\sqrt{\lambda_i}}|v_i|\eta_i e^{|\theta-\lambda_i|T}(1-e^{-(\lambda_i-\theta)T})\right], \\
&\quad \left(\text{where } \sum_{i=1}^\infty v_i^2 = \|f\|^2\right).
\end{aligned}
$$

Hence

$$\begin{aligned}
\left[E_\theta^{\varepsilon,N}\left(Z_{N,T,\varepsilon}^{1/2}(u)\right)\right]^4 &\le \exp\left[-C_3\frac{u^2}{Q_{N,T,\varepsilon}}\|f\|^2\right] \\
&\quad \times \prod_{i=1}^{N}\exp\left((3C_2/2)\frac{Tu^4\varepsilon^2}{Q_{N,T,\varepsilon}^2}\frac{1}{\lambda_i}v_i^2 e^{2|\theta-\lambda_i|T}(1-e^{-(\lambda_i-\theta)T})^2\right), \\
&\quad \left(\text{since } E_\theta \exp\left\{l\sup_{0\le t\le T}|W_i(t)|\right\} \le 2e^{(3/2)Tl^2},\ \text{cf. Kutoyants (1998)}\right) \\
&\le \exp\left[-C_3\frac{u^2}{Q_{N,T,\varepsilon}}\|f\|^2\right]\exp\left[C_4\frac{u^4\varepsilon^2}{Q_{N,T,\varepsilon}^2}\|f\|^2 e^{2(N^2\pi^2-\theta)T}\right],
\end{aligned}$$

Hence

$$E_\theta^{\varepsilon,N}\left(Z_{N,T,\varepsilon}^{1/2}(u)\right) \le C_5 e^{-C_6 u^2}$$

(the constants C_5 and C_6 depend on $\varepsilon, \theta, N, T$ and $\|f\|$).

Proof of Lemma 3.2 : Note that

$$\begin{aligned}
&E_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_1) - Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_2)\right\}^2 \\
&= E_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}(u_1)+Z_{N,T,\varepsilon}(u_2)\right\} - 2E_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_1)Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_2)\right\} \\
&= 2\left[1-E_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_1)Z_{NT\varepsilon}^{\frac{1}{2}}(u_2)\right\}\right] \\
&\quad (\text{since } E_\theta^{\varepsilon,N} Z_{N,T,\varepsilon}(u) = E_\theta^{\varepsilon,N}\left[\exp\left\{\sum_{i=1}^{N}\left(\frac{u\sqrt{\lambda_i+1}}{Q_{N,T,\varepsilon}^{\frac{1}{2}}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right.\right. \\
&\quad \left.\left.\left. -\frac{1}{2}\frac{u^2(\lambda_i+1)}{Q_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}^2(t)dt\right)\right\}\right] = 1).
\end{aligned}$$

Denote

$$\begin{aligned}
V_{N,T,\varepsilon} &= \left(\frac{dP_{\theta_2}^{\varepsilon,N}}{dP_{\theta_1}^{\varepsilon,N}}\right)^{1/2} \quad \text{where } \theta_1 = \theta + \frac{\varepsilon u_1}{Q_{N,T,\varepsilon}^{\frac{1}{2}}} \text{ and } \theta_2 = \theta + \frac{\varepsilon u_2}{Q_{N,T,\varepsilon}^{\frac{1}{2}}} \\
&= \exp\left\{\frac{1}{2}\sum_{i=1}^{N}\frac{(u_2-u_1)\sqrt{\lambda_i+1}}{Q_{N,T,\varepsilon}^{1/2}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right. \\
&\quad \left. -\frac{1}{4}\sum_{i=1}^{N}\frac{(u_2-u_1)^2(\lambda_i+1)}{Q_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}^2(t)dt\right\}.
\end{aligned}$$

Now

$$
\begin{aligned}
& E_\theta^{\varepsilon,N}\left\{\left(Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_1)\right)\left(Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_2)\right)\right\} \\
= \quad & E_\theta^{\varepsilon,N}\left\{\left(\frac{dP^{\varepsilon,N}_{\theta+\varepsilon u_1 Q^{-1/2}_{N,T,\varepsilon}}}{dP^{\varepsilon,N}_\theta}\right)^{1/2}\left(\frac{dP^{\varepsilon,N}_{\theta+\varepsilon u_2 Q^{-1/2}_{N,T,\varepsilon}}}{dP^{\varepsilon,N}_\theta}\right)^{1/2}\right\} \\
= \quad & \int\left(\frac{dP^{\varepsilon,N}_{\theta_1}}{dP^{\varepsilon,N}_\theta}\right)^{1/2}\left(\frac{dP^{\varepsilon,N}_{\theta_2}}{dP^{\varepsilon,N}_\theta}\right)^{1/2} dP^{\varepsilon,N}_\theta \\
= \quad & \int\left(\frac{dP^{\varepsilon,N}_{\theta_2}}{dP^{\varepsilon,N}_{\theta_1}}\right)^{1/2} dP^{\varepsilon,N}_{\theta_1} = E^{\varepsilon,N}_{\theta_1}(V_{N,T,\varepsilon}) \\
= \quad & E^{\varepsilon,N}_{\theta_1}\left[\exp\left\{\frac{1}{2}\sum_{i=1}^N \frac{(u_2-u_1)\sqrt{\lambda_i+1}}{Q^{\frac{1}{2}}_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
& \left.\left. -\frac{1}{4}\sum_{i=1}^N \frac{(u_2-u_1)^2(\lambda_i+1)}{Q_{N,T,\varepsilon}}\int_0^T u^2_{i\varepsilon}(t)dt\right\}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2\left\{1 - E_\theta^{\varepsilon,N}\left(Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_1)Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_2)\right)\right\}\\
&= 2\left[1 - E_{\theta_1}^{\varepsilon,N}\exp\left\{\frac{1}{2}\sum_{i=1}^N \frac{(u_2-u_1)\sqrt{\lambda_i+1}}{Q^{1/2}_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right.\\
&\quad\left.\left.-\frac{1}{4}\sum_{i=1}^N \frac{(u_2-u_1)^2(\lambda_i+1)}{Q_{N,T,\varepsilon}}\int_0^T u^2_{i\varepsilon}(t)dt\right\}\right]\\
&\le 2\left[1 - \exp E_{\theta_1}^{\varepsilon,N}\left\{\frac{1}{2}\sum_{i=1}^N \frac{(u_2-u_1)\sqrt{\lambda_i+1}}{Q^{1/2}_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right.\\
&\quad\left.\left.-\frac{1}{4}\sum_{i=1}^N \frac{(u_2-u_1)^2(\lambda_i+1)}{Q_{N,T,\varepsilon}}\int_0^T u^2_{i\varepsilon}(t)dt\right\}\right] \text{ (by Jensen's inequality)}\\
&= 2\left[1-\exp\left\{\sum_{i=1}^N \frac{-(u_2-u_1)^2(\lambda_i+1)}{4Q_{N,T,\varepsilon}}E_{\theta_1}\int_0^T u^2_{i\varepsilon}(t)dt\right\}\right]\\
&\le 2\left[\sum_{i=1}^N \frac{(u_2-u_1)^2}{4Q_{N,T,\varepsilon}}(\lambda_i+1)E_{\theta_1}^{\varepsilon,N}\int_0^T u^2_{i\varepsilon}(t)dt\right] \text{ (since } 1-e^{-x}\le x)\\
&= \sum_{i=1}^N \frac{(u_2-u_1)^2}{2Q_{N,T,\varepsilon}}\left\{\frac{\lambda_i+1}{\lambda_i-\theta}v_i^2\left(1-e^{-2(\lambda_i-\theta)T}\right)+\frac{\varepsilon^2}{2}\sum_{i=1}^N\frac{1}{\lambda_i-\theta}\left(T-\frac{1-e^{-2(\lambda_i-\theta)T}}{2(\lambda_i-\theta)}\right)\right\}\\
&\quad \text{(cf. Hübner et. al (1993), p.152)}\\
&= \frac{(u_2-u_1)^2}{Q_{N,T,\varepsilon}}C\left[\sum_{i=1}^N\left\{v_i^2\left(1-e^{-2(\lambda_i-\theta)T}\right)+\frac{\varepsilon^2}{2}\sum_{i=1}^N\left\{\frac{1}{\lambda_i-\theta}\left(T-\frac{1-e^{-2(\lambda_i-\theta)T}}{2(\lambda_i-\theta)}\right)\right\}\right.\right]\\
&\le \frac{(u_2-u_1)^2}{Q_{N,T,\varepsilon}}C\left(\sum_{i=1}^N v_i^2+\frac{T\varepsilon^2}{\lambda_i-\theta}\right)\\
&\le \frac{(u_2-u_1)^2}{Q_{N,T,\varepsilon}}C\left(\|f\|^2+T\varepsilon^2\right)\\
&\le C\,(u_2-u_1)^2
\end{aligned}
$$

for some constant C depending on $\theta, N, T, \varepsilon$ and $||f||$.

Proof of Theorem:

Denote $U_\varepsilon = \{u : \theta + u\varepsilon^{-1} \in \Theta\}$, where $\Theta \subset R$. Let Γ_γ be the interval $H+\gamma \le |u| \le H+\gamma+1$.

We use the following inequality to prove our theorem:

$$P_\theta^{\varepsilon,N}\left\{\sup_{\Gamma_\gamma} Z_{N,T,\varepsilon}(u) \ge 1\right\} \le C_1(1+H+\gamma)^{\frac{1}{2}}e^{-\frac{1}{4}(H+\gamma)^2}. \tag{3. 3}$$

So

$$
\begin{aligned}
& P_\theta^{\varepsilon,N}\left\{|\sqrt{Q_{N,T,\varepsilon}}\varepsilon^{-1}\left(\hat\theta_{N,T,\varepsilon}-\theta\right)|>H\right\} \\
\le\ & P_\theta^{\varepsilon,N}\left\{\sup_{|u|>H,u\in U_\varepsilon} Z_{N,T,\varepsilon}(u)\ge Z_{N,T,\varepsilon}(0)\right\} \\
\le\ & \sum_{\gamma=0}^{\infty} P_\theta^{\varepsilon,N}\left\{\sup_{\Gamma_\gamma} Z_{N,T,\varepsilon}(u)\ge 1\right\} \\
\le\ & C_2\sum_{\gamma=0}^{\infty} e^{-C_3(H+\gamma)^2} \\
\le\ & C_4 e^{-C_5H^2}.
\end{aligned}
$$

This proves Theorem 3.1.

We now prove the inequality (3.3).

We divide the interval Γ_γ into N sub-intervals each with length at most h. The number of sub-intervals $N\le\left[\frac{1}{h}\right]+1$. Choose $u_j\in\Gamma_\gamma^{(j)}$, $1\le j\le N$.

$$
\begin{aligned}
P_\theta^{\varepsilon,N}\left\{\sup_{\Gamma_\gamma} Z_{N,T,\varepsilon}(u)\ge 1\right\} &\le \sum_{j=1}^{N} P_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}(u)\ge\frac{1}{2}\right\} \\
&+P_\theta^{\varepsilon,N}\left\{\sup_{|u-v|\le h}\left|Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)-Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)\right|\ge\frac{1}{2}\right\} \\
&(\text{when } |u|,|v|\le H+\gamma+1). \qquad (3.\ 4)
\end{aligned}
$$

From the Chebyshev's inequality and in view of Lemma 3.1 it follows that

$$
P_\theta^{\varepsilon,N}\left\{Z_{N,T,\varepsilon}^{\frac{1}{2}}(u_j)\ge\frac{1}{2}\right\}\le Ce^{-(H+\gamma)^2}.
$$

By Lemma 3.2 with $\zeta(u)=Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)$ and using Lemma 3.3 we obtain

$$
E_\theta^{\varepsilon,N}\sup_{\substack{|u-v|\le h\\ |u|,|v|\le(H+\gamma+1)}}\left|Z_{N,T,\varepsilon}^{1/2}(u)-Z_{N,T,\varepsilon}(v)\right|\le C(H+\gamma+1)^{\frac{1}{2}}h^{1/2}\log(h^{-1}).
$$

Hence

$$
P_\theta^{\varepsilon,N}\left\{\sup_{\Gamma_\gamma} Z_{N,T,\varepsilon}(u)\ge 1\right\}\le C\{\frac{1}{h}e^{-(H+\gamma)^2}+(H+\gamma+1)^{\frac{1}{2}}h^{\frac{1}{2}}\log(h^{-1})\}\ (\text{by using }\ (3.4)).
$$

Considering $h=e^{-(H+\gamma)^2/2}$, we prove the inequality in Theorem 3.1.

Proof of Theorem 3.2 : Observe that the conditions (1) and (2) in Theorem 5.2 of Ibragimov and Khasminskii (1981) are satisfied by Lemmas 3.1 and 3.2. In view of the conditions on the loss function mentioned in Section 2 with $\alpha=2$ and $g(u)=u^2$, we can prove the Theorem 3.2 by using Theorem 5.2. We omit the details.

4 Stochastic PDE with linear drift (Singular case)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_{i\varepsilon}(t,x)$, $0 \leq x \leq 1$, $0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\varepsilon(t,x) = \theta \Delta u_\varepsilon(t,x)dt + \epsilon(1-\Delta)^{-1/2}dW(t,x) \tag{4. 1}$$

where $\theta > 0$ satisfying the initial and boundary conditions

$$u_\epsilon(0,x) = f(x), 0 < x < 1, f \epsilon L_2[0,1],$$

$$u_\epsilon(t,0) = u_\epsilon(t,1) = 0, \quad 0 \leq t \leq T \tag{4. 2}$$

Here I is the identity operator, $\Delta = \frac{\partial^2}{\partial x^2}$ as defined in the Section 3 and the process $W(t,x)$ is the cylindrical Brownian motion in $L_2[0,1]$. In analogy with the discussion following the stochastic differential equation given by (2.5) in Section 2, it can be checked that the Fourier coefficients $u_{i\epsilon}(t)$ satisfy the stochastic differential equation

$$du_{i\epsilon}(t) = -\theta\lambda_i u_{i\epsilon}(t)dt + \frac{\epsilon}{\sqrt{\lambda_1 + 1}}dW_i(t), \quad 0 < t \leq T \tag{4. 3}$$

with

$$u_{i\epsilon}(0) = v_i \quad \text{where} \quad v_i = \int_0^1 f(x)e_i(x)dx. \tag{4. 4}$$

Let P_θ^ϵ be the measure generated by the process u_ϵ on $C[0,T]$ when θ is the true parameter. It can be shown that the family of measures $\{P_\theta^\epsilon, \ \theta \in \Theta\}$ do not form a family of equivalent probability measures. In fact P_θ^ϵ is singular with respect to $P_{\theta'}^\epsilon$, when $\theta \neq \theta'$ in Θ (cf. Hübner et. al [3]).

Let $u_\epsilon^{(N)}(t,x)$ be the projection of $u_\epsilon(t,x)$ onto the subspace spanned by $\{e_1, e_2, \ldots, e_N\}$ in $L_2[0,1]$. In other words,

$$u_\epsilon^{(N)}(t,x) = \sum_{i=1}^{N} u_{i\epsilon}(t)e_i(x). \tag{4. 5}$$

Let $P_\theta^{\epsilon,N}$ be the probability measure generated by the process $u_\epsilon^{(N)}$ on the subspace spanned by $\{e_1, \ldots e_N\}$ in $L_2[0,1]$. It can be shown that the measures $\left\{P_\theta^{\epsilon,N}, \ \theta \in \Theta\right\}$ form an equivalent family and

$$
\begin{aligned}
&\log \frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{(\epsilon,N)}}(u_\epsilon^{(N)}) \\
&= \quad -\frac{1}{\epsilon^2}\sum_{i=1}^{N}\lambda_i(\lambda_i+1)\Bigg[(\theta-\theta_0)\int_0^T u_{i\epsilon}(t)(du_{i\epsilon}(t)+\theta_0\lambda_i u_{i\epsilon}(t)dt) \\
&\qquad +\frac{1}{2}(\theta-\theta_0)^2\lambda_i\int_0^T u_{i\epsilon}^2(t)dt\Bigg] \qquad (4.\ 6)
\end{aligned}
$$

It can be checked that the MLE $\hat{\theta}_{N,T,\varepsilon}$ of θ based on $u_\epsilon^{(N)}$ satisfies the likelihood equation

$$
\alpha_{\epsilon,N} = \epsilon^{-1}\left(\hat{\theta}_{N,T,\varepsilon}-\theta_0\right)\beta_{\epsilon,N} \qquad (4.\ 7)
$$

when θ_0 is the true parameter,

$$
\alpha_{\epsilon,N} = \sum_{i=1}^{N}\lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)
$$

and

$$
\beta_{\epsilon,N} = \sum_{i=1}^{N}(\lambda_i+1)\,\lambda_i^2\int_0^T u_{i,\epsilon}^2(t)dt.
$$

Define

$$
R_{N,T,\epsilon} = \sum_{i=1}^{N}\frac{(\lambda_i)(\lambda_i+1)}{2\theta}\{v_i^2(1-e^{-2\theta\lambda_i T})+T\frac{\epsilon^2}{\lambda_i+1}\}.
$$

Then

$$
\sqrt{R_{N,T,\epsilon}}(\hat{\theta}_{N,T,\varepsilon}-\theta_0) = \frac{\epsilon\left\{\sum_{i=1}^{N}\lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)\right\}/\sqrt{R_{N,T,\varepsilon}}}{\left\{\sum_{i=1}^{N}(\lambda_i+1)\lambda_i^2\int_0^T u_{i\epsilon}^2(t)dt\right\}/R_{N,T,\varepsilon}}
$$

It can be checked that

$$
E_{\theta_0}\int_0^T u_{i\varepsilon}^2(t)dt < \infty
$$

We define the Bayes Estimator $\tilde{\theta}_{N,T,\varepsilon}$ as in Section 2 and assume that the conditions D(i) - D(v) stated there for the loss function $L(.,.)$ hold.

Main results :

Theorem 4.1 : Suppose $\theta > 0$. Under the conditions stated above, there exists positive constants C_1 and C_2, depending on $\theta, \varepsilon, N, T$ and $\|f\|$, such that for every $H > 0$,

$$
P_\theta^{\varepsilon,N}\left\{\left|R_{N,T,\varepsilon}^{\frac{1}{2}}\varepsilon^{-1}\left(\hat{\theta}_{N,T,\varepsilon}-\theta\right)\right| > H\right\} \le C_1 e^{-C_2 H^2},
$$

where $\hat{\theta}_{N,T,\varepsilon}$ is the MLE of the parameter and

$$R_{N,T,\varepsilon} = \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i+1)}{2\theta} \left\{ v_i^2 \left(1 - e^{-2\theta\lambda_i T}\right) + T\frac{\varepsilon^2}{\lambda_i+1} \right\}.$$

Theorem 4.2 : Suppose $\theta > 0$. Under the conditions stated above,there exists positive constants C_1 and C_2, depending on $\theta, \varepsilon, N, T$ and $\|f\|$, such that for every $H > 0$,

$$P_\theta^{\varepsilon,N} \left\{ \left| R_{N,T,\varepsilon}^{\frac{1}{2}} \varepsilon^{-1} \left(\tilde{\theta}_{N,T,\varepsilon} - \theta\right) \right| > H \right\} \leq C_1 e^{-C_2 H^2}$$

where $\tilde{\theta}_{N,T,\varepsilon}$ is the BE of the parameter with respect to the prior $\lambda(.)$ and the loss function $L(.,.)$ satisfies the conditions D(i)-D(v).

For proofs of Theorems given above, we need the following lemmas and Lemma 3.3. Define

$$Z_{N,T,\varepsilon}(u) = \frac{dP^{\varepsilon,N}_{\theta+\varepsilon u/\sqrt{R_{N,T,\varepsilon}}}}{dP_\theta^{\varepsilon,N}}.$$

Lemma 4.1 : Under the conditions stated above, there exists positive constants C_1 and C_2 such that

$$E_\theta^{\varepsilon,N}[Z^{\frac{1}{2}}_{N,T,\varepsilon}(u)] \leq C_1 e^{-C_2 u^2}$$

for $-\infty < u < \infty$.

Lemma 4.2 : Under the conditions stated above, there exists positive constant C_2 such that

$$E_\theta^{\varepsilon,N} \left\{ Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_1) - Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_2) \right\}^2 \leq C_2(u_1-u_2)^2$$

for $-\infty < u_1, u_2 < \infty$.

Proofs of these two lemmas are given below.

Proof of the Theorem 4.1 is similar to that of Theorem 3.1 following the procedure in Ibragimov and Khasminskii (1981).

Proof of the Theorem 4.2 can be given following the same remarks made earlier for Theorem 3.2.

Proof of Lemma 4.1: Observe that

$$
\begin{aligned}
Z_{N,T,\varepsilon}(u) &= \frac{dP^{\varepsilon,N}_{\theta+\varepsilon u/\sqrt{R_{NT\varepsilon}}}}{dP^{\varepsilon,N}_{\theta}} \\
&= \exp\left[-\left\{\sum_{i=1}^{N} \frac{\lambda_i\sqrt{\lambda_i+1}u}{\sqrt{R_{N,T,\varepsilon}}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
&\quad \left.\left. +\frac{1}{2}\frac{u^2}{R_{N,T,\varepsilon}}\sum_{i=1}^{N}\lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_\theta^{\varepsilon,N}\left(Z^{\frac{1}{2}}_{N,T,\varepsilon}(u)\right) &= E_\theta^{\varepsilon,N}\exp\left[-\left\{\sum_{i=1}^{N} \frac{\lambda_i\sqrt{\lambda_i+1}u}{2\sqrt{R_{N,T,\varepsilon}}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
&\quad \left.\left. +\frac{1}{4}\frac{u^2}{R_{N,T,\varepsilon}}\sum_{i=1}^{N}\lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right].
\end{aligned}
$$

Proceeding as in Lemma 3.1, we get

$$
\begin{aligned}
\left[E_\theta^{\varepsilon,N}\left(Z^{\frac{1}{2}}_{N,T,\varepsilon}(u)\right)\right]^4 &\leq E_\theta^{\varepsilon,N}\exp\left[-\frac{1}{3}\left\{\sum_{i=1}^{N}\frac{u^2}{R_{N,T,\varepsilon}}\int_0^T\lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right] \\
&\leq E_\theta^{\varepsilon,N}\exp\left[\frac{u^2}{3R_{N,T,\varepsilon}}\sum_{i=1}^{N}\lambda_i^2(\lambda_i+1)\left\{\int_0^t -u_{i0}^2(t)dt\right.\right. \\
&\quad \left.\left. +\int_0^T |2u_{i0}(t)|\,|u_{i\varepsilon}(t)-u_{i0}(t)|dt\right\}\right]. \qquad (4.\ 8)
\end{aligned}
$$

From the equation (4.3) and by the Ito's lemma, we get that

$$d\left(u_{i\varepsilon}(t)e^{\theta\lambda_i t}\right) = \frac{\epsilon}{\sqrt{\lambda_i+1}}e^{\theta\lambda_i t}dW_i(t)$$

or

$$u_{i\varepsilon}(t)e^{\theta\lambda_i t} - v_i = \int_0^t \frac{\epsilon}{\lambda_i+1}e^{\theta\lambda_i s}dW_i(s).$$

An application of Lemma 1.13 of Kutoyants (1998), p.26 shows that

$$\sup_{0\leq t\leq T}|u_{i\varepsilon}(t)-u_{i0}(t)| \leq e^{|\theta\lambda_i|T}\frac{\varepsilon}{\sqrt{\lambda_i+1}}\sup_{0\leq t\leq T}|W_i(t)|.$$

Note that

$$u_{i0}(t) = v_i e^{-\theta\lambda_i t}.$$

and hence

$$\int_0^T |u_{i0}(t)|\,|u_{i\varepsilon}(t) - u_{i0}(t)|dt \leq e^{|\theta\lambda_i|T} \int_0^T |v_i| e^{-\theta\lambda_i t} \frac{\epsilon}{\sqrt{\lambda_i + 1}} \sup_{0 \leq s \leq T} |W_i(s)| dt.$$

From (4.8), we obtain that

$$\begin{aligned}
& E_\theta^{\varepsilon,N}\left[Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)\right] \\
\leq\ & \exp\left[-C\sum_{i=1}^{N}\frac{u^2}{R_{N,T,\varepsilon}}\lambda_i^2(\lambda_i+1)v_i^2\left(\frac{1-e^{-2\theta\lambda_i T}}{2\theta\lambda_i}\right)\right] \\
& \times E_\theta^{\varepsilon,N}\left\{\exp\left[2\sum_{i=1}^{N}\frac{u^2}{\theta\lambda_i R_{N,T,\varepsilon}}\lambda_i^2(\lambda_i+1)|v_i|\frac{\varepsilon}{\sqrt{\lambda_i+1}}\eta_i e^{|\theta\lambda_i|T}(1-e^{-\theta\lambda_i T})\right]\right\} \\
\leq\ & \exp\left[-C\sum_{i=1}^{N}\frac{u^2}{R_{N,T,\varepsilon}}\frac{\lambda_i(\lambda_i+1)}{2\theta}v_i^2\left(1-e^{-2\theta\lambda_i T}\right)\right] \\
& \times E_\theta^{\varepsilon,N}\exp\left[\frac{2u^2}{\theta R_{N,T,\varepsilon}}\sum_{i=1}^{N}\lambda_i(\lambda_i+1)^{\frac{1}{2}}|v_i|\varepsilon\eta_i e^{|\theta\lambda_i|T}(1-e^{-\theta\lambda_i T})\right], \\
& (\text{where } \ \eta_i = \sup_{0\leq t\leq T}|W_i(t)|) \\
\leq\ & \exp\left[-C_1\sum_{i=1}^{N}\frac{u^2}{R_{N,T,\varepsilon}}\lambda_i(\lambda_i+1)v_i^2\right] \\
& \times\prod_{i=1}^{N}E_\theta^{\varepsilon,N}\left(\exp\left(e^{|\theta\lambda_i|T}(1-e^{-\theta\lambda_i T})\frac{u^2}{R_{N,T,\varepsilon}}\lambda_i(\lambda_i+1)^{\frac{1}{2}}|v_i|\varepsilon\eta_i\right)\right) \\
\leq\ & \exp\left[-C_1\sum_{i=1}^{N}\frac{u^2}{R_{N,T,\varepsilon}}i^4v_i^2\right]\exp\left[\frac{u^4}{R_{N,T,\varepsilon}^2}T\varepsilon^2\sum_{i=1}^{N}e^{2|\theta\lambda_i T|}(1-e^{-\theta\lambda_i T})^2 i^6 v_i^2\right] \\
\leq\ & e^{-C_2u^2}\exp\left[\frac{u^4}{R_{N,T,\varepsilon}^2}T\varepsilon^2\sum_{i=1}^{N}i^6v_i^2e^{2|\theta\lambda_i|T}(1-e^{-\theta\lambda_i T})^2\right]
\end{aligned}$$

for some positive constant C_2 depending on $\theta, \varepsilon, T, N$ and $||f||$ since

$$R_{N,T,\varepsilon} \geq C\sum_{i=0}^{N} i^4 v_i^2.$$

Hence the above inequality implies that

$$E_\theta^{\varepsilon,N}[Z_{N,T,\varepsilon}^{\frac{1}{2}}(u)] \leq C_3 e^{-C_2u^2}.$$

Proof of Lemma 4.2:

Proceeding as in Lemma 3.2, we have

$$
\begin{aligned}
& E_\theta^{\varepsilon,N}\left\{Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_1) - Z^{\frac{1}{2}}_{N,T,\varepsilon}(u_2)\right\}^2 \\
\le\ & 2\left[1 - E_\theta^{\varepsilon,N}\left\{Z^{\frac{1}{2}}_{NT\varepsilon}(u_1) Z_{NT\varepsilon}(u_2)\right\}\right] \\
=\ & 2\left[1 - E_{\theta_1}^{\varepsilon,N}\exp\left\{-\frac{1}{2}\sum_{i=1}^{N}\frac{(u_2-u_1)\lambda_i\sqrt{\lambda_i+1}}{R^{\frac{1}{2}}_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right. \\
& \left.\left. -\frac{1}{4}\frac{(u_2-u_1)^2}{R_{N,T,\varepsilon}}\sum_{i=1}^{N}\int_0^T \lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right] \\
\le\ & 2\left[1 - \exp\left\{E_{\theta_1}^{\varepsilon,N}\left\{-\frac{1}{2}\sum_{i=1}^{N}\frac{(u_2-u_2)\lambda_i\sqrt{\lambda_i+1}}{R^{\frac{1}{2}}_{N,T,\varepsilon}}\int_0^T u_{i\varepsilon}(t)dW_i(t)\right.\right.\right. \\
& \left.\left.\left. -\frac{1}{4}\frac{(u_2-u_1)^2}{R_{N,T,\varepsilon}}\sum_{i=1}^{N}\int_0^T \lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right]\right. \quad \text{(by Jensen's inequality)} \\
=\ & 2\left[1 - \exp\left\{-\frac{(u_2-u_1)^2}{4R_{N,T,\varepsilon}}E_{\theta_1}^{\varepsilon,N}\sum_{i=1}^{N}\lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right\}\right] \\
\le\ & 2\left(\frac{(u_2-u_1)^2}{4R_{N,T,\varepsilon}}E_{\theta_1}^{\varepsilon,N}\sum_{i=1}^{N}\int_0^T\lambda_i^2(\lambda_i+1)u_{i\varepsilon}^2(t)dt\right) \\
& \text{(since } 1-e^{-x}\le x\text{)} \\
=\ & \frac{(u_2-u_1)^2}{2R_{N,T,\varepsilon}}\sum_{i=1}^{N}\lambda_i^2(\lambda_i+1)\int_0^T Eu_{i\varepsilon}^2(t)dt \\
\le\ & \frac{(u_2-u_1)^2}{2\theta R_{N,T,\varepsilon}}\left(\sum_{i=1}^{N}\lambda_i(\lambda_i+1)v_i^2(1-e^{-2\theta\lambda_i T}) + T\sum_{i=1}^{N}\lambda_i\right) \\
& \text{(following Hübner et. al. (1993), p.158)} \\
\le\ & C_1(u_2-u_1)^2\frac{\sum_{i=1}^{N} i^4 v_i^2(1-e^{-2\theta\lambda_i T}) + Ti^3}{\sum_{i=1}^{N} i^4 v_i^2(1-e^{-2\theta\lambda_i T})} \\
& \text{(since } R_{N,T,\varepsilon} = \sum_{k=1}^{n}\frac{\lambda_k(\lambda_k+1)}{2\theta}v_k^2(1-e^{-2\theta\lambda_k T}) + \frac{T\varepsilon^2}{\lambda_k+1} \ge \sum_{k=1}^{N}k^4 v_k^2(1-e^{-2\theta\lambda_k T})\text{)} \\
\le\ & C_2(u_2-u_1)^2
\end{aligned}
$$

for some positive constant C_2 depending on $\theta, \varepsilon, N, T$ and $||f||$.

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