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Hajek-Renyi type inequality for some nonmonotonic functions of associated random variables

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Abstract

Let $\{Y_n, n \geq 1\}$ be a sequence of nonmonotonic functions of associated random variables. We derive a Newman and Wright (1981) type of inequality for the maximum of partial sums of the sequence $\{Y_n, n \geq 1\}$ and a Hajek-Renyi type inequality for nonmonotonic functions of associated random variables under some conditions.

Key Words and phrases : Associated sequences, nonmonotonic functions, Hajek-Renyi type inequality.

1 Introduction

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of associated random variables defined on it. A finite collection $\{X_1, X_2, \ldots, X_n\}$ is said to be associated if for every pair of functions $h(\mathbf{x})$ and $g(\mathbf{x})$ from R^n to R, which are nondecreasing componentwise,

$$\text{Cov}(h(\mathbf{X}), \mathbf{g}(\mathbf{X})) \geq \mathbf{0},$$

whenever it is finite, where $\mathbf{X} = (X_1, X_2, ..., X_n)$. The infinite sequence $\{X_n, n \geq 1\}$ is said to be associated if every finite subfamily is associated.

Associated random variables are of considerable interest in reliability studies (cf. Esary, Proschan and Walkup (1967), Barlow and Proschan (1981)), statistical physics (cf. Newman (1980, 1983)) and percolation theory (cf. Cox and Grimmet (1984)). For an extensive review of several probabilistic and statistical results for associated sequences, see Roussas (1999) and Prakasa Rao and Dewan (2001).

Newman and Wright (1981) proved an inequality for maximum of partial sums and Prakasa Rao (2002) proved the Hajek-Renyi type inequality for associated random variables. Esary et al (1967) proved that monotonic functions of associated random variables are associated. Hence one can easily extend the above mentioned inequalities to monotonic functions of associated random variables. We now generalise above results to some nonmonotonic functions of associated random variables.

In Section 2, we discuss some preliminaries. Two inequalities are proved for nonmonotonic functions of associated random variables in Section 3. Some applications of these inequalities are discussed in Section 4.

2 Preliminaries

Let us discuss some definitions and results which will be useful in proving our main results.

DEFINITION 2.1 : (Newman (1984)) Let f and f_1 be two complex-valued functions on R^n. Then we say that $f << f_1$ if $f_1 - Re(e^{i\alpha} f)$ is componentwise nondecreasing for all real α.

REMARK 2.2 : (Newman (1984)) Let f and f_1 be two real-valued functions. Then $f << f_1$ if and only if $f_1 + f$ and $f_1 - f$ are both nondecreasing componentwise. In particular, if $f << f_1$ and $f,\ f_1$ are functions of a single variable, then f_1 will be nondecreasing.

Dewan and Prakasa Rao (2001) observed the following.

REMARK 2.3 : Suppose that f is real-valued function. Then $f << f_1$ for f_1 real iff for $x < y$,

$$f(y) - f(x) \leq f_1(y) - f_1(x) \tag{2.1}$$

and

$$f(x) - f(y) \leq f_1(y) - f_1(x). \tag{2.2}$$

It is clear that these relations hold iff, for $x < y$,

$$|f(y) - f(x)| \leq f_1(y) - f_1(x). \tag{2.3}$$

REMARK 2.4 : Suppose $f << \tilde{f}$. Following the Remark 2.3, we must have, for $x < y$,

$$|f(y) - f(x)| \leq \tilde{f}(y) - \tilde{f}(x). \tag{2.4}$$

Let $\tilde{f}(x) = cx$ for some constant $c > 0$. Then $f << \tilde{f}$ iff, for $x < y$,

$$|f(y) - f(x)| \leq c(y - x) \tag{2.5}$$

which indicates that f is Lipschitzian. A sufficient condition for (2.5) to hold is that

$$\sup_x |f'(x)| \leq C. \tag{2.6}$$

Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables. Let

$$\begin{array}{lrcl}
(i) & Y_n & = & f_n(X_1, X_2, \ldots), \\
(ii) & \tilde{Y}_n & = & \tilde{f}_n(X_1, X_2, \ldots), \\
(iii) & f_n & << & \tilde{f}_n, \text{ and} \\
(iv) & E(Y_n^2) < \infty & , & E(\tilde{Y}_n^2) < \infty, \text{ for } n \in N.
\end{array} \tag{2.7}$$

The functions $f_n, \tilde{f}_n$ are assumed to be real and depend only on a finite number of $X_n's$. Let $S_n = \sum_{k=1}^n Y_k, \tilde{S}_n = \sum_{k=1}^n \tilde{Y}_k$. Matula (2001) proved the following result which will be useful in proving our results. He used them to prove the strong law of large numbers and the central limit theorem for nonmonotonic functions of associated random variables.

Lemma 2.5 : Suppose the conditions stated above in (2.7) hold. Then

$$\begin{array}{lrcl}
(i) & Var(f_n) & \leq & Var(\tilde{f}_n), \\
(ii) & |Cov(f_n, \tilde{f}_n)| & \leq & Var(\tilde{f}_n), \\
(iii) & Var(S_n) & \leq & Var(\tilde{S}_n), \\
(iv) & f_1 + f_2 + \ldots + f_n & << & \tilde{f}_1 + \tilde{f}_2 + \ldots + \tilde{f}_n, \\
(v) & Cov(f_1 + \tilde{f}_1, f_2 + \tilde{f}_2) & \leq & 4\ Cov(\tilde{f}_1, \tilde{f}_2), \text{ and} \\
(vi) & Cov(\tilde{f}_1 - f_1, \tilde{f}_2 - f_2) & \leq & 4\ Cov(\tilde{f}_1, \tilde{f}_2).
\end{array} \tag{2.8}$$

It is easy to see that

$$4\ Var(\tilde{f}_1) = Var(\tilde{f}_1 + f_1) + 2\ Cov(\tilde{f}_1 + f_1, \tilde{f}_1 - f_1) + Var(\tilde{f}_1 - f_1).$$

Note that the covariance term in the above equation is nonnegative since $\tilde{f}_1 + f_1$ and $\tilde{f}_1 - f_1$ are nondecreasing functions of associated random variables. Hence

$$Var(f_1 + \tilde{f}_1) \leq 4 \ \ Var(\tilde{f}_1)$$

and

$$Var(\tilde{f}_1 - f_1) \leq 4 \ \ Var(\tilde{f}_1).$$

For completeness, we now state the inequalities due to Newman and Wright (1981) and Prakasa Rao (2002) for associated random variables.

Lemma 2.6 : (Newman and Wright) : Suppose $X_1, X_2, \ldots, X_m$ are associated, mean zero, finite variance random variables and $M_m^* = \max(S_1^*, S_2^*, \ldots, S_m^*)$, where $S_n^* = \sum_{i=1}^n X_i$. Then

$$E((M_m^*)^2) \leq Var(S_m^*). \tag{2.9}$$

Remark 2.7 : Note that if $X_1, X_2, \ldots, X_m$ are associated random variables, then $-X_1, -X_2, \ldots, -X_m$ also form a set of associated random variables. Let $M_m^{**} = \max(-S_1^*, -S_2^*, \ldots, -S_m^*)$ and $\tilde{M}_m^* = \max(|S_1^*|, |S_2^*|, \ldots, |S_m^*|)$. Then $\tilde{M}_m^* = \max(M_m^*, M_m^{**})$ and $(\tilde{M}_m^*)^2 \leq (M_m^*)^2 + (M_m^{**})^2$ so that

$$E((\tilde{M}_m^*)^2) \leq 2 \ \ Var(S_m^*). \tag{2.10}$$

Lemma 2.8 : (Prakasa Rao): Let $\{X_n, n \geq 1\}$ be an associated sequence of random variables with $Var(X_j) = \sigma_j^2$ and $\{b_n, n \geq 1\}$ be a positive nondecreasing sequence of real numbers. Then, for any $\epsilon > 0$,

$$P(\max_{1 \leq k \leq n} |\frac{1}{b_n} \sum_{i=1}^k (X_i - E(X_i))| \geq \epsilon) \leq \frac{4}{\epsilon^2} [\sum_{j=1}^n \frac{Var(X_j)}{b_j^2} + \sum_{1 \leq j \neq k \leq n} \frac{Cov(X_j, X_k)}{b_j b_k}]. \tag{2.11}$$

3 Main results

We now extend the Newman and Wright's (1981) result to nonmonotonic functions of associated random variables satisfying conditions (2.7).

Theorem 3.1 : Let $Y_1, Y_2, \ldots, Y_m$ be as defined in (2.7) with zero mean and finite variances . Let $M_m = \max(|S_1|, |S_2|, \ldots, |S_m|)$. Then

$$E(M_m^2) \leq 20 \ \ Var(\tilde{S}_m). \tag{3.1}$$

Proof: Observe that

$$\begin{aligned} & \max_{1 \leq k \leq m} |S_k| \\ = & \max_{1 \leq k \leq m} |\tilde{S}_k - S_k - E(\tilde{S}_k) - \tilde{S}_k + E(\tilde{S}_k)| \\ \leq & \max_{1 \leq k \leq m} |\tilde{S}_k - S_k - E(\tilde{S}_k)| + \max_{1 \leq k \leq m} |\tilde{S}_k - E(\tilde{S}_k)| \end{aligned} \tag{3.2}$$

Note that $\tilde{S}_k - E(\tilde{S}_k)$ and $\tilde{S}_k - S_k - E(\tilde{S}_k)$ are partial sums of associated random variables each with mean zero. Hence using the results of Newman and Wright (1981), we get

$$\begin{aligned}
& E(M_m^2) \\
=\leq\ & E(\max_{1\leq k\leq m} |S_k|)^2 \\
\leq\ & 2[E(\max_{1\leq k\leq m} |\tilde{S}_k - S_k - E(\tilde{S}_k)|)^2 + E(\max_{1\leq k\leq m} |\tilde{S}_k - E(\tilde{S}_k)|)^2] \\
\leq\ & 4[Var(\tilde{S}_m - S_m) + Var(\tilde{S}_m)] \text{ (by Remark 2.7)} \\
\leq\ & 4[Var(2\tilde{S}_m) + Var(\tilde{S}_m)] \\
=\ & 20Var(\tilde{S}_m). \qquad (3.3)
\end{aligned}$$

We have used the fact that

$$\begin{aligned}
Var(2\tilde{S}_n) &= Var(\tilde{S}_n - S_n + \tilde{S}_n + S_n) \\
&= Var(\tilde{S}_n - S_n) + Var(\tilde{S}_n + S_n) + 2\ Cov(\tilde{S}_n + S_n, \tilde{S}_n - S_n).
\end{aligned} \qquad (3.4)$$

Since $\tilde{S}_n + S_n$ and $\tilde{S}_n - S_n$ are nondecreasing functions of associated random variables , it follows that $Cov(\tilde{S}_n + S_n, \tilde{S}_n - S_n) \geq 0$. Hence $Var(2\tilde{S}_n) \geq Var(\tilde{S}_n - S_n)$.

We now prove a Hajek-Renyi type inequality for some nonmonotonic functions of associated random variables satisfying conditions (2.7).

Theorem 3.2 : Let $\{Y_n, n \geq 1\}$ be sequence of nonmonotonic functions of associated random variables as defined in (2.7). Suppose that $Y_n << \tilde{Y}_n$, $n \geq 1$. Let $\{b_n, n \geq 1\}$ be a positive nondecreasing sequence of real numbers. Then for any $\epsilon > 0$,

$$P(\max_{1\leq k\leq n} |\frac{1}{b_n}\sum_{i=1}^{k}(Y_i - E(Y_i))| \geq \epsilon) \quad \leq \quad (80)\epsilon^{-2}[\sum_{j=1}^{n}\frac{Var(\tilde{Y}_j)}{b_j^2} + \sum_{1\leq j\neq k\leq n}\frac{Cov(\tilde{Y}_j, \tilde{Y}_k)}{b_j b_k}]. \qquad (3.5)$$

Proof: Let $T_n = \sum_{j=1}^{n}(Y_j - E(Y_j))$. Note that

$$\begin{aligned}
& P[\max_{1\leq k\leq n} |\frac{T_k}{b_k}| \geq \epsilon] \\
=\ & P[\max_{1\leq k\leq n} |\frac{\tilde{T}_k - T_k - E(\tilde{T}_k) - \tilde{T}_k + E(\tilde{T}_k)|}{b_k}| \geq \epsilon] \\
\leq\ & P[\max_{1\leq k\leq n} |\frac{\tilde{T}_k - T_k - E(\tilde{T}_k)}{b_k}| \geq \frac{\epsilon}{2}] + P[\max_{1\leq k\leq n} |\frac{\tilde{T}_k - E(\tilde{T}_k)|}{b_k}| \geq \frac{\epsilon}{2}] \\
\leq\ & (16)\epsilon^{-2}[\sum_{j=1}^{n}\frac{Var(\tilde{Y}_j - Y_j)}{b_j^2} + \sum_{1\leq j\neq k\leq n}\frac{Cov(\tilde{Y}_j - Y_j, \tilde{Y}_k - Y_k)}{b_j b_k}]
\end{aligned}$$

$$+ \quad (16)\epsilon^{-2}[\sum_{j=1}^{n} \frac{Var(\tilde{Y}_j)}{b_j^2} + \sum_{1\leq j\neq k\leq n} \frac{Cov(\tilde{Y}_j, \tilde{Y}_k)}{b_j b_k}]. \tag{3.6}$$

The result follows by applying the inequalities

$$Var(\tilde{Y}_j - Y_j) \leq 4 \ \ Var(\tilde{Y}_j)$$

and

$$Cov(\tilde{Y}_j - Y_j, \tilde{Y}_k - Y_k) \leq 4 \ \ Cov(\tilde{Y}_j, \tilde{Y}_k).$$

4 Applications

Let C denote a generic positive constant.

Theorem 4.1 : Let $\{Y_n, n \geq 1\}$ be sequence of nonmonotonic functions of associated random variables satisfying condition (2.7). Assume that

$$\sum_{j=1}^{\infty} Var(\tilde{Y}_j) + \sum_{1\leq j\neq k<\infty} Cov(\tilde{Y}_j, \tilde{Y}_k) < \infty. \tag{4.1}$$

Then $\sum_{j=1}^{\infty}(Y_j - EY_j)$ converges almost surely.

Proof: Without loss of generality, assume that $EY_j = 0$ for all $j \geq 1$. Let $\epsilon > 0$. Using Theorem 3.2 is easy to see that

$$\begin{aligned} P(\sup_{k,m\geq n}|T_k - T_m| \geq \epsilon) &\leq P(\sup_{k\geq n}|T_k - T_n| \geq \frac{\epsilon}{2}) + P(\sup_{m\geq n}|T_m - T_n| \geq \frac{\epsilon}{2}) \\ &\leq C \ \lim_{N\to\infty} P(\sup_{n\leq k\leq N}|T_k - T_n| \geq \frac{\epsilon}{2}) \\ &\leq C \ \epsilon^{-2}[\sum_{j=n}^{\infty} Var(\tilde{Y}_j) + \sum_{n\leq j\neq k<\infty} Cov(\tilde{Y}_j, \tilde{Y}_k)]. \end{aligned} \tag{4.2}$$

The last term tends to zero as $n \to \infty$ because of (4.1). Hence the sequence of random variables $\{T_n, n \geq 1\}$ is Cauchy almost surely which implies that T_n converges almost surely.

The following theorem proves the strong law of large numbers for nonmonotonic functions of associated random variables.

Theorem 4.2 : Let $\{Y_n, n \geq 1\}$ be sequence of nonmonotonic functions of associated random variables satisfying condition (2.7). Suppose that

$$\sum_{j=1}^{n} \frac{Var(\tilde{Y}_j)}{b_j^2} + \sum_{1\leq j\neq k\leq n} \frac{Cov(\tilde{Y}_j, \tilde{Y}_k)}{b_j b_k} < \infty$$

Then $\frac{1}{b_n}\sum_{j=1}^{n}(Y_j - EY_j)$ converges to zero almost surely as $n \to \infty$.

Proof: The proof is an immediate consequence of Theorem 4.2 and Kronecker Lemma (Chung (1974)).

For any random variable X and any constant $k > 0$, define $X^k = X$ if $|X| \leq k, X^k = -k$ if $X < -k$ and $X^k = k$ if $X > k$. The following theorem is an analogue of the three series theorem for nonmonotonic functions of associated random variables.

Theorem 4.3 : Let $\{Y_n, n \geq 1\}$ be sequence of nonmonotonic functions of associated random variables satisfying condition (2.7). Further suppose that there exists a constant $k > 0$ such that $Y_n^k << \tilde{Y}_n^k$ and

$$\sum_{n=1}^{\infty} P[|Y_n| \geq k] < \infty, \tag{4.3}$$

$$\sum_{n=1}^{\infty} E(Y_n^k) < \infty, \tag{4.4}$$

$$\sum_{j=1}^{\infty} Var(\tilde{Y}_j^k) + \sum_{1 \leq j \neq j' < \infty} Cov(\tilde{Y}_j^k, \tilde{Y}_{j'}^k) < \infty. \tag{4.5}$$

Then $\sum_{n=1}^{\infty} Y_n$ converges almost surely.

Theorem 4.4: Let $\{Y_n, n \geq 1\}$ be sequence of nonmonotonic functions of associated random variables satisfying condition (2.7). Suppose

$$\sum_{j=1}^{n} \frac{Var(\tilde{Y}_j)}{b_j^2} + \sum_{1 \leq j \neq k \leq n} \frac{Cov(\tilde{Y}_j, \tilde{Y}_k)}{b_j b_k} < \infty. \tag{4.6}$$

Then, for any $0 < r < 2$,

$$E[\sup_n (\frac{|T_n|}{b_n})^r] < \infty. \tag{4.7}$$

Proof: Note that

$$E[\sup_n (\frac{|T_n|}{b_n})^r] < \infty.$$

if and only if

$$\int_1^{\infty} P(\sup_n (\frac{|T_n|}{b_n})^r > t^{1/r}) dt < \infty.$$

The above condition holds because of Theorem 3.2 and condition (4.6) and hence the result in (4.7) holds.

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